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An Extension of Partial Differential Equations to Metric Spaces

Zohra Farnana

Mathematics Department, Faculty of Education-Tripoli, University of Tripoli, Tripoli, Libya

Email: z.farnana@uot.edu.ly

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المخلص:

في هذا البحث نعرض طريقة لتعميم مصطلح المشتقة الجزئية في الفضاء المتري. هذا التعميم يمكننا من دراسة مسألة ديريشليت والعديد من المسائل الأخرى في الفضاءات المترية. كذلك عرضنا بعض الأمثلة التي يكون فيها هذا التعميم ممكن.

الكلمات المفتاحية: المشتقة، المشتقة الضعيفة، الفضاء المتري، فضاءات سوبوليف، التدرج العلوي، الفضاءات النيوتونية، معادلة لابلاس، مسألة ديرتشليت..

A B S T R A C T:

The We present the way of extending the notion partial derivative to metric measure space equipped with a doubling Borel measure supporting the *p*-Poincare inequality. This extension makes it possible to study the Dirichlet problem, and many other problems, in metric spaces. We also present some examples where this extension is applied.

Keywords: derivative, weak derivative, metric space, Sobolev spaces, upper gradient, Newtonian spaces, Laplace equation, Dirichlet problem.

1. Introduction:

Solving partial differential equations on metric spaces is an active area of research where the theory is currently under development.

During the last two decades, people have shown the possibility to solve partial differential equations in general metric spaces by extending the concept of derivative. The Sobolev spaces in R^n are defined using the weak derivatives which are the first generalization of the partial derivatives. In metric measure spaces we do not have partial derivatives nor weak derivatives. Therefore, the concept of an upper gradient was introduced by Heinonen-Koskela [6] as a substitute of the usual gradient. This makes it possible to extend the Sobolev spaces to metric measure space which have been used to apply calculus of variations and study the *p*-Laplace equation as a minimizer of the *p*-Dirichlet integral.

For more details about upper gradients and Newtonian spaces see, e.g., Shanmugalingam [9], Björn-Björn [1] and Farnana [3]. For minimizers in metric spaces, see e.g., Shanmugalingam [10], Kinnunen-Shanmugalingam [7], Farnana [4,5].

In this paper we let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a positive complete Borel measure μ which is doubling, i.e., there exists a constant $C > 0$ such that for all $B = B(x, r) := \{y \in X : d(x, y) < r\}$

$$0 < \mu(2B) \le C\, \mu(B) < \infty,$$

where $2B = B(x, 2r)$.

This paper is organized as follows. In Section 2, we give some definitions with examples needed in the rest of the paper. In Section 3, we define the Dirichlet problem and show that it can be solved in the classical sense, where the solution is a C^2-function. Another way of solving the Dirichlet problem is to minimize the variational integral, where the solution not necessarily twice differentiable. In Section 4, we show that, one can instead have a solution of the Dirichlet problem that has only a weak derivative. Moreover, in Section 5, for a metric measure space we use the upper gradient as substitute of the usual gradient. This makes it possible to define and solve the Dirichlet problem, based on the variational integral, in metric spaces. Finally, in Section 6, we present some examples of metric spaces in which the extension of partial derivatives is possible.

2. Preliminaries and definitions

In this section we recall basic definitions needed in our study.

Definition 2.1 (Metric space, metric)

A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have

(M1) d is real-valued, finite and nonnegative

(M2) $d(x, y) = 0$ if and only if $x = y$

(M3) $d(x, y) = d(y, x)$

(M4) $d(x, y) \le d(x, z) + d(z, y)$

Examples 2.2

The following examples of metric spaces are from Kreyszig [8], the last three examples shows that the metric spaces are not necessarily spaces of numbers.

