The Libyan Journal of Science (An International Journal): Volume 23, 2020.

The Strong Coarse Homotopy Groups

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Abstract

In this Article the cofibration category machinery (see [1], [2]) is used to define strong coarse homotopy groups (for preliminaries see [3],[4],[5],[6]) in the pointed quotient coarse category *PQcrs* via the notion of strong coarse homotopy as defined in [7] which we proved to be a Baues cofibration category in [8], [9]. In general, our definition is the same as the abstract definition. Later we also show that there is a surjective homomorphism between these groups and the n-dimensional coarse spheres.

Keywords: Homotopy Theory, Baues Cofibration Category, Coarse Geometry Version.

المستخلص

في هذه الورقة تم إستخدام آلية فئة الكوفايبريشن في تعريف زمر الهوموتوبي الخشن القوي في فئة حاصل الضرب المدببة الخشنة *PQcrs* بإستخدام تعريف الهوموتوبي الخشن القوي ، والذي أثبتنا في [8] ,[9] أنه فئة الكوفايبريشن للعالم الرياضي باوز (Baues). بصورة عامة التعريف المستخدم هو التعريف المجرد. وسوف أثبت أنه يوجد تشاكل تقابلي بين هذه الزمر وبين الكرة الخشنة نونية البعد.

Preliminaries

This definition comes from [10].

Definition a.1. Let X be a set. Then X is called *a unital coarse space* if it is equipped with a coarse structure, defined to be a collection ε of subsets M of $X \times X$ called entourages satisfying the following axioms:

(a.1.1): If $M \in \varepsilon$ and $M' \subseteq M$, then $M' \in \varepsilon$.

(a.1.2): Let $M_1, M_2 \in \varepsilon$ then $M_1 \cup M_2 \in \varepsilon$ and $M_1 M_2 \in \varepsilon$ where $M_1 M_2 = \{(x, z): (x, y) \in M_1, (y, z) \in M_2 \text{ for some } y\}$. We call $M_1 M_2$ the composite of M_1 and M_2.

(a.1.3): $\Delta_X \in \varepsilon$ where $\Delta_X = \{(x, x) : x \in X\}$.

(a.1.4): $\bigcup_{M\in\varepsilon} M = X \times X$.

Accepted for publication: 11/8/2020

(a.1.5): If $\in \varepsilon$, $M^t = \{(y,x):(x,y) \in M\} \in \varepsilon$.
We can use (X,ε) to refer to a coarse space when we need to emphasize the collection of *entourages.*
A subset M is called *symmetric* if $M = M^t$.
A non-unital coarse space is a coarse space defined as above, but we drop the axiom where Δ_X must be an entourage.

The following definition comes from [11].
Definition a.2. Let R be the topological space $[0,\infty)$ equipped with a coarse structure compatible with the topology (see (1.5) in [9]). We call the space R *a generalised ray* if the following conditions hold:

(i) The sum $M + N = \{(u + x, v + y) | (u,v) \in M, (x,y) \in N\}$ is an entourage for any entourage $M, N \subseteq R \times R$.
(ii) The set $M^s = \{(u,v) \in R \times R | x \le u, v \le y, (x,y) \in M\}$ is an entourage. for any entourage $M \subseteq R \times R$.
(iii) The set $a + N = \{(a + x, a + y) | (x,y) \in N\}$ for any entourage $N \subseteq R \times R$, and any $a \in R$.

The following definitions are prompted from [12].
Definition a.3. Let X, Y be coarse spaces and $f : X \to Y$ a map.
(a.3.1) We call f *a locally proper map* if $f|_{X'}$ is proper whenever $X' \subseteq X$ is a unital coarse subspace, that is, the inverse image of a bounded set $B \subseteq Y$ under the map $f|_{X'}$ is bounded.
(a.3.2) We call f *a coarse map between non-unital coarse spaces* if it is a controlled (see (1.2) in [9]) and locally proper map. And we mean by f being *a controlled map* if for every entourage $M \subseteq X \times X$, the image $f[M] = \{(f(x), f(y)) : (x,y) \in M\}$ is an entourage.

