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Water Infiltration into Uniform and Stratified Soils

I. Review and use of an approximate theory [1]

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ABSTRACT

The equations of the approximate infiltration theory of Green and Ampt are much simpler than the mathematical forms of the diffusivity-conductivity theory of soil-water movement, and also involve parameters of characterization that are easier to measure experimentally. For cumulative water infiltrated into uniform soil as a function of time, these parameters are: the initial water content of the soil, the constant mean water content behind the wet front, a soil constant akin to the hydraulic conductivity of nearly saturated soil, and an equivalent constant head of water arising from the capillary properties of the soil. For fitting of the Green and Ampt equation to experimental infiltration data, a least-squares approach was developed, and was found to work very satisfactorily for measurements made on sand-silt mixtures.

The approximate theory was applied to infiltration into stratified soils, thus involving the parameters of characterization of each stratum. Analysis was carried out for two and three strata, and can be continued for as many strata as desired. The resulting equations, however, become increasingly complicated, especially for the larger values of time at which the wet front has advanced into successively deeper strata.

INTRODUCTION

Theoretical analyses of infiltration of water into soil have generally been made for physical conditions represented in terms of a vertical column of unsaturated porous material to which water is applied at the top, often by ponding. In principle, an exact mathematical solution for water inflow into such a column is obtainable for a great variety of conditions, in terms of the diffusivity and conductivity functions of the porous material. Hence, the prediction of infiltration, as well as of other liquid-flow soil-water processes, becomes possible if the diffusivity and conductivity functions can be measured. Utilization of this so-called exact approach, however, as done by Green, Hanks,

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and Larson (8), is greatly complicated by the experimental difficulties attendant to the measurement of the diffusivity and conductivity functions. Steady-state flow methods for determining the conductivity function are time consuming. Furthermore, the pressure-plate outflow (or inflow) method for measuring both the diffusivity and conductivity functions is not only lengthy, but the results are subject to 7- to 10-fold variations in precision (1, 9).

In view of these difficulties, a somewhat different approach would appear to be justified. Specifically, it would seem worthwhile to investigate the validity and utility of infiltration equations of simpler mathematical form, involving parameters of characterization that are easier to measure experimentally. Admittedly, such an attack would be less capable of rigorous physical interpretation, and the soil-characterizing parameters therefrom might be of little if any utility for describing other types of soil-water transfer. But, if a workable and self-consistent approximate theory and method could be put together successfully just for the infiltration process alone, the result would still be of considerable merit and utility. In the present study, attention is directed to an approximate theory of water infiltration in which the parameters appear to be much simpler to measure.

THEORETICAL CONSIDERATIONS

Uniform Systems

One-dimensional downward infiltration is diagrammed in Figure 1A. To the infinitely deep uniform porous material of initial constant volumetric water content θ_0, the constant depth H of ponded water is applied at time $t = 0$. If θ_0 is on the order of air dryness, a distinct visual wet front will be present, the depth of which is designated z, and which increases with time $t > 0$. The first analysis of this flow problem was given by Green and Ampt (7), considering the soil to behave in an idealized fashion which implied complete water saturation behind the wet front, so that the mean volumetric water content $\overline{\theta}$ behind the wet front (for all z) would equal the total porosity. This assumption was first shown to be untenable by Kirkham and Feng (10). Philip (12) in effect relaxed the requirement of total saturation. He assumed for all z that $\overline{\theta}$ was constant, but left open the possibility of it being less than the total porosity. But the infiltration equation he derived was still in essence the same functional relation as that of Green and Ampt. We shall develop this equation in brief fashion here, but the basic limitations and approximations will remain as in the result of Philip (12). Childs (3, pp. 107–108) was the first to point out clearly that Green and Ampt's approach need not necessarily be restricted to a capillary-tube model.

The Darcy equation can be written as

$$v = dy/dt = Ki \tag{1}$$

where v is the flux of water transmitted, y is the volume of water transmitted per unit cross sectional area of soil column, t is the time, K is the hydraulic conductivity, and i is the total hydraulic gradient. It is granted that certain questions about the ultimate validity of the Darcy equation are as yet unresolved (19). But, since equation (1) will here be used in a development which from the outset is admitted to be approximate, the existence of some possible approximate character in the basic equation is not considered forbiddingly serious, especially since the final results would need to be assessed experimentally.

