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Certain Applications of Analytic Functions Associated in Complex BB Differential Equations

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تطبيقات معينة للدوال التحليلية بالمعادلات التفاضلية BB المعقدة

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الملخص:

في هذه الورقة البحثية، نقدم بعض التطبيقات من المؤثر التفاضلي المعمم في حقل من نظرية الدوال الهندسية باستعمال المفاهيم التفاضل والتكامل الكسري. بهذا المؤثر التفاضلي $(I^m(\lambda_1,\lambda_2,l,n)f(z))$ نعرف فصل جديد من الدوال التحليلية بواسطة المؤثر التفاضلي، وكذلك نحدد حل للمعادلة التفاضلية المركبة Briot–Bouquet باستعمال المؤثر المعرف. الغرض الرئيسي من هذه الورقة هو حل المعادلة التفاضلية المركبة BB باستخدام المؤثر ونناقش بعض التطبيقات من المؤثر التفاضلي في قرص الوحدة المفتوح.

الكلمات المفتاحية: التفاضل والتكامل الكسرى، الدوال التحليلية، المؤثر التفاضلي، قرص الوحدة، معادلة BB.

A B S T R A C T:

In this paper, we introduced some applications of generalized derivative operator in the field of geometric function theory by using the concepts of fractional calculus. With this operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$, we defined a new class of normalized analytic functions and establishes the solution of the complex Briot–Bouquet differential equation by using this operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$. The main purpose of this paper is to solve the complex BB differential equation by using operator and we discussed some applications of a generalized derivative operator in the open disk.

Keywords: Analytic functions, BB equation, Fractional calculus, Generalized derivative operator and Unit disk.

Introduction:

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (1)$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \ ; \ |z| < 1\},$$

Moreover, satisfy the normalization condition

$$f(0) = 0 \quad , \quad \acute{f}(0) = 1.$$

Furthermore, we denote by S the subclass of $\mathcal{A}$ consisting of functions of the form (1) , which are also univalent in $\mathbb{D}$.

For two functions, f , $g \in \mathcal{A}$ we say that f subordinated to g , written as:

$$f(z) \prec g(z),$$

or equivalently

$$f(z) = g\big(w(z)\big),$$

where , $w(z)$ is the schwarz function in $\mathbb{D}$ along with the condition $w(0) = 0$ and $|w(z)| < 1$. If g is univalent in $\mathbb{D}$, then $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. (see (Goodman, 1983))

The class of starlike functions (S^*):

$$Re\left(\frac{z\acute{f}(z)}{f(z)}\right) > 0 \ , z \in \mathbb{D}.$$

And convex functions ($\mathcal{C}$):

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \ , z \in \mathbb{D}.$$

Related to classes S^* and $\mathcal{C}$, we define the class $\mathcal{P}$ of analytic functions

$p \in \mathcal{P}$, which are normalized by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k.$$

Such that

$$Re\, p(z) > 0 \text{ in } \mathbb{D} \text{ and } p(0) = 1.$$

The convolution $(*)$ of f and g defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ ,$$

where,

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \ \ , \ (z \in \mathbb{D}).$$

A function $h \in \mathcal{A}$ is called bounded turning if it satisfies the condition

$$Re\big(f'(z)\big) > 0.$$

For $0 \leq t < 1$, let $w(t)$ denote the class of functions f of the form (1), so that $Re\,\big(f'(z)\big) > t$ in $\mathbb{D}$. The functions in $w(t)$ are called functions of bounded turning(Goodman, 1983). Nashiro-warschowski Theorem (Goodman, 1983) stated that the functions in $w(t)$ are univalent and also close to convex in $\mathbb{D}$ .Now recall the definition of $\mathcal{T}$ of bounded turning functions and can be defined as:

$$\mathcal{T} = \left\{f \in \mathcal{A}\,;\ f'(z) < \frac{1+z}{1-z} \ \ ; \ z \in \mathbb{D}\right\}.$$

Definition 1 (Amer & Darus, 2011)

The shifted factorial $(c)_k$ can be defined as:

$$(c)_k = c(c+1) \ldots\ldots (c+k-1) \ \ if \ k \in \mathbb{N} = \{1,2,3, \ldots\}, c \in \mathbb{C} - \{0\}.$$

and

$$(c)_k = 1 \ \ if \ k = 0.$$

Definition 2 (Amer & Darus, 2011).