Definition a.4. Let f; $g : X \to Y$ be two coarse maps between non-unital coarse spaces. We say that f *is close to* g if for any unital subspace $X' \subseteq X$, we have $f|_{X'}$ is close to $g|_{X'}$, that is; $\{(f(s), g(s)): s \in X'\}$ is an entourage. For details see [8].
We call f *a coarse equivalence between non-unital coarse spaces* if $f|_{X'}$ is a coarse equivalence whenever $X' \subseteq X$ is a unital coarse subspace, that is; there is coarse map The $g: Y \to X'$ such that the compositions $f|_{X'} \circ g$ and $g \circ f|_{X'}$ are close to the identitymaps 1_Y and $1_{X'}$ respectively. For details see [8].
Let X be a topological space. The product $X \times [0,1]$ is called *a cylinder* on X.
We need to define a coarse version of the topological cylinder in order to define a coarse version of homotopy.

The following definition comes from [13].
Definition a.5. Let X be a coarse space, R be a generalised ray, and $p: X \to R$ be some controlled map. Then we define the *p-cylinder of X*:

$$I_p X = \{(x,t) \in X \times R \mid t \le p(x) + 1\}$$

The p-cylinder is a coarse space. We define the projection $p': I_pX \to R$ by the formula $p'(x,t) = p(x) + t$ and we define coarse maps $i_0, i_1 : X \to I_pX$ by the formula $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x) + 1)$ respectively.

Our aim in this work is to define coarse homotopy theory using Baues cofibration category technique on the category of non-unital coarse spaces. The above definition yields ideas of homotopy and mapping cylinder which are vital to the construction.

Definition a.6. Let $f_0, f_1: X \to Y$ be unital coarse maps. *A coarse homotopy* between f_0, f_1 is a coarse map $H: I_pX \to Y$ for some controlled map $p: X \to R$ such that $f_0 = H \, o \, i_0$ and $f_1 = H \, o \, i_1$ respectively.

We say the maps $f_0, f_1: X \to Y$ are *coarsely homotopic between non-unital coarse spaces* if $f_0|_{X'}$ is coarsely homotopic to $f_1|_{X'}$ whenever $X' \subseteq X$ is a unital coarse subspace.
A coarse map $f : X \to Y$ is termed *coarse homotopy equivalence* if there is a coarse map $g : Y \to X$ such that the compositions $g \, of$ and $f \, og$ are coarsely homotopic to the identities 1_X and 1_Y respectively.

The following definition is from [10].
Definition a.7: Let X be a subspace of the unit sphere S^{n-1} . Then we define *the open cone* of X to be the metric space

$$CX = \{\lambda x: \lambda \in \mathbb{R}^+, x \in X\} \subseteq \mathbb{R}^n$$

The open cone CX is a coarse space. The coarse structure is defined by the Eucli-dean metric on $\mathbb{R}^n$.

The cone of S^{n-1} is the Euclidean space $\mathbb{R}^n$, and the n-cell D^n can be viewed as the upper hemisphere in the cone of S^n , so its cone is $\mathbb{R}^n \times \mathbb{R}^+$

The following definition comes from [10] .
Definition a.8:. Let X and Y be coarse spaces. Then we define the *disjoint union* to be the set $X \sqcup Y$ equipped with the coarse structure given by defining the entourages to be subsets of unions of the form

$$M \cup N \cup (B_X \times B_Y) \cup (B'_Y \times B'_X)$$

where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages, and $B_X, B'_X \subseteq X$ and $B_Y, B'_Y \subseteq Y$ are bounded subsets. We denote this disjoint union by $X \sqcup Y$.

The following result is easy to check.
Proposition a.9: Let X and Y be coarse spaces, R be a generalised ray. Let $p_X : X \to R$ and $p_Y : Y \to R$ be controlled maps. Then $X \sqcup Y$ is a coarse space and the map $p_{X \sqcup Y} : X \sqcup Y \to R$ defined by the formula

$$p_{X \sqcup Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map.■

The following definition comes from [14] and [15].

