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Computation of the MLE of the Non-Centrality Parameter of the Non-Central χ^2 Distribution

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Abstract

In this paper, a Matlab computer program is presented for finding the maximum likelihood estimate of the non-centrality parameter of the non-central chi squared distribution with ν degrees of freedom.

المستخلص

في هذه الورقة يتم تقديم وعرض برنامج كمبيوتر بلغة الماتلاب لإيجاد تقدير الأرجحية العظمى للمعلمة اللامركزية لتوزيع مربع كاي اللامركزي بدرجات حرية ν.

Keywords: Non-central chi-squared distribution; non-centrality parameter; maximum likelihood estimate (MLE).

Introduction

Let $X_1, X_2, \cdots, X_\nu$ be ν independent random variables. If X_i is distributed as $N(\mu_i, 1)$ for $i = 1, 2, \cdots, \nu$, then the random variable X, defined by $X = \sum_{i=1}^{\nu} X_i^2$, is termed a non-central chi-squared having ν degrees of freedom and non-centrality parameter $\delta^2 = \sum_{i=1}^{\nu} \mu_i^2$.

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The non-central chi squared distribution arises in various statistical analyses and the estimation of the non-centrality parameter is of importance in some problems. The computation of the maximum likelihood estimate (MLE) of δ^2 is a difficult task. An algorithm was presented by Dwivedi and Pandey (1975) for finding the MLE of δ^2.

The main objectives are: 1) to write a Matlab program for computing the MLE of δ^2, using Dwivedi and Pandey algorithm, to be available for Matlab users, and, 2) to create a table of the values of the MLE of δ^2.

The probability density function (pdf) of X that follows a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2, was first given by Fisher (1928) and may be written as:

$$f(x;\nu,\delta^2)=\sum_{j=0}^{\infty}\frac{\exp(-\delta^2/2)}{j!}\,\frac{(-\delta^2/2)^j\,x^{\nu/2+j-1}\exp(-x/2)}{\Gamma(\nu/2+j)\,2^{\nu/2+j}}\quad for\ \ x>0 \qquad (1)$$

Where we define $(\delta^2/2)^j=1$ when $\delta^2=0, j=0.$

The random variable X that follows a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2 is usually denoted by $\chi^2_\nu(\delta^2)$. Some texts refer to the non-central chi-squared distribution as the generalized Rayleigh, Rayleigh-Rice, or Rice distribution.

The pdf of X can be written in the following way:

$$f(x;\nu,\delta^2)=\frac{1}{2}\exp(-(\delta^2+x)/2)\,(\sqrt{x/\delta^2})^{\nu/2-1}\,I_{(\nu/2-1)}(\sqrt{\delta^2 x}),\quad for\ \ x>0 \qquad (2)$$

Where $I_{(k)}(u)$ stands for the modified Bassel function for the first kind and order k (e.g. Johnson and Kots, 1970).

A good discussion of the non-central chi-squared distribution can be found in Johnson and Kotz (1970).

Computation of pdf of a non-central chi-squared random variable

It is important to compute the pdf of a non-central chi-squared distribution in order to compute the maximum likelihood estimate of the non-centrality parameter δ^2. The Matlab function **ncx2pdf**; which is built into Matlab R2011, (Brani, 2011), computes the probability that a random variable X takes on a value equal to x, when X follows a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2.

The plots of the non-central chi-squared pdfs (Fig. 1) show that, for large values of ν, the pdf is approximately symmetric about its mean $\nu+\delta^2$.

Maximum Likelihood estimator of δ^2

Power calculations for a variety of research designs used in behavior genetics require the determination of a non-centrality chi-squared parameter.

It was mentioned, earlier, that the pdf of a non-central chi-squared random variable X can be expressed in terms of modified Bassel function.

Now we put $x = y^2$ in equation (2), then the pdf of y is $h(y;\nu,\delta^2)$, where

$$h(y;\nu,\delta^2) = \delta(y/\delta)^{\nu/2}\exp(-(\delta^2+y^2)/2)\,I_{(\nu/2-1)}(\delta y) \quad (3)$$

Let $y_1, y_{2,} \cdots, y_n$ be an observed random sample of size n from (3), and let $\hat{\delta}$ denote the maximum likelihood estimate of δ. Dwivedi and Pandey (1975) showed that $\hat{\delta}$ is,