To obtain an expression for i of equation (1) for the column in Figure 1A, we envision

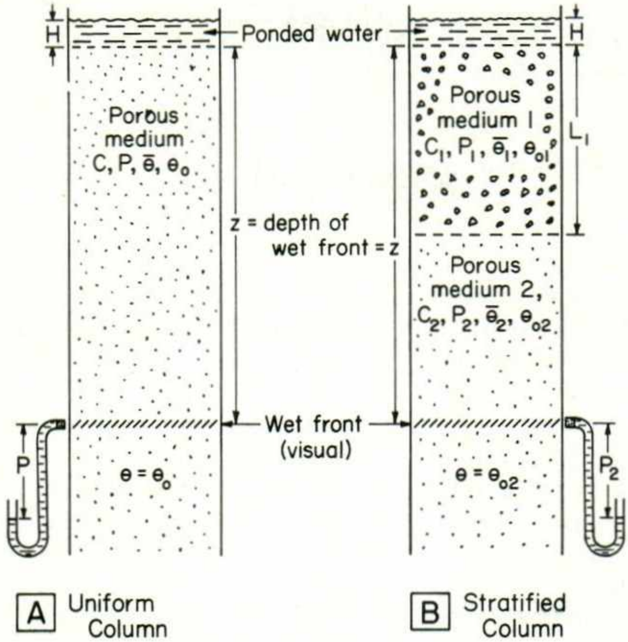

Fig. 1. Diagrams of water infiltration into columns of porous media, (A) a uniform column, and (B) a stratified column.

a fictitious tensiometer to be inserted at or just slightly behind the wet front and to be capable of moving along with it. The water level in this fictitious moving tensiometer is considered to stand at the constant distance P below the wet front. Philip (12) terms $-P$ as 'the capillary potential of the wetting front.' Taking the top of the column as an arbitrary datum plane for hydraulic head, the total hydraulic head h_{wf} at the wetting front becomes $h_{wf} = -(P + z)$ while that at the inlet end (top) of the column becomes $h_{inlet} = H$. Forming the total hydraulic head differences yields $h_{inlet} - h_{wf} = H + P + z$. We express gradient i as $(h_{inlet} - h_{wf})/z = (H + P + z)/z$. We now put this result into equation (1), but at the same time replace K by another constant C to allow for any distinction from the Darcian K; the result becomes

$$dy/dt = C(P + H + z)/z \tag{2}$$

Since C fills an analogous role to K and has the same units, it may at this stage be of some interest to speculate whether C might ultimately be interpretable as the value of K corresponding to the water content $\bar{\theta}$. This is somewhat akin to van Duin's (20) suggestion that throughout the transmission zone the hydraulic conductivity is nearly constant and is associated with a water content of about 80% pore-space saturation.

Making use of Philip's (12) assumption of a constant $\bar{\theta}$ behind the wet front, we can write

$$(\bar{\theta} - \theta_0)z = y = Mz \tag{3}$$

where $\overline{\theta} - \theta_0 = M$ is introduced merely as an economy of symbols. Solving equation (3) for z and putting the result into equation (2), separating variables, integrating, and using the condition $y = 0$ at $t = 0$, yields

$$(y/a) - \ln(1 + y/a) = Ct/a \tag{4}$$

as the form of the Green and Ampt equation, where

$$a = M(H + P) = (\overline{\theta} - \theta_0)(H + P) \tag{5}$$

Although the derivation just given does not impart rigor to equation (4), Philip (13) has considered an alternative derivation based upon diffusivity-conductivity approaches, and reports that the form of equation (4) will result from assuming that the diffusivity is given by the Dirac delta function. Physically, however, this implies a porous medium of coarse texture and low initial water content, and, if these special conditions are not met, the validity of equation (4) is still left approximate. In spite of this, Philip (13,14,15) has reported fairly good success in comparing equation (4) with the general results of his more exact analysis, although Childs (3) has criticized some of the comparisons as not being particularly meaningful. Nevertheless, both Gardner (5) and Childs (3) have encouraged a greater use of the Green and Ampt equation to characterize infiltration properties.