Definition a.10: Let R be a generalized ray, $n \in \mathbb{N}$. Write $S_R^{n-1} = (\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}}$, $D_R^n = (\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}} \times \mathrm{R}$ where R $\sqcup$ R means two disjoint copies of R. We call S_R^{n-1} a coarse R-sphere of dimension $n-1$, D_R^n a coarse R-cell of dimension n, and the coarse R-sphere $\{(x,0) \in D_R^n : x \in S_R^{n-1}\}$ is called *the boundary of the coarse R-cell* D_R^n, i.e. $\partial D_R^n = S_R^{n-1} \times \{0\}$.

The Main Structure

Definition 1: Let X and Y be non-unital pointed coarse spaces. Then we write $[\mathrm{X}, \mathrm{Y}]_{\mathrm{R}}$ to denote *the set of strong coarse homotopy classes of pointed coarse maps from X to Y relative to R*.

Note that we have a canonical base (trivial) element of the set $[\mathrm{X}, \mathrm{Y}]_{\mathrm{R}}$ defined by the strong coarse homotopy class of the pointed coarse map relative to a generalized ray R

$$\mathrm{X} \overset{p_{\mathrm{X}}}{\to} \mathrm{R} \overset{i_{\mathrm{Y}}}{\to} Y$$

where $\mathrm{p_x}$ is some controlled map, and $\mathrm{i_Y}$ is the basepoint in the space Y.
If $i : \mathrm{R} \to \mathrm{A}$ is the basepoint inclusion, we set $\mathrm{I_R A} = \mathrm{I}_{i[\mathrm{R}]}\mathrm{A}$.

Definition 2: For a given based object A in the category *PQcrs*, we define the torus $\sum_{\mathrm{R}} \mathrm{A}$, where R $\subseteq$ A by the push out diagram

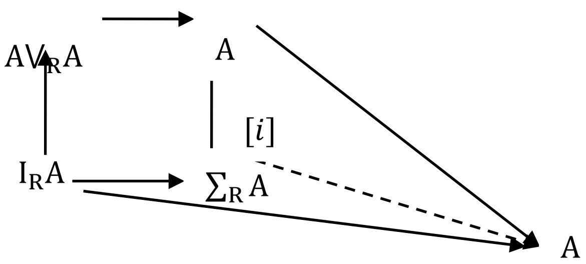


Figure 1: torus $\sum_{\mathrm{R}} \mathrm{A}$

Here the space $\sum_{\mathrm{R}} \mathrm{A}$ is a based object. To draw an explicit picture, we take $\mathrm{A} = \mathrm{R} \sqcup \mathrm{R}$ as an example and the torus can be seen by lemma (3.10) in [7] to be coarsely equivalent to the following space

$$\Big(\sum\nolimits_{\mathrm{R}} (\mathrm{R} \sqcup \mathrm{R})\Big)_{\mathrm{Glue}} = \{(x,t) \in (\mathrm{R} \sqcup \mathrm{R})^2 : -|x| - 1 \leq t \leq |x| + 1\}/\sim$$

where $(s,t) \sim (s,-t)$ for all s $\in$ R and $-s - 1 \leq t \leq s + 1$, and $(x, |x| + 1) \sim (x, -|x| - 1)$ for all $x \in \mathrm{R} \sqcup \mathrm{R}$. Geometrically it can be viewed as follows

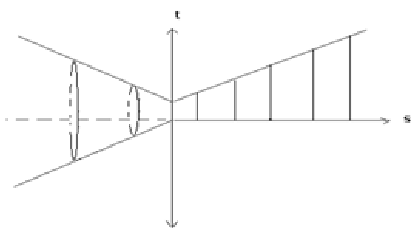
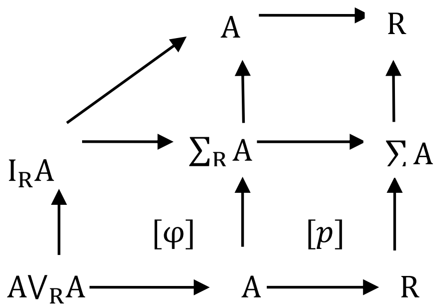


Figure 2. $(\sum_{\mathrm{R}}(\mathrm{R} \sqcup \mathrm{R}))_{\mathrm{Glue}}$.