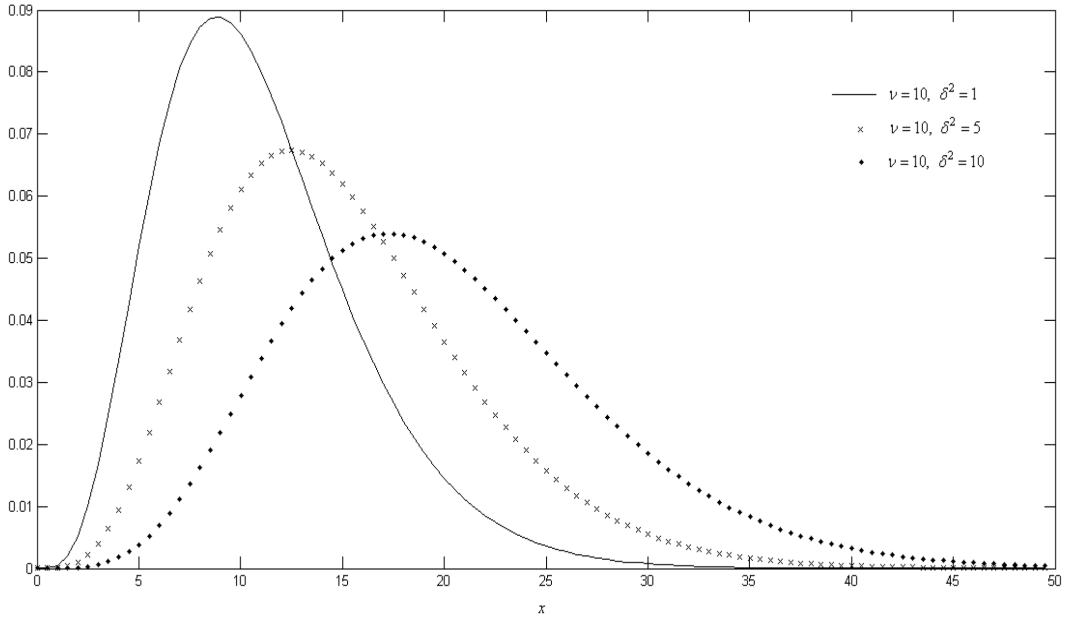
i) Zero, if $\left(\sum_{i=1}^{n} y_i^2 / \nu\right) \le n$.

ii) The zero of $-n+\sum_{i=1}^{n}\frac{I_{(\nu/2)}(\delta\, y_i)}{\delta\, y_i\, I_{(\nu/2-1)}(\delta\, y_i)}\, y_i^2,\ \ if \ \ \sum_{i=1}^{n} y_i^2/\nu > n$.

Matlab Program

Using the results of Dwivedi and Pandely (1975), the Matlab computer program **mledelta** (see Appendix A) has been written in order to compute the maximum likelihood estimate of δ^2 for a given n observations $x_1, x_{2,} \cdots, x_n$, from a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2.

For selected values of ν, Appendix B shows the value of maximum likelihood estimate of δ^2 for a given single observation x, where X has a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2.



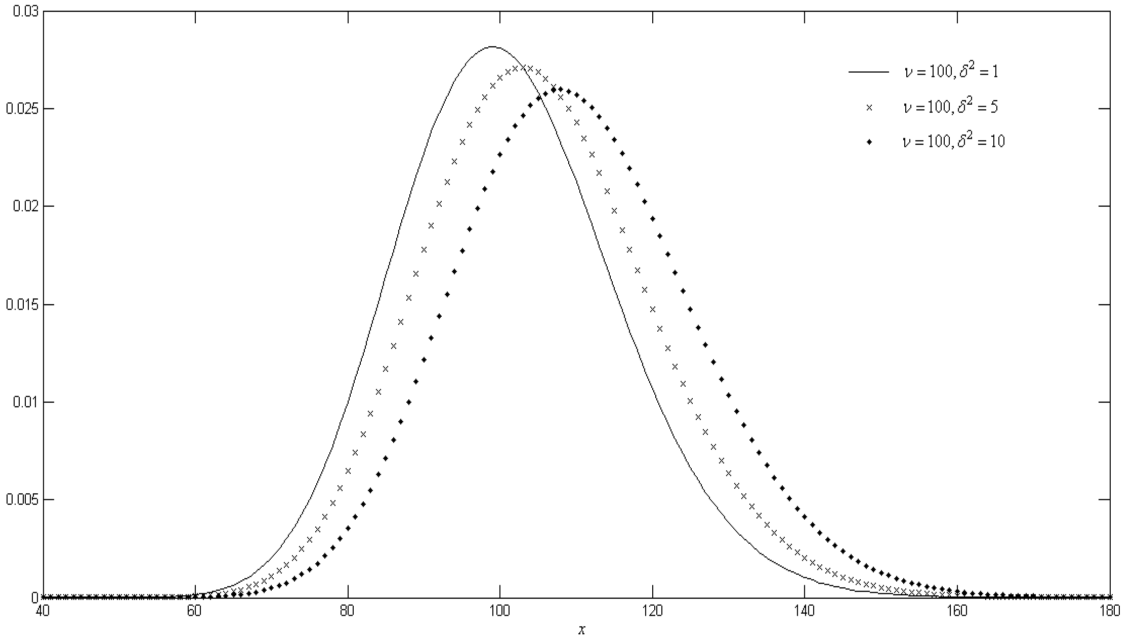


Fig. 1**.** Non-central chi-squared pdfs

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Appendix A
The Matlab Computer Program mledelta

