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On the System of Nonlinear Elliptic Equations with Nonlocal Boundary Conditions of Neumnna Type

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Abstract

In this paper, we consider a system of nonlinear elliptic equations with nonlocal boundary conditions of Neumman type of the form:
* $\partial v^{*}u = h_{1,j}\left(x, u\big(\emptyset(x)\big)\right)$, $\partial v^{*}u = \sum_{k=1}^{n} f_j^k(x,u,\partial u)\, v_k$
Where, $\Omega \in IR^n$ is a bounded domain $\partial\Omega$ and belongs to c^1; v_k denote the coordinates of the normal unit vector on $\partial\Omega$. Also we prove the existence of a weak solution for such problem.

المستخلص

في هده الورقة سوفه يتم دراسة منظومة المعادلات الجزئية اللاخطيه من النوع ألناقصي مع شروط حديه غير محليه من نوع نويمان، والنطاق Ω والحدود $\partial\Omega$ محدودة وتنتمي إلي الفضاء C^1 كذلك سوف يتم برهان وجود الحل الضعيف لتلك المسالة .

Introduction

The aim of this paper is to prove the existence of a weak solution $u = (u_1, u_2, \ldots\ldots, u_M)$ of the system,

$$-\sum_{k=1}^{n} \partial_k\left[f_j^k(x,u,\partial u)\right] + f_j^0(x,u,\partial u) + \mathrm{g}_j(x,u) = F_j \quad in\ \Omega, j = 1,2,\ldots\ldots,\mathrm{M} \qquad (1)$$

With nonlinear and nonlocal boundary condition of the form:

$$\partial v^{*}u = h_{1,j}\left(x, u\big(\emptyset(x)\big)\right) \qquad (2)$$

Where, $\Omega \in IR^n$ is a bounded domain (may be unbounded),
$\partial\Omega$ is bounded and belongs to c^1;
$\partial v^{*}u = \sum_{k=1}^{n} f_j^k(x,u,\partial u)\, v_k$
v_k denote the coordinates of the normal unit vector on $\partial\Omega, \emptyset$ is c^1;
_diffeomorphism in a neighbourhood of $\partial\Omega$ such that $\emptyset(\partial\Omega) \subset \bar{\Omega}$

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It must be emphasized that the term $g_j\ (x, u(x))$ no growth restriction is imposed but it is supposed that the function g_j satisfies the sign condition $g_j(\ x, \eta)\ \ \eta \geq 0$. The existence of weak solution of (1) and (2) will be proved using the arguments of [1] . Weak solution of (1) and (2) will be defined as follows:
Assuming that u is a classical solution of (1) an (2) By Gauss-Ostrogrdsdskij theorem and by using an integral transformation we obtain:

$$\sum_{j=1}^{M} \left\{ \begin{matrix} \sum_{k=1}^{n} f_j^k \int_{\Omega} (x, u, \partial_1 u, \dots, \partial_n u) \partial_k v_j + \int_{\Omega} f_j^0\ (x, u, \partial_1 u \dots, \partial_n u) v_j \\ - \int_{s} \widetilde{h_{1,J}}(x, u) v_j(\emptyset^{-1}(x)) d\sigma_x + \int_{\Omega} g_j\ (x, u(x)) v_j \end{matrix} \right\} \quad (3)$$

$$= \sum_{j=1}^{M} \int_{\Omega} F_j v_j$$

For all $v_j \in C^1(\bar{\Omega})$, with compact support, $S = \emptyset(\partial\Omega) \subset \bar{\bar{\Omega}}$. Thus weak solution $u = (\ u_1\ , u_2\ , \dots\dots\dots\ , u_M\ )$ of (1) , (2) will be defined by (3) .
The nonlocal linear boundary value problems have been considered e .g in [2] and [3] and the importance of nonlocal in [4]. In [4], [5] and [6] nonlocal and nonlinear value problems have been studied.