Definition 3: Given a based object A in the category *PQcrs* and let $[\varphi]: \mathrm{AV_R A} \rightarrow \mathrm{A}$ be the folding class. Then the suspension, $\sum$A, is defined by the commutative diagram



where the lower two squares are push out diagrams. Here the spaces $\mathrm{I_R A}$, $\sum_{\mathrm{R}}$A, and $\sum$A are based objects.

Explicitly, and by lemma (3.10) in [7] we define the suspension of our above example to be coarsely equivalent to the following space again;
$(\sum(\mathrm{R} \sqcup \mathrm{R}))_{\mathrm{Glue}} = \{(x,t) \in (\mathrm{R} \sqcup \mathrm{R})^2 : -|x| - 1 \leq t \leq |x| + 1\}/\sim$ where $(s,t) \sim (s,-t)$ for all $s \in \mathrm{R}$, $-s - 1 \leq t \leq s + 1$, $(x,|x|+1) \sim (x,-|x|-1)$ for all $x \in \mathrm{A}$, and $(x,|x| + 1) \sim (-x,|x| + 1)$ for all $x \in \mathrm{R}$.

The above space will be seen as the following:

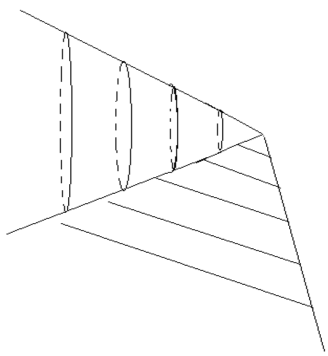


Figure 3. $(\sum(\mathrm{R} \sqcup \mathrm{R}))_{\mathrm{Glue}}$

By lemma (2.13) in [5], since $\mathrm{I}_p\mathrm{A}$ is coarsely homotopy equivalent to R, and any bounded subset is coarsely equivalent to a point, our suspension is coarsely homotopic to the space in the following figure, which is isomorphic to the space $(\mathrm{R} \sqcup \mathrm{R})^2$

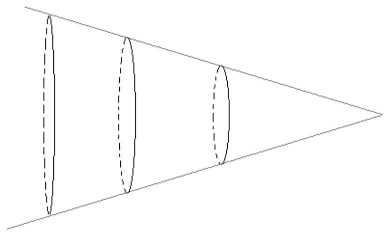


Figure 4. $(\sum(\mathrm{R} \sqcup \mathrm{R}))_{\mathrm{Glue}} \simeq (\mathrm{R} \sqcup \mathrm{R})^2$

This implies by the above that the space $\sum(\mathrm{R} \sqcup \mathrm{R})$ equipped with the Coequalizer coarse structure is coarsely homotopic equivalent to the coarse sphere $\mathrm{S}^1_{\mathrm{R}} = (\mathrm{R} \sqcup \mathrm{R})^2$.

The suspension $\sum \mathrm{A}$ depends on the choice of the coarse map p. Since $\sum \mathrm{A}$ is a based object, we can define inductively $\sum^n \mathrm{A} = \sum(\sum^{n-1} \mathrm{A}), n \geq 1, \sum^0 \mathrm{A} = \mathrm{A}$.

The following theorem gives an important key to prove the main result.

Theorem 4: $\sum(R \sqcup R)^k$ is coarsely homotopy equivalent to $(R \sqcup R)^{k+1}$ for $k \geq 1$.

Proof. First let $k = 1$. Then the statement is true by the above calculation.