1. Real line $\boldsymbol{R}$. This is the set of all real numbers, taken with the usual metric defined by

$$d(x,y) = |x-y|$$

2. Euclidean plane $\boldsymbol{R}^2$. The metric space $\boldsymbol{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers, written $x=(\xi_1,\xi_2)$, $y=(\eta_1,\eta_2)$ and the Euclidean metric defined by

$$d(x,y) = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}$$

3. Euclidean space $\boldsymbol{R}^n$. This space is obtained if we take the set of all ordered n-tuples of real numbers written as

$$x=(\xi_1,\xi_2,\cdots,\xi_n), \qquad y=(\eta_1,\eta_1,\cdots,\eta_n),$$

and the Euclidian metric defined by

$$d(x,y) = \sqrt{(\xi_1-\eta_1)^2+\cdots+(\xi_n-\eta_n)^2}$$

4. Sequence space $\boldsymbol{l}^\infty$. This space consists of all bounded sequences of complex numbers; that is, every element of $\boldsymbol{l}^\infty$is a complex sequence

$$x=(\xi_1,\xi_2,\cdots) \quad \text{briefly} \quad x=(\xi_j)$$

such that we have $|\xi_j| \le c_x$, where c_x is a real number which may depend on x but does not depend on j. We choose the metric defined by

$$d(x,y) = \sup_{j\in \boldsymbol{N}} |\xi_j-\eta_j|$$

where $y=(\eta_j)\in \boldsymbol{l}^\infty$ and $\boldsymbol{N}=\{1,2,3,\cdots\}$

5. Function space $C[a,b]$. The set of all real-valued functions $x, y, \cdots$ which are functions of an independent variable t and are defined and continuous on a given closed interval $\mathrm{J}=[a,b]$. Choosing the metric defined by

$$d(x,y) = \max_{j\in J} |x(t)-y(t)|$$

Remark 2.3. As we see in the last two examples the metric space in general is not necessarily a space of numbers therefore, it is not clear how to define the derivative of a function defined on a metric space.

Definition 2.4.

1. A function $f: X \to \boldsymbol{R}$ is said to be in the space $C^k(X)$ if f has a derivative of order k.

2. Let $\Omega \subset \boldsymbol{R^n}$, the space $C_c^\infty(\Omega)$ is defined to be the space of all functions $\phi: \Omega \to \boldsymbol{R}$ such that ϕ is infinitely differentiable with compact support in Ω. We will sometimes call a function belonging to $C_c^\infty(\Omega)$ a test function.

Definition 2.5. We say that $f: X \to \overline{\boldsymbol{R}}$ belongs to the space $L^p(X)$ if f is measurable and

$$\int_X |f|^p \, d\mu < \infty.$$

The space $L^p(X)$ is equipped with norm

$$\| f \|_{L^p(X)} = \left(\int_X |f|^p \, d\mu\right).^{1/p}$$

A function belongs to the space $L^p(X)$ is often referred to as *p*-integrable on X.

3. The Dirichlet problem

The Laplace equation is defined by

$$\Delta u = \nabla . \nabla u = \sum_{i=1}^n u_{x_i x_i} = 0$$

The classical Dirichlet problem is to find a solution of the Laplace equation with given boundary values i.e., a function u that satisfies

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \qquad (1)$$

Definition 3.1 Let $\mathcal{A} = \{w \in C^2(\overline{\Omega}): w = f \text{ on } \partial\Omega\}$, we define the energy functional $I(w) \coloneqq \int_\Omega |\nabla w|^2 \, dx, w \in \mathcal{A}$.

The following theorem shows that the Dirichlet problem can be characterized as a minimizer of an appropriate functional, it is from Evants [2].

Theorem 3.2 (Dirichlet's principle).

Assume that $u \in C^2(\overline{\Omega})$ such that u solves Equation (1) (a solution of the Dirichlet problem). Then $I(u) = \min_{w \in \mathcal{A}} I(w)$ i.e.,

$$\int_\Omega |\nabla u|^2 \, dx = \min_{v \in \mathcal{A}} \int_\Omega |\nabla v|^2 dx \qquad (2)$$

Conversely, if $u \in \mathcal{A}$ satisfies Equation (2), then u is a solution of the Dirichlet problem.

In other words, if $u \in \mathcal{A}$, the partial differential equation $\Delta u = 0$ is equivalent to the statement that u minimizes energy integral (2).

Proof

Let $v \in \mathcal{A}$. Then Equation (1) implies that

$$0 = \int_\Omega \Delta u \, (u - v) \, dx.$$

An integration by part yields

$$0 = \int_\Omega -\nabla u . \nabla(u - v) dx,$$

And there is no boundary term since $u - v = f - f = 0$ on $\partial\Omega$. Hence, using the Cauchy-Schwarz and Cauchy inequalities, we get

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega \nabla u . \nabla v \, dx \leq \int_\Omega \frac{1}{2} |\nabla u|^2 \, dx + \int_\Omega \frac{1}{2} |\nabla v|^2 \, dx,$$

which implies that

$$\int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx,$$

since $v \in \mathcal{A}$ was arbitrary, this implies that $\int_\Omega |\nabla u|^2 \, dx = \min_{v \in \mathcal{A}} \int_\Omega |\nabla v|^2 dx$.