Existence Theorem

Let $X := W_p^1(\Omega) \times \dots \times W_p^1(\Omega)$. Then X is a reflexive Banach Space.
Denote by X' the dual space of X. The points
$\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n) \in IR^{(n+1)M}$ $\left(\zeta_j = \left(\zeta_j^1, \zeta_j^2, \dots, \zeta_j^M\right) \in IR^M\right)$ Will be written in the form $\zeta = (\eta, \zeta)$ where $\eta = \zeta_0, \zeta = (\zeta_0, \zeta_1, \dots, \zeta_n)$
Assume that:

a) Function $f_j^k : \Omega \times IR^{(n+1)M} \to IR$ satisfies Carathodoary conditions, i.e. they are measurable for every fixed $\zeta \in IR^{(n+1)M}$ and continuous in ζ for a.e.x. Similarly function $h_{1,j} : s \times IR^M \to IR$ satisfies the Carathrodory conditions:

b) There exists a constant $c_1 > 0$ and a function $k_1 \in L^q(\Omega)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ such that $\left|f_j^k(x, \zeta)\right| \leq c_1 |\zeta|^{p-1} + k_1(x),\ \ j = 1,2, \dots, M, k = 1,2, \dots, n$

c) For all (η, ζ) , (η, ζ') in $IR^{(n+1)M}$ with $\zeta \neq \zeta'$

$$\sum_{j=1}^{M} \sum_{k=1}^{n} \left[f_j^k(x, \eta, \zeta) - f_j^k(x, \eta,)\zeta'\ \right] \left(\zeta_j - \zeta_j'\right) > 0$$

d) There exists a constant c_2 and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{j=1}^{M} \sum_{k=1}^{n} f_j^k(x, \zeta) \zeta_j^k \ \geq c_2 |\zeta|^p - k_2(x)$$

e) For any s> 0 , there exits $g_{j,s} \in L^1(\Omega)$such that for a.e. x $\in \Omega$

$$|g_j(x,\eta)| \leq g_{j,s}(x) \quad if \quad |\eta| \leq s, \qquad j = 1,2,\dots.M$$

f) For any $\eta \in IR^M$,and for a.e. x $\in \Omega$

$$g_j(x,\eta)\eta_j \geq 0 \quad , \quad j = 1,2,\dots\dots, M$$

g) There exist a constants $c_3 > 0$ and a fixed function $k_3 \in L^{1+\frac{1}{\rho}}(s)$ such that

$$|h_{1,j}(x,\eta)| \leq c_3|\eta|^\rho + k_3(x)$$

where 0$< \rho < p - 1$

Results

The main result of this paper is the following:
Theorem: Suppose that the assumptions a-g are satisfied, then for any

$$F_j \in \left(w_p^1(\Omega)\right)'$$

There exits $u \in X$ such that $g_j(x,u) \in L^1(\Omega)$, $g_j(x,u)u_j \in L^1(\Omega)$, and (3) holds for all $v \in X$ Satisifying $v \in L^\infty(\Omega) \times \dots \times L^\infty(\Omega)$, $v|_{\partial\Omega} \in L^\infty(\partial\Omega) \times \dots \times L^\infty(\partial\Omega)$ and for $v = u$.
For any $u, v \in X$ let$\langle T(u), v\rangle := \sum_{j=1}^{M}\sum_{k=1}^{n}\int_\Omega f_j^k(x,u,\partial_1 u \dots, \partial_n u)\partial_k v_j +$
$+\sum_{j=1}^{M}\int_\Omega f_j^0(x,u,\partial_1 u \dots., \partial_n u)v_j - \sum_{j=1}^{M}\int_s \widetilde{h_{1,J}}(x,u)v_j\big(\emptyset^{-1}(x)\big)d\sigma_x$ (4)

And for any $\mu > 0$ Let

$$g_j^\mu(x,\eta) := \begin{cases} g_{j(x,\eta)} & if \ |\eta| \leq \mu \ ,|x| \leq \mu \\ \mu g_{j(x,\eta)} & if \ |\eta| > \mu \ ,|x| < \mu \\ |g_{j(x,\eta)}| & \\ 0 & if \ |x| < \mu \end{cases} \tag{5}$$