Now let $k = 2$. Then $(\mathrm{R} \sqcup \mathrm{R})^3 = (\mathrm{R} \sqcup \mathrm{R})^2 \times (\mathrm{R} \sqcup \mathrm{R})$. By the above this is coarsely homotopy equivalent to $\sum(\mathrm{R} \sqcup \mathrm{R}) \times (\mathrm{R} \sqcup \mathrm{R})$. It is enough to show that $\sum(\mathrm{R} \sqcup \mathrm{R}) \times (\mathrm{R} \sqcup \mathrm{R})$ is coarsely homotopic
equivalent to $\sum(\mathrm{R} \sqcup \mathrm{R})^2$

We have $\mathrm{I}_{\mathrm{R}}(\mathrm{R} \sqcup \mathrm{R})^2 \simeq \mathrm{I}_{\mathrm{R}}(\mathrm{R} \sqcup \mathrm{R})^2 \sqcup \mathrm{I}_{\mathrm{R}}(\mathrm{R} \sqcup \mathrm{R})^2/\sim$ such that $(a, 0, t) \sim (a, 0, -t)$, $a \in \mathrm{R}$, $p: (\mathrm{R} \sqcup \mathrm{R})^2 \rightarrow R$ is some controlled map defined by $p(x,t) = ||(x,t)||, x, t \in R \sqcup R$. By lemma (2.13) in [9], $\mathrm{I}_{\mathrm{R}}(\mathrm{R} \sqcup \mathrm{R})^2$ is coarsely homotopy equivalent to $(\mathrm{R} \sqcup \mathrm{R})^2$.

Define a map $q : \mathrm{R} \sqcup \mathrm{R} \rightarrow R$ by $q(x) = p(j(x)) = p(x, 0) = |x|$ where $j : \mathrm{R} \sqcup \mathrm{R} \rightarrow (\mathrm{R} \sqcup \mathrm{R})^2$ is the inclusion, then q is a controlled map, and again by lemma (2.13) in [9], $I_q(\mathrm{R} \sqcup \mathrm{R})$ is coarsely homotopy equivalent to $(\mathrm{R} \sqcup \mathrm{R})$. Therefore $I_p(R \sqcup R)^2$ is coarsely homotopy equivalent to $I_q(R \sqcup R) \times (R \sqcup R)$.

Define a map $f : I_q(R \sqcup R) \times (R \sqcup R) \rightarrow I_p(R \sqcup R)^2$ as follows

$$f(x,t,y) = \begin{cases} (\mathrm{x},\mathrm{y},\mathrm{t}) & \mathrm{t} \leq \mathrm{p}(\mathrm{x},\mathrm{y}) + 1 \\ (x, y, \mathrm{p}(\mathrm{x},\mathrm{y}) + 1) & \mathrm{t} \geq \mathrm{p}(\mathrm{x},\mathrm{y}) + 1 \end{cases}$$

We have another map $\mathrm{g} : I_p(R \sqcup R)^2 \rightarrow I_q(R \sqcup R) \times (R \sqcup R)$ defined as follows

$$g(x,t,y) = \begin{cases} (\mathrm{x},\mathrm{t},\mathrm{y}) & \mathrm{t} \leq \mathrm{q}(\mathrm{x}) + 1 \\ (x, \mathrm{q}(\mathrm{x}) + 1, y) & \mathrm{t} \geq \mathrm{q}(\mathrm{x}) + 1 \end{cases}$$

Then f, g are coarse maps, also we have

$$fog(x,y,t) = \begin{cases} (x,y,t) & \mathrm{t} \leq \mathrm{q}(\mathrm{x}) + 1 \\ (x, y, \mathrm{q}(\mathrm{x}) + 1) & \mathrm{q}(\mathrm{x}) + 1 \leq \mathrm{t} \leq \mathrm{p}(\mathrm{x},\mathrm{y}) + 1 \\ (x, y, \mathrm{p}(\mathrm{x},\mathrm{y}) + 1) & \mathrm{t} \geq \mathrm{p}(\mathrm{x},\mathrm{y}) + 1 \end{cases}$$

and

$$gof(x,t,y) = \begin{cases} (x,t,y) & t \leq q(x) + 1 \\ (x, q(x) + 1, y) & q(x) + 1 \leq t \leq p(x,y) + 1 \\ (x, q(x) + 1, y) & t \geq p(x,y) + 1 \end{cases}$$

It is very easy to verify that fog and gof are coarsely homotopy equivalent to the identities.

Consider the space

$$I_R(R \sqcup R) \times (R \sqcup R) = (I_q(R \sqcup R) \sqcup Iq(R \sqcup R)/\sim) \times (R \sqcup R)$$

where $(a,t) \sim (a,-t)$, $a \in R$.