Conversely, suppose that Equation (2) holds. Let $\varphi \in C_c^\infty(\Omega)$, then $v = u + t\varphi \in \mathcal{A}$

By the minimizing property of u we have $I(u) \leq I(v) = I(u + t\varphi)$ for all $t \in R$.

This means that $h(t) \coloneqq I(u + t\varphi)$ has a global maximum at $t = 0$ i.e., $h(0) \leq h(t)$ for all $t \in R$.

It follows that $h'(0) = 0$ provided that the derivative exists. But

$$h(t) = \int_\Omega |\nabla(u + t\varphi)|^2 \, dx = \int_\Omega |\nabla u + t\nabla\varphi|^2 \, dx$$

$$= \int_\Omega \ |\nabla u|^2 + 2t \, \nabla u \, \nabla\varphi + t^2 |\nabla\varphi)|^2 \, dx.$$

Consequently

$$0 = h'(0) = \int_\Omega \ \nabla u \, \nabla\varphi \, dx = \int_\Omega \ -\Delta u. \varphi \, dx.$$

This identity holds for each $\varphi \in C_c^\infty(\Omega)$ and so $\Delta u = 0$ in Ω.

4. The weak derivative and Sobolev spaces.

Before turning to the weak derivative, we first mention that some functions do not have partial derivative, even continuous functions may not have partial derivatives. For this reason, we need to weaken the notion of partial derivatives.

4.1 Motivation for the definition of weak derivatives.

For $u \in \boldsymbol{C}^1(\Omega)$ and if $\phi \in C_c^\infty(\Omega)$, we see from the integration by parts formula that

$$\int_\Omega \ u(x) \, \phi_{x_i}(x) \, dx = -\int_\Omega \ u_{x_i}(x)\phi(x) \, dx \quad (i = 1,2, \dots, n).$$

There are no boundary terms, since ϕ vanishes on $\partial\Omega$.

Definition 4.1. Let $\Omega \subset \boldsymbol{R^n}$ and $u \in L^1 \ (\Omega)$. We say that v is a weak derivative of

u in the direction x_i if

$$\int_\Omega \ u(x) \, \phi_{x_i}(x) \, dx = -\int_\Omega \ v(x)\phi(x) \, dx$$

for all test functions $\phi \in C_c^\infty(\Omega)$. We write $v = D_i u$.

Lemma 4.2. (Uniqueness of weak derivatives)

A weak partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.

Proof.

Let $v = (v_1, v_2, \ldots, v_n) \in \boldsymbol{L^1}(\Omega)$ and $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n) \in \boldsymbol{L^1}(\Omega)$ be two weak derivatives of u, then for all $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u\phi_{x_i} dx = -\int_\Omega v_i \phi\, dx = -\int_\Omega \tilde{v}_i\, \phi\, dx, \quad i = 1,2, \ldots, n.$$

Then

$$\int_\Omega (v_i - \tilde{v}_i) dx, \quad i = 1,2, \ldots, n$$

for all $\phi \in C_c^\infty(\Omega)$. This means that $v_i - \tilde{v}_i$=0, a.e., $i = 1,2, \ldots, n$. Thus $v = \tilde{v}$ a.e.

4.2. Sobolev spaces

The Sobolev space $W^{1,2}(\Omega)$ consists of all functions $u \in L^2(\Omega)$ that have weak derivatives $D_i u \in L^2(\Omega)$ in all directions. It is equipped with the norm

$$\| u \|_{W^{1,2}(\Omega)} = \left(\int_\Omega |u(x)|^2 + \sum_{i=1}^n |D_i u(x)|^2 dx \right)^{1/2}$$