If characterizing infiltration were the only goal, then it would not particularly matter whether C and P were precisely interpretable in terms of the commonly accepted basic soil-physics concepts of unsaturated conductivity and soil-water suction, as long as both C and P were dependable and self-consistent characterizers of the infiltration process. It would be particularly convenient if both parameters were determined only by the soil material and $\overline{\theta}$. This would seem a reasonable possibility if, in accord with van Duin's (20) suggestion, C were closely associated with the hydraulic conductivity of the transmission zone, and if P were essentially the hydraulic head loss through this zone. As pointed out by Swartzendruber (18), Mein and Larson (11) have in effect taken P as the negative of Bouwer's (2) critical pressure head, a quantity that is independent of the initial water content θ_0. It can also be argued, however, that P should be related in some inverse fashion to the mean radius of curvature of the air-water interface in the soil and this implies P to decrease as θ_0 increases. Resort to experiment would seem to be the most meaningful way of resolving the matter.

An infiltration equation has been given by Philip (13) in the form

$$y = St^{1/2} + Gt \tag{6}$$

where S and G are constants for flooding applications of water and a given constant initial water content. Strictly speaking, the assumptions giving rise to equation (6) do not allow time t to increase without limit (13). Nevertheless, equations (4) and (6) can be matched in the limits $t \to 0$ and $t \to \infty$ by taking $S = (2aG)^{1/2}$ and $G = C$. Using such matching conditions, Swartzendruber and Youngs (19) compared equations (4) and (6) over the complete time range, and found the maximum relative difference between them to be only 15.1%. This relatively small difference suggests that both equations describe infiltration in about the same way. Nevertheless, the Green and Ampt approach would seem to possess some advantages for our present purposes. Parameter a, for example, depends on θ_0 in a much less complicated fashion than does S. Furthermore, the considerations which produce equation (6) are not simply extendable to stratified porous media, whereas the approximate reasoning used to derive equation (4) can also be applied straightforwardly to stratified media, a matter to which we now turn.

Stratified Systems

Consider the system of Figure 1B, where a stratum of infiltration properties C_1, P_1, $\overline{\theta}_1$, and θ_{01}, and of length L_1, is underlain by an infinitely deep stratum of infiltration properties C_2, P_2, $\overline{\theta}_2$, and θ_{02}. For $z < L_1$, that is, before the wet front reaches the junction between the two strata, the infiltration behavior is simply that of the Green and Ampt expression [equation (4)], namely

$$(y/a_1) - \ln(1 + y/a_1) = C_1 t/a_1 \quad (7)$$

where $a_1 = M_1(H + P_1) = (\overline{\theta} - \theta_{01})(H + P_1)$. When the wet front penetrates into the lower stratum, for $z > L_1$ as actually depicted in Figure 1B, a somewhat different expression eventually will emerge. The equivalent of equation (2) becomes

$$dy/dt = C_e(H + P_2 + z)/z \quad (8)$$

where C_e is analogous to the equivalent conductivity of the two regions of C_1 and C_2 spanned by the wet-front depth z. C_e is thus expressed (16) as $C_e = z/[L_1/C_1 + (z - L_1)/C_2]$. Combining this with equation (8) and rearranging yields

$$dy/dt = C_1C_2(H + P_2 + z)/[L_1(C_2 - C_1) + C_1 z] \quad (9)$$

For later simplification, it is convenient to reckon y and t of equation (9) from the point at which the wet front strikes th junction of the two strata ($z = L_1$), using y_2 and t_2 for this purpose but remembering that they are still variables. The counterpart of equation (3) is then written

$$y_2 = M_2(z - L_1) \quad (10)$$

Setting $y = y_2$ and $t = t_2$ in equation (9) and combining with equation (10) to eliminate z will result in a differential equation in which the variables can be separated and the differentials integrated. Using $y_2 = 0$ at $t_2 = 0$ in the integrated result enables the final result to be written