Define operator S_μ by

$$\langle S_\mu(u), u\rangle := \sum_{j=1}^{M}\int_\Omega g_j^\mu(x,u)\, v_j \tag{6}$$

Lemma 1. Operator $T + S_\mu \ : X \to X'$ is pseudomontate
Proof $T + S_\mu$ is bounded. Opertator T can be written in the form $T = A - B$. Where

$$\langle A(u), v\rangle := \sum_{j=1}^{M}\sum_{k=1}^{n}\int_\Omega f_j^k(x,u,\partial_1 u \dots, \partial_n u)\partial_k v_{j+}$$

$$+\sum_{j=1}^{M}\int_\Omega f_j^0(x,u,\partial_1 u \dots., \partial_n u)v_j$$

$$\langle B(u), v\rangle := \sum_{j=1}^{M} \int_s \widetilde{h_{1,S}}(x,u) v_j\big(\emptyset^{-1}(x)\big) d\sigma_x$$

From conditions a-d it follows that A is pseudomonotone (See [7]). Let (u^l) be a sequence such that (u^l) converge weakly to u in X and $\lim_{l\to\infty} sup < T(u^l), u^l - u > \quad \leq 0$

Now, by compact of trace operator there exists a subsequence $(\tilde{u}_j^l)$ of (u_j^l) such that for all $j, \tilde{u}_j^l|_{\partial\Omega}$ converges to $u_j$in $L\tilde{q}(\partial\Omega)$. Where $\tilde{q} := \rho + 1 < p$. be condition (g) and holder is inequality

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ we have:

$$lim\, sup < B(u^l), u^l - u >= 0 \qquad (7)$$

And (7) is true also for the original sequrnce. Further we prove that

$$B(u^l) \overset{w'}{\to} B(u) \text{in } X' \qquad (8)$$

i.e. For All $v \in X$ $\langle B(u^l), v\rangle \overset{w'}{\to} \langle B(u), v\rangle$

We have seen that there exists a subsequence $(\tilde{u}^l)$of (u^l)such that $(\tilde{u}^l)|_{\partial\Omega}$ converges to u in $L\tilde{q}(\partial\Omega)$ and converges a.e. to u on $\partial\Omega$. Thus

$$h_{1,j}(x, \tilde{u}^l) \to h_{1,j}(x,u) \text{ a.e.on } \partial\Omega$$

By Holder s inequality and the bounded of the trace operator we have:

$$\lim_{l\to\infty} < B(\tilde{u}^l), v > = < B(u), v >$$

and it is easy to show that the above equality is true also for the original sequence i.e. we have proved (8), thus we have shown that if (u^l) converges weakly to u in X and

$$lim\, sup < T(u^l), u^l - u > \leq 0, lim\, sup < B(u^l), u^l - u >= 0 \qquad (9)$$

and

$$B(u^l) \overset{w'}{\to} B(u) \text{ in } X'. \qquad (10)$$

From (9) it follows that

$$\lim_{l\to\infty} sup < A(u^l), u^l - u > \leq 0$$

Since A is pseudomonotone thus

$$A(u^l) \overset{w'}{\to} A(u) \text{ in } X'.$$

And by (10)

$$T(u^l) \overset{w'}{\to} T(u) \text{ in } X'.$$

By (9) we have

$$\lim_{l\to\infty} < T(u^l), u^l - u > = 0$$

So we have shown that T is pseudomonotone operator. Now we shall prove that $T + S_\mu$ is pseudomonotone operator.