This can be written as:

$(I_q(R \sqcup R) \times (R \sqcup R) \sqcup Iq(R \sqcup R) \times (R \sqcup R))/\sim$ where $(a,t,0) \sim (a,-t,0)$, $a \in R$. The above shows that $I_R(R \sqcup R) \times (R \sqcup R)$ is coarsely homotopy equivalent to $I_p(R \sqcup R)^2 \sqcup I_p(R \sqcup R)^2/\sim$ where $(a,0,t) \sim (a,0,-t)$, $a \in R$, and the later space is exactly $I_R(R \sqcup R)^2$.

Hence $I_R(R \sqcup R) \times (R \sqcup R)$ is coarsely homotopy equivalent to $I_R(R \sqcup R)^2$.

Now, look at the space

$\sum_R(R \sqcup R) \times (R \sqcup R) = (I_R(R \sqcup R) \sqcup (R \sqcup R))/\sim \times (R \sqcup R)$ where

$(x, q(x) + 1) \sim (x, -q(x) - 1), x \in R \sqcup R$. This is equal to $I_R(R \sqcup R) \times (R \sqcup R) \sqcup (R \sqcup R)^2$ /~where $(x, q(x) + 1, y) \sim (x, -q(x) - 1, y)$, $x, y \in R \sqcup R$.

Again the above shows that $\sum_R(R \sqcup R) \times (R \sqcup R)$ is coarsely homotopy equivalent to $I_R(R \sqcup R)^2 \sqcup (R \sqcup R)^2/\sim$ where $(x,y,p(x,y)+1) \sim (x,y,-p(x,y)-1), x,y \in R \sqcup R$, and this is exactly $\sum_R(R \sqcup R)^2$. Therefore

$$\sum\nolimits_R (R \sqcup R) \times (R \sqcup R) \simeq \sum\nolimits_R (R \sqcup R)^2$$

Finally, consider

$$\sum (R \sqcup R) \times (R \sqcup R) = (\sum\nolimits_R (R \sqcup R) \sqcup R)/\sim) \times (R \sqcup R)$$

where $(x, q(x)+1) \sim (y, q(y)+1)$ if $q(x) = q(y), x, y \in R \sqcup R$. This is equal to

$$\sum\nolimits_R (R \sqcup R) \times (R \sqcup R) \sqcup R \times (R \sqcup R)$$

where $(x, q(x)+1, z) \sim (y, q(y)+1, z)$if $q(x) = q(y), x, y, and\ z \in (R \sqcup R)$.

By the same technique used in lemma (1.8) in [7] we can prove that $R \times (R \sqcup R)$ is coarsely homotopic to R. All that show

$$\sum (R \sqcup R) \times (R \sqcup R) \simeq \sum\nolimits_R (R \sqcup R)^2 \sqcup R$$

where$(x,z,p(x,z) + 1) \sim (y,z,p(y,z)+1)$ if $p(x,z) = p(y,z),\ x, y, and\ z \in R \sqcup R$.

This is exactly $\sum(R \sqcup R)^2$. Therefore

$$\sum(R \sqcup R) \times (R \sqcup R) \simeq \sum(R \sqcup R)^2.$$

Similarly we can prove that $\sum(R \sqcup R)^k \times (R \sqcup R)$ is coarsely homotopic equivalent to $\sum(R \sqcup R)^{k+1}$ for $k > 2$.

In this stage we have proved that $(\mathrm{R} \sqcup \mathrm{R})^3$ is coarsely homotopic equivalent to $\sum(\mathrm{R} \sqcup \mathrm{R})^2$. Therefore the statement $(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}}$ is coarsely homotopic to $\sum(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}-1}$ is true for n = 1, 2, and 3.
Now suppose that $(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}}$ is coarsely homotopic equivalent to $\sum(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{n}-1}$ for $n = k$. We need to prove the statement is true for $n = k + 1$.
By the above it is easy to see that:

$$(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{k}+1} = (\mathrm{R} \sqcup \mathrm{R})^{\mathrm{k}} \times (R \sqcup R) \simeq \sum(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{k}-1} \times (\mathrm{R} \sqcup \mathrm{R}) \simeq \sum(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{k}}$$