The $(c)_k$ can be expressed in terms of the Gamma function as:

$$(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)} \ , (k \in \mathbb{N}).$$

In order to derive the generalized derivative operator (Amer & Darus, 2011), we define the analytic function

$$\phi^m(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \qquad (2)$$

where $m \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and $\lambda_2 , \lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$.

Now , In (Amer & Darus, 2011) the authors introduce the generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ As the following:

Definition 3 (Amer & Darus, 2011).

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n): A \longrightarrow A$

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = \phi^m(\lambda_1,\lambda_2,l)(z) * R^n f(z) \quad , (z \in \mathbb{D}) \qquad (3)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,\dots.\}$ and $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator (Ruscheweyh, 1975) ,and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k \ , (n \in \mathbb{N}_0, z \in \mathbb{D}),$$

where

$$c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$$

If f is given by (1) , then we easily find from the equality (3) that

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\big(1+\lambda_2(k-1)\big)^m} c(n,k) a_k z^k,$$

where $n,m \in \mathbb{N}_0 = \{0,1,2,\dots.\}$, $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$,
$$c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$$

Special cases of this operator includes:

- the Ruscheweyh derivative operator (Ruscheweyh, 1975) in the cases:

$$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n)$$
$$\equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n,$$

- the Salagean derivative operator (Salagean, 1983)

$$I^{m+1}(1,0,0,0) \equiv S^n,$$

- The generalized Ruscheweyh derivative operator (Shaqsi & Darus, 2008):

$$I^2(\lambda_1,0,0,n) \equiv R_\lambda^n \ ,$$

- The generalized Salagean derivative operator introduced by
Al-Oboudi (Al-Oboudi, 2004):

$$I^{m+1}(\lambda_1, 0,0,0) \equiv S_\beta^n \,,$$

- The generalized Al-Shaqsi and Darus derivative operator (Al-Shaqsi & Darus, 2008):

$$I^{m+1}(\lambda_1, 0,0, n) \equiv R_{\lambda,\beta}^n \,,$$

- The Al-Abbadi and Darus generalized derivative operator (Al-Abbadi & Darus, 2009):

$$I^m(\lambda_1, \lambda_2, 0, n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m},$$

and finally

- The Catas derivative operator (Catas & Borsa, 2009):

$I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda, \beta, l).$

Using simple computation one obtains the next result.

$$(1+l)I^{m+1}(\lambda_1, \lambda_2, l, n)f(z) = (1+l-\lambda_1)[I^m(\lambda_1, \lambda_2, l, n) * \phi^1(\lambda_1, \lambda_2, l)(z)]f(z)$$
$$+\lambda_1 z[(I^m(\lambda_1, \lambda_2, l, n) * \phi^1(\lambda_1, \lambda_2, l)(z)]'.$$

Where $(z \in \mathbb{D})$ and $\phi^1(\lambda_1, \lambda_2, l)(z)$ an analytic function and form (2) given by

$$\phi^1(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k.$$

Definition 4.

A function $f \in \mathcal{A}$**,** is the class $S^*(\lambda_1, \lambda_2, l, n, m, \delta)$ if and only if

$$S^*(\lambda_1, \lambda_2, l, n, m, \delta) = \left\{ f \in \mathcal{A} : \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \delta(z) \ , \ \delta(0) = 1 \right\}.$$

Lemma 1(Miller & Mocanu, 2000):

For $q\epsilon\mathbb{C}$ and a positive integer m , the class of analytic functions is given by

$$\mathcal{G}(f, m) = \{f : f(z) = q + q_m z^m + q_{m+1} z^{m+1} + \cdots\}.$$

i. Let $j \in R$ .Then

$$Re\big(f(z) + jzf'(z)\big) > 0 \longrightarrow Re\big(f(z)\big) > 0$$

Moreover, $j > 0$ and $f\epsilon\mathcal{G}(1, m)$, then there is constant $\alpha > 0$ and $w > 0$ such that