$$(y_2/b) - (1 - c/b)\ln(1 + y_2/b) = C_2 t_2/b \quad (11)$$

where

$$b = M_2(H + P_2 + L_1) = a_2 + M_2 L_1 \quad (12)$$

and

$$c = L_1 M_2 C_2/C_1 \quad (13)$$

Note the similarity of equation (11) with equation (4). In particular, if $L_1 = 0$ then Figure 1B becomes a uniform system for the subscript 2 material, whereupon $c = 0$, $b = M_2(H + P_2) = a_2$, and, as it should, equation (11) reduces to the Green and Ampt equation for the subscript 2 material. Equation (11) was first presented by van Duin (20), but without details of derivation. Childs (3, 4) and Childs and Bybordi (5) have also applied the Green and Ampt approach to stratified porous media, but without reference to van Duin (20).

To construct a complete curve of water infiltrated versus time for a column with $L_1 > 0$, equation (7) is used until the wet front reaches the stratum junction. Thereupon, let y and t from equation (7) be designated as y_0 and t_0, respectively. As the wet front penetrates into the lower stratum, equation (11) becomes applicable, and the overall accumulative curve, reckoned from the same origin as in equation (7), becomes $y_0 + y_2$ versus $t_0 + t_2$.

The foregoing general approach can readily be extended to as many strata as desired, and the basic form of the resulting equation remains that of equation (11). Consider, for example, that the L_1 stratum of Figure 1B be underlain by a stratum of properties C_2, P_2, and M_2, and of length L_2, and that this in turn be underlain by an infinitely deep stratum of properties C_3, P_3, and M_3. Then, for the wet front in the subscript 3 layer, the resulting equation will take the form of equation (11), but with the b of it and equation (12) replaced by $M_3(H + P_3 + L_1 + L_2)$, and the c of equations (11) and (13) replaced by $M_3(L_1 + L_2)C_3/C_{e12}$, where $C_{e12} = (L_1 + L_2)/(L_1/C_1 + L_2/C_2)$ is the equivalent C for the two top strata. Also, C_2, and y_2, and t_2 of equation (11) are replaced by C_3, y_3, and t_3, respectively, where y_3 and t_3 are reckoned from the stage at which the wet front strikes the junction between the subscript 2 and subscript 3 strata ($z = L_1 + L_2$).

FITTING OF CONSTANTS FOR UNIFORM SYSTEMS

The fitting of equation (4) to data of cumulative infiltration y versus time t is complicated by the presence of the constant a in the logarithmic term. Experimental data, however, can be plotted on translucent log-log graph paper used as an overlay, which is then matched by horizontal and vertical translation to an underlay plot of equation (4) prepared on the same kind of log-log graph paper. This method, however, is workable only if the experimental data follow the theoretical curve very precisely; otherwise, subjective errors of fitting by eye are encountered. Because of this weakness, the log-log overlay-underlay graphical technique was eventually discarded, even though a number of attempts had been made to use it.

To fit equation (4) by least squares, the sum of squares of the t deviations is minimized, since t is given as an explicit function of y rather than vice versa. The sum of squares thus formulated is

$$\sum_{j=1}^{p} (t_j - \hat{t}_j)^2 = \sum_{j=1}^{p} \left[t_j - (y_j/C) + (a/C)\ln(1 + y_j/a)\right]^2 \tag{14}$$

where $\hat{t}_j$ is simply the t of equation (4) with a subscript j, and p is the total number of experimental pairs of y_j and t_j. The sum in equation (14) is minimized in the conventional way by taking partial derivatives with respect to C and a and setting equal to zero. Doing this for C and solving for C yields

$$C = \frac{\sum_{j=1}^{p} a\left[(y_j/a) - \ln(1 + y_j/a)\right]^2}{\sum_{j=1}^{p} t_j\left[(y_j/a) - \ln(1 + y_j/a)\right]} \tag{15}$$

Taking the partial derivative of equation (14) with respect to a and setting equal to zero yields

$$\sum_{j=1}^{p} \left[Ct_j - y_j + a\ln(1 + y_j/a)\right]\left[\ln(1 + y_j/a) - y_j/(y_j + a)\right] = T = 0 \tag{16}$$

where T simply denotes the summation expression.