Hence by induction the statement is true for all n, and we are done. ■

Corollary 5: $\sum(\mathrm{R} \sqcup \mathrm{R})^{\mathrm{k}}$is coarsely homotopy equivalent to $\sum^K(R \sqcup R)$ for $k \geq 0$.
Proof. Straightforward by the above theorem and the properties of the suspension. ■

Definition 6: Let A and X be non-unital pointed coarse spaces. Let $n \geq 0$. Then we define *the n-th coarse homotopy group with respect to A to* be the set of coarse homotopy classes of pointed coarse maps $\sum^n A \to X$ relative to R, and denoted by $\pi_n^A(X)$ where

$$\pi_n^A(X) = [\sum^n A, X]_R$$

Proposition 7: If $A = R \sqcup R$, then $\pi_n^{R \sqcup R}(X)$ isomorphic to the group $\pi_n^{Pcrs}(X, R)$ in definition (2.4) in [7].
Proof. : Straightforward by theorem (4). ■

Definition 8: Let A and X be non-unital pointed coarse spaces. Let $n \geq 0$. Then we define *the n-th strong coarse homotopy group* to be the set of strong coarse homotopy classes of pointed coarse maps $\sum^n A \to X$ relative to R

$$\pi_n^{A,Strong}(X) = [\sum^n A, X]_R^{Strong}$$

In particular if $A = R \sqcup R$, we have:

$$\pi_0^{R \sqcup R,Strong}(X) = [\mathrm{R} \sqcup \mathrm{R}, \mathrm{X}]_{\mathrm{R}}^{\mathrm{Strong}} = [\mathrm{R}, \mathrm{X}]^{\mathrm{Strong}}$$

so we define the set $\pi_0^{R \sqcup R,Strong}(X)$ to be the set of strong coarse homotopy classes (not relative homotopy classes) of coarse maps $R \to X$, and we define the higher coarse homotopy groups by writing

$$\pi_n^{R \sqcup R,Strong}(X) = [\sum^n (\mathrm{R} \sqcup \mathrm{R}), \mathrm{X}]_{\mathrm{R}}^{\mathrm{Strong}}$$

Corollary 9: We have a well defined surjective homomorphism

$$\alpha: \pi_n^{A,Strong}(X) \to \pi_n^A(X)$$

Proof. Let $f:\sum^n A \to X$ be a pointed coarse map. By theorem (3.15) in [9], if $[f]_{\text{Strong}} = [g]_{\text{Strong}}$ then $[f] = [g]$. So we have a homomorphism

$$\alpha: \pi_n^{A,Strong}(X) \to \pi_n^{A}(X)$$

defined by $\alpha([f]_{\text{Strong}}) = [f], and\ \alpha$ is clearly surjective. ■

Definition 10: Let $[i] : A \hookrightarrow X$ be a coarse cofibration class. We write $[I_R A; X]_R^0$ to denote *the set of relative coarse homotopy classes* of coarse maps $F : I_R A \to X$ such that the map F restricts to the base element

$$\mathrm{A} \xrightarrow{p_{\mathrm{A}}} \mathrm{R} \xrightarrow{i_{\mathrm{X}}} X$$

at the ends of the cylinder.

There is a canonical map from the set $[\sum A, X]_R$ to the set $[I_R A; X]_{\mathrm{R}}^0$ arising from the maps on the top row in the push-out diagram in the category *PQcrs*

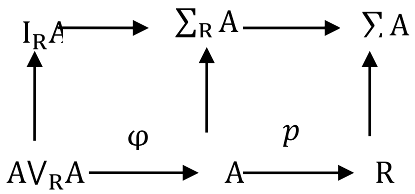


Figure 5: The suspension $\sum$ A

used to define the suspension.