$w = w(j, \alpha, m)$, and

$$f(z) + jzf'(z) \prec \left(\frac{1+z}{1-z}\right)^w \longrightarrow f(z) = \left(\frac{1+z}{1-z}\right)^\alpha.$$

ii. For $f\epsilon\mathcal{G}(1, m)$, and for fixed real number $j > 0$ and let $d \in [0,1)$, so that

$$Re\left(f^2(z) + 2f(z)\left(zf'(z)\right)\right) > d \ \longrightarrow Re\left(f(z)\right) > j$$

iii. Let $f\epsilon\mathcal{G}(f, m)$, with Re(f)>0 , then

$$Re(f(z) + zf'(z) + z^2 f"(z)) \ > 0 \,,$$

or for $k: \mathbb{D} \longrightarrow R$, such that

$$Re\left(f(z) + \left(\frac{zf'(z)}{f(z)}\right) k(z)\right) > 0 \,.$$

Then

$$Re\left(f(z)\right) > 0 \,.$$

Now we find the upper bounds of the operator $\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}$ by using the exponential integral in $\mathbb{D}$, which provided $z \in S^*\,(\lambda_1, \lambda_2, l, n, m, \delta)$.

Theorem 1:

Let $z \in S^*\,(\lambda_1, \lambda_2, l, n, m, \delta)$,where $\delta(z)$ is convex in$\mathbb{D}$. Then,

$$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \prec z\, exp \int_0^z \frac{\delta(w(\beta))-1}{\beta}\, d\beta \,, \qquad (4)$$

where, $w(z)$ is analytic in $\mathbb{D}$ having condition $w(0) = 0$ and $|w(z)| \ < \ 1$.

Furthermore, for $|z| = \rho$,we have

$$\exp \int_0^1 \frac{\delta(w(-\beta))-1}{\rho}\, d\rho \ \le \left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right| \le \exp \int_0^1 \frac{\delta(w(\beta))-1}{\rho}\, d\rho \,.$$

Proof.

From definition 4, we get

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \delta(z) \,,$$

from definition subordination, we get

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \delta\left(w(z)\right), z \in \mathbb{D} \,,$$

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1 = \delta\left(w(z)\right) - 1 \,,$$

$$z\left[\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - \frac{1}{z}\right] = \delta\left(w(z)\right) - 1 \,,$$

by divided z

$$\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}-\frac{1}{z}=\frac{\delta(w(z))-1}{z},$$

$$\int_0^z \frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\,dz-\int_0^z \frac{1}{z}\,\mathrm{dz}=\int_0^z \frac{\delta(w(z))-1}{z}\,dz,$$

$$\ln I^m(\lambda_1,\lambda_2,l,n)f(z)-\ln z=\int_0^z \frac{\delta(w(z))-1}{z}\,dz,$$

$$\ln\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}=\int_0^z \frac{\delta(w(z))-1}{z}\,dz \qquad (5)$$

$$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}=\exp\int_0^z \frac{\delta(w(z))-1}{z}\,dz.$$

By the definition of subordination, we get

$$I^m(\lambda_1,\lambda_2,l,n)f(z)\prec z\exp\int_0^z \frac{\delta(w(z))-1}{z}\,dz.$$

Hence (4) is proved.

Note that the function $\delta(z)$ convex and symmetric with respect to real axis, where , $0<|z|<\rho<1$. That is

$$\delta(-\rho\,|z|)\le \mathrm{Re}\{\delta(w(\rho\,z))\}\le \delta(\rho\,|z|)\quad,\quad (0<\rho<1\ ,z\in\mathbb{D}).$$