The function **mledelta** computes the maximum likelihood estimate of the non-central chi-squared parameter δ^2. The Matlab function **mledelta** calls the function **equmle**.

```
function [MLEd2]=mledelta(xi,v)
n=max(size(xi));
 E=.0000001;
xbar=sum(xi)/n;
if  xbar<=v
          MLEd2=0;
           k=5;
else
          k=0;
                    lambdaL0=0;
   lambdaU0=1000;
end
while k~=5
                    d2_est=.5*(lambdaL0+lambdaU0);
```

```
Q=equmle(xi,v,d2_est);
if Q < n-E/2
                    lambdaL0=lambdaL0;
lambdaU0=d2_est;
elseif Q >n+E/2

                    lambdaL0=d2_est;
lambdaU0=lambdaU0;
elseif (abs(Q-n) <= E/2 )
                    MLEd2=d2_est;
                    k=k+1;
end
end
```

```
function Q=equmle(xi,v,d2)

d2=d2+1E-201;
if v >= 2
    m1=v/2;
m2=(v/2)-1;
Q=0;
for k=1:max(size(xi))
dy=sqrt(d2*xi(k));
      Q=Q+(xi(k)/sqrt(d2*xi(k)))*besseli(m1,dy)/besseli(m2,dy);
end
elseif v==0
      m1=0;
m2=1;
Q=0;
for k=1:max(size(xi))
dy=sqrt(d2*xi(k));
      Q=Q+(xi(k)/sqrt(d2*xi(k)))*besseli(m1,dy)/besseli(m2,dy);
end
elseif v==1
      m1=1/2 ;
m2=-1/2;
Q=0;
for k=1:max(size(xi))
dy=sqrt(d2*xi(k));
```

```
   Q=Q+(xi(k)/sqrt(d2*xi(k)))*besseli(m1,dy)/besseli(m2,dy);
end
end
```

Appendix B

The value of the maximum likelihood estimate of δ^2 for a given single observation χ, where X has a non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter δ^2.

χ	ν = 1	2	3	5	10	15	20	25	30	35	40
1	0	0	0	0	0	0	0	0	0	0	0
3	2.9695	1.5732	0	0	0	0	0	0	0	0	0
5	4.9991	3.8441	2.6345	0	0	0	0	0	0	0	0
10	10	8.9405	7.873	5.7071	0	0	0	0	0	0	0
15	15	13.963	12.923	10.832	5.5184	0	0	0	0	0	0
20	20	18.973	17.944	15.882	10.687	5.4099	0	0	0	0	0
25	25	23.979	22.956	20.909	15.768	10.585	5.3392	0	0	0	0
30	30	28.982	27.964	25.926	20.816	15.68	10.509	5.2895	0	0	0
35	35	33.985	32.97	30.937	25.847	20.74	15.61	10.451	5.2526	0	0
40	40	38.987	37.974	35.946	30.869	25.781	20.677	15.553	10.405	5.2241	0
45	45	43.989	42.977	40.952	35.886	30.81	25.724	20.623	15.506	10.367	5.2014
50	50	48.99	47.979	45.957	40.899	35.833	30.759	25.675	20.578	15.467	10.336
60	60	58.991	57.983	55.965	50.917	45.865	40.808	35.744	30.673	25.594	20.505
70	70	68.993	67.985	65.97	60.93	55.887	50.84	45.789	40.733	35.672	30.605
80	80	78.994	77.987	75.974	70.94	65.903	60.863	55.821	50.775	45.725	40.672
90	90	88.994	87.989	85.977	80.947	75.915	70.88	65.844	60.805	55.764	50.719
100	100	98.995	97.99	95.979	90.952	85.924	80.894	75.862	70.828	65.792	60.754
110	110	109	107.99	105.98	100.96	95.931	90.904	85.876	80.846	75.815	70.782
120	120	119	117.99	115.98	110.96	105.94	100.91	95.888	90.861	85.833	80.804
130	130	129	127.99	125.98	120.96	115.94	110.92	105.9	100.87	95.848	90.822
140	140	139	137.99	135.99	130.97	125.95	120.93	115.91	110.88	105.86	100.84
150	150	149	147.99	145.99	140.97	135.95	130.93	125.91	120.89	115.87	110.85
160	160	159	157.99	155.99	150.97	145.95	140.94	135.92	130.9	125.88	120.86
170	170	169	167.99	165.99	160.97	155.96	150.94	145.92	140.91	135.89	130.87
180	180	179	177.99	175.99	170.97	165.96	160.94	155.93	150.91	145.9	140.88
190	190	189	187.99	185.99	180.98	175.96	170.95	165.93	160.92	155.9	150.88
200	200	199	197.99	195.99	190.98	185.96	180.95	175.94	170.92	165.91	160.89
210	210	209	208	205.99	200.98	195.97	190.95	185.94	180.93	175.91	170.9
220	220	219	218	215.99	210.98	205.97	200.95	195.94	190.93	185.92	180.9
230	230	229	228	225.99	220.98	215.97	210.96	205.94	200.93	195.92	190.91
240	240	239	238	235.99	230.98	225.97	220.96	215.95	210.94	205.92	200.91
250	250	249	248	245.99	240.98	235.97	230.96	225.95	220.94	215.93	210.92