Proposition 11: The above canonical map $[\sum A, X]_R \to [I_R A; X]_{\mathrm{R}}^0$ is a bijection.
Proof. By construction of abstract cofibration categories, every object in the pointed quotient coarse category is both cofibrant and based. It follows that the quotient map $I_R A \to \sum_{\mathrm{R}} \mathrm{A}$ induces a bijection $[\sum_R A, X]_R \to [I_R A, X]_{\mathrm{R}}^0$ from results in section (2) of chapter (II) of [1]. Also from [1], sections 5 and 6 of chapter (II), we have $[I_R A, X]_{\mathrm{R}}^0$
is a group, and by proposition (2.11) (b)) in [1], the quotient map $\sigma: \sum_R A \to \sum A$ yields a bijection $[\sum A, X]_R =^{\sigma^*} [\sum_R A, X]_R$. Since the composite of bijections is a bijection, we are done. ■

The abstract proof of the following proposition can be found in [1].
Proposition 12: Let $n \geq 1$. Then the set $\pi_n^{A,Strong}(X)$ is a group. The operation is defined by composition of strong coarse homotopies using the last proposition. The identity element is the strong coarse homotopy class of closeness class of the base map

$$I_R A \xrightarrow{p_{I_R A}} \mathrm{R} \xrightarrow{i_{\mathrm{X}}} X$$

Further. For n $\geq$ 2, the strong coarse homotopy group $\pi_n^{A,Strong}(X)$ is abelian. ■

Theorem 13. Let X be a non-unital pointed coarse space. Then there is a surjective homomorphism $\beta: \pi_n^{R\sqcup R, Strong}(X) \to [S_R^n, X]_R$.

Proof. First, by corollary (9), we have a surjective homomorphism

$$\alpha: \pi_n^{R\sqcup R, Strong}(X) \to \pi_n^{R\sqcup R}(X)$$

and by proposition (7) we have $\pi_n^{R\sqcup R}(X)$ is isomorphic to the group $[S_R^n, X]_R$. So it is enough to show that $\sum^n(R \sqcup R)$ is coarsely homotopic to $(R \sqcup R)^{n+1}$ which is so by theorem (4), and we are done. ■

References

[1] Baues, H. J. (1989). Algebraic homotopy, volume 15 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge.

[2] Baues, H. J. (1999). Combinatorial Foundation of Homology and Homotopy. Springer Monographs in Mathematics. Springer-Verlag, Berlin.

[3] Roe, J. (1993). Coarse Cohomology and Index Theory on Complete Riemannian Manifolds, American Mathematical Society Memoirs. American Mathematical Society. V.497.

[4] Roe, J. (1996). Index Theory, Coarse Geometry, and Topology of Manifolds (Vol. 90). American Mathematical Soc..

[5] Roe, J. (2003). Lectures on Coarse Geometry (No. 31). American Mathematical Soc..

[6] Selick, P. (2008). Introduction to homotopy theory (Vol. 9). American Mathematical Soc..

[7] Gheith, N. E. M. (Submitted for publishing). Relative Coarse Homotopy.

[8] Gheith, N. E. M. (2013). Coarse Version of Homotopy Theory (Axiomatic Structure). PhD thesis, University of Sheffield.

[9] Gheith, N. E. M. and Mitchener, P. D. (2020) Coarse examples of cofibration category. Sarajevo Journal of Mathematics، **16** (29) (1) 83-103..

[10] Mitchener, P. D. Norouzizadeh, B. and Schick, T. (2020) Coarse homotopy groups.Mathemtische Nachrichten. 293 (6) :1515-1533.

[11] Mitchener, P. D. (2001). Coarse homology theories. Algebr. Geom. Topol, **1**, 271-297 (electronic).

[12] Luu, V. T. (2006). Coarse categories i: Foundations. Unpublished, Found in: http://arxiv.org/abs/0708.3901,.

[13] Mitchener, P. D. (2010). The general notion of descent in coarse geometry. Algebr. Geom. Topol., **10**(4), 2419-2450.

[14] Grave, B. (2006). Coarse Geometry and Asymptotic Dimension. PhD thesis, Georg-August universitat Gottingen.

[15] Mitchener, P. D. (2001). Addendum to: "Coarse homology theories" [Algebr. Geom. Topol. 1:271-297 (electronic); mr1834777]. [Algebr. Geom. Topol., 3:1089{1101 (electronic), 2003].