Then we have the inequalities

$$-|z|>-1 \qquad \rightarrow\ |z|<1,$$

$$\delta(-\rho)\le\delta(-\rho\,|z|)\ ,\ \delta(\rho\,|z|)\le\delta(\rho).$$

Consequently, we get

$$\delta(w(-\rho\,|z|))\le \mathrm{Re}\{\delta(w(\rho\,z))\}\le \delta(w(\rho\,|z|)),$$

$$\delta(w(-\rho\,|z|))-1\le \mathrm{Re}\{\delta(w(z))\}-1\le \delta(w(\rho\,|z|))-1,$$

$$\frac{\delta(w(-\rho\,|z|))-1}{\rho}\le\frac{\mathrm{Re}\{\delta(w(z))\}-1}{\rho}\le\frac{\delta(w(\rho\,|z|))-1}{\rho},$$

$$\int_0^1 \frac{\delta(w(-\rho\,|z|))-1}{\rho}\,d\rho\le\int_0^1 \frac{\mathrm{Re}\{\delta(w(z))\}-1}{\rho}\,d\rho\le\int_0^1 \frac{\delta(w(\rho\,|z|))-1}{\rho}\,d\rho,$$

from (5), we obtain

$$\int_0^1 \frac{\delta(w(-\rho\,|z|))-1}{\rho}\,d\rho\le \ln\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right|\le\int_0^1 \frac{\delta(w(\rho\,|z|))-1}{\rho}\,d\rho,$$

$$\exp\int_0^1 \frac{\delta(w(-\rho\,|z|))-1}{\rho}\,d\rho\le\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right|\le\exp\int_0^1 \frac{\delta(w(\rho\,|z|))-1}{\rho}\,d\rho,$$

hence , we have

$$\exp\int_0^1 \frac{\delta(w(-\rho))-1}{\rho}\,d\rho\le\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right|\le\exp\int_0^1 \frac{\delta(w(\rho))-1}{\rho}\,d\rho.$$

If δ is convex univalent and $\delta(0) = 1$, then we find a condition on f to be in the class $S^*(\lambda_1, \lambda_2, l, n, m, \delta)$.

Theorem 2.

If $f \in \mathcal{A}$ satisfy the subordination condition

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left[2+\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}-\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right] \prec \delta(z),$$

Then , $f \in S^*(\lambda_1, \lambda_2, l, n, m, \delta)$.

Proof. Let

$$p(z) = \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)},$$

and let $p(z) = 1$ and from lemma (1) , part (i)

$$p(z) + p(z)z\left(p(z)\right)' \prec \delta(z),$$

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + z\left[\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)\left[z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''+\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'\right]}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)^2}\right] -$$
$$z\left[\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)^2}\right] \prec \delta(z),$$

Then

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)} + \frac{z^2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)} + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)} - \frac{z^2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'^2}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)^2} =$$
$$\frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \cdot \left[2+\frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)''}{I^m(\lambda_1,\lambda_2,l,n)f(z)'}-\frac{zI^m(\lambda_1,\lambda_2,l,n)f(z)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right] \prec \delta(z),$$

this implies that δ

$$\operatorname{Re} p(z) > 0 \Rightarrow p(z) = \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} \prec \delta(z),$$

that is

$$f \in S^*(\lambda_1, \lambda_2, l, n, m, \delta).$$

For example:

Let:

$$\frac{z f'(z)}{f(z)} = \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}.$$

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=\frac{z}{(1-z)^2} \quad ,f(z)\in\mathcal{A}.$$

Then the solution of $\frac{z\,f'(z)}{f(z)}=\frac{1+z}{1-z}$ is formulated as follows:

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=\frac{z}{(1-z)^2} \quad ,f(z)\in\mathcal{A}$$

We have $I^m(\lambda_1,\lambda_2,l,n)f(z)=\frac{z}{(1-z)^2}=\sum_{k=1}^{\infty}kz^k.$

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=\frac{\sum_{k=1}^{\infty}k^2z^k}{\sum_{k=1}^{\infty}kz^k},$$

$$\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=1+2z+2z^2+2z^3+\cdots$$

$$\therefore\ \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=1+2\sum_{k=1}^{\infty}z^k=\frac{1+z}{1-z} \quad .$$