Remark 4.3 The more general (non-linear) p-Laplace equation is defined by $\nabla.(|\nabla u|^{p-2}\, \nabla \mathrm{u}) = 0,\ 1 < p < \infty$ and the corresponding Dirichlet problem is to find a function u that satisfies the

$$\begin{cases} \nabla.(|\nabla u|^{p-2}\, \nabla \mathrm{u}) = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \qquad (3)$$

which reduces to the classical Dirichlet problem for $p = 2$. It can be shown, as in Theorem 3.2, that a solution of Equation (3) is a minimizer of the energy integral

$$\int_\Omega |\nabla v(x)|^p\, dx,$$

When solving the Dirichlet problem for p-Laplace equation, $1 < p < \infty$, one can look for a solution that has a weak derivative instead of the usual derivative i.e., a solution in the Sobolev space $W^{1,p}(\Omega)$, the space of all functions $u \in L^p(\Omega)$ that have weak derivatives $D_i u \in L^p(\Omega)$ in all directions. It is equipped with the norm

$$\| u \|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p + \sum_{i=1}^n |D_i u(x)|^p dx \right)^{1/p}$$

5.The Newtonian spaces (Sobolev spaces in metric spaces)

In a metric measure space X we do not have partial derivatives nor weak derivatives but we have an upper gradient as a substitute of the usual gradient.

Definition 5.1

Let $u: X \to \overline{\boldsymbol{R}}$ be a function. A non-negative Borel function $g: X \to [0, \infty]$ is said to be an upper gradient of u if for all rectifiable curves $\gamma: [0, l_\gamma] \to X$ parametrized by the arc length we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g\, ds \qquad (4)$$

Whenever both $u(\gamma(0))$ and $u(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if Equation (4) holds for p-almost every curve then g is a p*-weak upper gradient* of u.

By saying that Equation (4) holds for p-almost every curve we mean that it is satisfied expect for a curve family of zero p-modulus, see Definition 2.1 in Shanmugalingam [9] or Definition 2.4 in Farnana [3].

One can see from Equation (4) that if g is an upper gradient of u, then every Borel function greater than g will be another upper gradient of g, i.e., the upper gradient is not unique. However, if u has an upper gradient in L^p(X), then it has a unique *minimal* p*-weak upper gradient* $g_u \in L^p(X)$ such that $g_u \leq g$ for all p-weal upper gradients $g \in L^p(X)$, see Corollary 3.7 in Shanmugalingam [10].

Remark 5.2. Notice that, in $\boldsymbol{R}^n$, it follows from the fundamental theorem of calculus that, Equation (4) holds when g is replaced by $|\nabla u|$ and in fact, $|\nabla u|$ is the smallest function that satisfies Equation (4), which means that $g_u = |\nabla u|$, see e.g., Proposition 4.3 in Farnana [3].

As we see, the minimal p-weak upper gradient is a replacement of the modulus of the usual gradient. This makes it possible to define Sobolev spaces in metric spaces called Newtonian spaces, see Shanmugalingam [9] and Definition 2.1 in Farnana [3].

Definition 5.3

Let $u \in L^p(X)$, then we define

$$\| u \|_{N^{1,p}(X)} = \left(\int_X |u|^p d\mu + \int_X g_u^p \, d\mu \right)^{1/p}$$

where g_u is the minimal p-weak upper gradient of u. The Newtonian space is the quotient space

$$N^{1,p}(X) = \{u: \| u \|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\| u - v \|_{N^{1,p}(X)} = 0$.

6. Some examples of metric spaces.

The following examples of metric spaces satisfy the conditions: completeness, doubling and Poincare inequality, for more details see, Björn-Björn [1].

1. The Euclidean spaces $\boldsymbol{R}^n$.

2. Weighted Sobolev spaces on $\boldsymbol{R}^n$.

3. Uniform domains and power weights.

4. Graphs.

5. Heisenberg groups.

6. Riemannian manifolds with nonnegative curvature.

7. Results and Analysis

In this section we present and analyze the results of extending the partial differential equation to metric spaces. We first show that the weak partial derivative, in $\boldsymbol{R}^n$, is the right extension of the usual derivative. In particular we show that, for differentiable functions the two concepts coincide. Moreover, for a differentiable function f, the minimal *p*-weak upper gradient is equal to the modulus of the usual gradient. This means that, when we are restricted to $\boldsymbol{R}^n$ the Newtonian space $N^{1,p}(\boldsymbol{R}^n)$ is equal to the Sobolev space $W^{1,p}(\boldsymbol{R}^n)$. Furthermore, the extension of the derivative is unique.

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