Equation (16) with $T = 0$ cannot be solved explicitly for a, since the logarithmic term contains a. Note, however, that for a given set of p pairs of y_j and t_j, C of equations (15) and (16) becomes a function only of a, so that T defined by the first equality of equations (16) also becomes a function only of a. Hence, in equation (15) and the first of equations (16), trial values of a can be used until one is found that makes $T = 0$. This is the least-squares value of a, and its substitution into equation (15) yields the least-squares value of C. The computations just described are much too lengthy for a desk calculator, but can be handled with ease by electronic computer. In programming the calculations, we have found Newton's method of iterative approximation to be workable and effective.

In a subsequent paper, experimental results will be considered and analyzed in detail, including, for uniform sand-silt mixtures, twenty sets of data that were fitted to equation (4) by the least-squares approach just outlined. As a measure of goodness of fit, a mean-square deviation was calculated on the basis of the left-hand side of equation (14) as

$$\sum_{j=1}^{p} (t_j - \hat{t}_j)^2/p$$

Out of the twenty sets of data, the set with the largest value of this mean-square deviation (18.31 min^2), and hence the poorest fit, is shown graphically in Fig. 2. Even for this case of poorest fit, it is seen that the curve of equation (4) passes very acceptably through the data points. It is thus concluded that the Green and Ampt expression [equation (4)] describes the infiltration data very well, and that the least-squares fitting process represented by equations (14) through (16) is satisfactory.

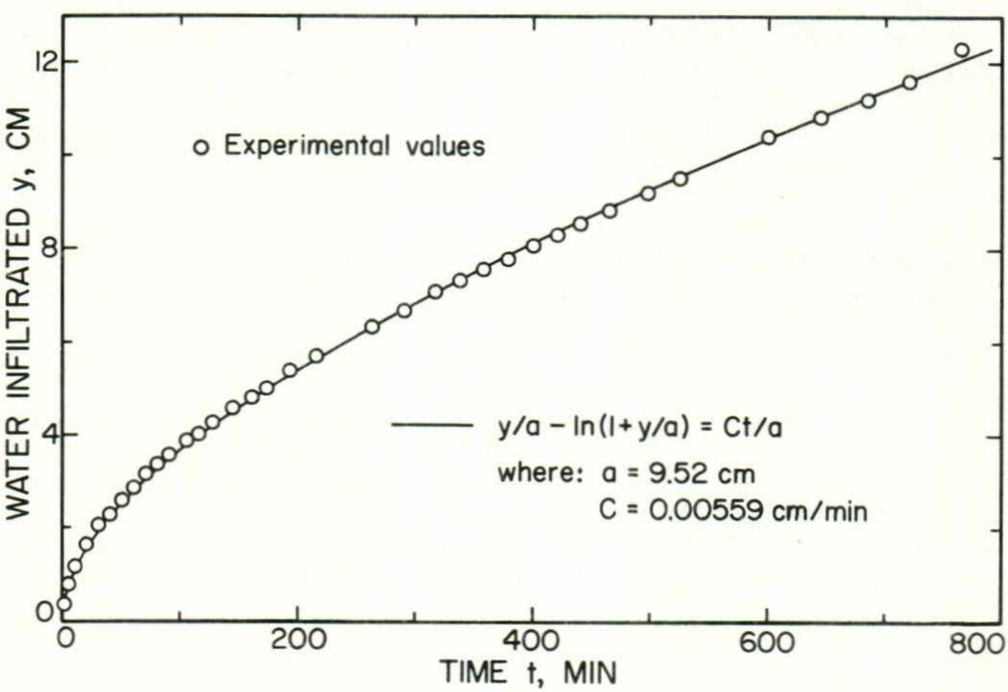

Fig. 2. Illustration of the least-squares fit of the Green and Ampt equation to data obtained for cumulative water infiltration into a uniform column of 25% silt and 75% fine sand, for a mean-square deviation of 18.31 min^2.

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