Moreover, the solution of the equation

$$f(z)+\frac{z\,f'(z)}{f(z)}=\frac{1+z}{1-z},$$

is approximated to

$$f(z)=\frac{z}{1-z}\ .$$

We have

$$f(z)=\frac{z}{1-z}=\sum_{k=1}^{\infty}z^k\ ,$$

$$f(z)+\frac{z\,f'(z)}{f(z)}=\sum_{k=1}^{\infty}z^k+\frac{z[\sum_{k=1}^{\infty}kz^{k-1}]}{\sum_{k=1}^{\infty}z^k}=\sum_{k=1}^{\infty}z^k+\frac{\sum_{k=1}^{\infty}kz^k}{\sum_{k=1}^{\infty}z^k}\ ,$$

$$=\ \sum_{k=1}^{\infty}z^k+1+\sum_{k=1}^{\infty}z^k\ ,$$

$$\therefore\ f(z)+\frac{z\,f'(z)}{f(z)}=1+2\sum_{k=1}^{\infty}z^k=\frac{1+z}{1-z}\ .$$

Applications of Generalized Derivative Operator:

The solution of complex Briot-Bouquet(BB) differential equation is established in (Miller & Mocanu, 2000). We produce a presentation of our results in complex BB differential equations, and the class of BB differential equations is a link of differential equations whose consequences are visible in the complex plane. The study new special functions as follows (Khan et al., 2023):

$$z\,f(z)+(1-z)\frac{zf'(z)}{f(z)}=\varphi(z)\,, \qquad (6)$$

$\varphi(z)=f(0)\ ,z\in[0,1]\,.$

In (Miller & Mocanu, 2000), many new applications of these equations in Geometric Function Theory have been discussed.

Now, we investigate (5) by using the operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ and find its solutions by applying the subordination relations. The operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ propagates the complex BB differential equation as follows:

$$zI^m(\lambda_1, \lambda_2, l, n)f(z) + (1-z)\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) = \varphi(z)\,, \qquad (7)$$

where,

$$\varphi(z) = f(0)\ , z \in \mathbb{D}.$$

A trivial solution of (7) is given when $z = 1$, our investigation concerns the case with $f \in \mathcal{A}$ and $z = 0$.

Theorem 3

Let we have equation (7) with $z = 0$.If $\delta(z)$ is convex in $\mathbb{D}$ .Then

$$\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z} \prec z\, exp \int_0^z \frac{\alpha(w(\beta))-1}{\beta}\, d\beta\,, \qquad (8)$$

where $w(z)$ is analytic in $\mathbb{D}$ and $w(0) = 0$ and $|w(z)| < 1$.

Proof.

From equation (7), and $f(z) \in \mathcal{A}$. Then, we get

$$Re\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) > 0.$$

From definition 3 we get

$$Re\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) > 0\,,$$

$$\Leftrightarrow Re\left(\frac{z\left(z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k\right)'}{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k}\right) > 0\,,$$

$$\Leftrightarrow Re\left(\frac{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}kc(n,k)a_k z^{k-1}\right)}{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k}\right) > 0\,,$$

$$\Leftrightarrow Re\left(\frac{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}kc(n,k)a_k z^k}{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k}\right) > 0\,,$$

$$\Leftrightarrow Re\left(\frac{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}kc(n,k)a_k z^{k-1}\right)}{z\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^{k-1}\right)}\right) > 0,$$

$$\Leftrightarrow Re\left(\frac{1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^{m}}kc(n,k)a_k}{1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^{m}}c(n,k)a_k}\right)>0\,, \quad z\longrightarrow 1^{+}$$

$$\Leftrightarrow Re\left(1+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^{m}}kc(n,k)a_k\right)>0.$$

Moreover ,by the definition of $I^m(\lambda_1,\lambda_2,l,n)f(z)$,we indicate that $I^m(\lambda_1,\lambda_2,l,n)f(z)=0$. Consequently,

$$\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\in p \ \Rightarrow f(z)\in S^*(\lambda_1,\lambda_2,l,n,m,\delta)\,.$$

Hence, in the light of theorem (3) , the result given in (8) is completed.

The numerous works already done on properties by several authors caused by its importance, see ((Amer & Darus, 2013; Alabbar et al., 2023; Amer, 2017; Amer, 2016; Amer & Darus, 2012)), and see (Abufares & Amer, 2024).

3. Conclusion

In this paper, we defined a new operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ of fractional calculus for a class of normalized functions in the open unit disk , and discussed some geometric properties of this newly defined operator. By using the BB equation and involving the operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$.

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