Suppose that (u^l) converges weakly to u in X and $(T+S_\mu)(u^l)$ converges weakly in X' to some y and

$$\lim_{l\to\infty} < (T+S_\mu)(u^l), u^l - u > \leq 0 \tag{11}$$

Then by compact imbedding theorems there is a subsequence (u^{l_k}) of (u^l) such that

$$\lim_{l\to\infty}(u^{l_k}) = u \text{ a.e. in } \Omega \text{ and on } \partial\Omega$$

Thus by Lebesguge's dominated convergence theorem

$$\lim_{l\to\infty}\left\|g_j^\mu(x,u^{l_k}) - g_j^\mu(x,u)\right\|_{L^q(\Omega)} = 0 \tag{12}$$

Where q defined by $\frac{1}{p}+\frac{1}{q}=1$, hence $\lim_{l\to\infty} S_\mu(u^{l_k}) = S_\mu(u)$ weakly in X' and so

$$\lim_{k\to\infty} T(u^{l_k}) = y - S_\mu \tag{13}$$

weakly in X' from equality

$$\langle S_\mu(u^{l_k}), u^l - u\rangle = < S_\mu(u^{l_k}) - S_\mu(u), u^{l_k} - u > + \quad + < S_\mu(u), u^{l_k} - u >$$

It follows that

$$\lim_{l\to\infty} < S_\mu(u^{l_k}), u^{l_k} - u > = 0$$

Because by (12), the boundness of $\|u^{l_k} - u\|_X$, $\left\|u_j{}^{l_k} - u_j\right\|_{L^p(\partial\Omega)}$

And Holder is inequality

$$\lim_{k\to\infty} < S_\mu(u^{l_k}) - S_\mu(u), u^{l_k} - u > = 0 \tag{14}$$

It is not difficult to show that (14), is true also for the original sequence. Therefore (11) implies.

$$\lim_{k\to\infty} sup \ \langle T(u^{l_k}), u^l - u\rangle \leq 0 \tag{15}$$

Since T is pseudomonotone thus by (13) and (15) we have $T(u) = y - S_\mu(u)$, i.e. $(T+S_\mu)(u) = y$

Further

$$\lim_{k\to\infty} < T(u^l), u^l - u > = 0$$
$$\lim_{l\to\infty} < (T+S_\mu)(u^l), u^l - u > = 0$$

Which completes the proof of lemma 1.

lemma 2. Assume that (u^l) converges weakly to u in X and there is a constant c such that

$$\sum_{j=1}^{M}\int_\Omega g_j(x,u^l)\,u_j \leq c \tag{16}$$

Then

$$g_{j(x,u)} \in L^1(\Omega) \ , \ g_{j(x,u)u_j} \in L^1(\Omega)$$

For all $j = 1,2, \dots, M$ and there is a subsequence (u^{l_k}) of (u^l) such that

$$\lim_{k\to\infty} u^{l_k} = u \text{ a.e in in } \Omega \text{ and on } \partial\Omega \quad (17)$$

Further

$$\lim_{k\to\infty} \left\| g_j^{l_k}(x, u^{l_k}) - g_j(x, u) \right\|_{L^1(\Omega)} = 0$$

Proof: As (u_j) converges weakly to u in X thus (by compact imbedding theorems) there exist a subsequence (u^{l_k}) of (u^l) such that

$$\lim_{k\to\infty} g^{l_k}(x, u^{l_k}) = g_j(x, u) \ \text{ for a.e } X \in \Omega \quad (18)$$

By (3),(4) and (16) and assumption (f) we have

$$\int_\Omega [g_j(x, u^l)] u_j{}^l \le c \quad (19)$$

Therefore by (18) and (f) implies

$$g_j \ (x, u) \ u_j \in L^1(\Omega)$$

For any $\delta > 0$ we have e)

$$\left| g_j^{l_k}(x, u^{l_k}) \right| \le g_j \delta^{-1}(x) + \delta \left| g_j^{l_k}(x, u^{l_k}) u_j{}^{l_k} \right|$$

This implies that $g^{l_k}(x, u^{l_k})$ is equiintegrable because by (19) $\int_E \left| g_j^{l_k}(x, u^{l_k}) \right| dx < \varepsilon$ if the measure of E is sufficiently small and there is a set $A\varepsilon$ of finite measure with

$$\int_{\Omega \times A_\varepsilon} \left| g_j^{l_k}(x, u^{l_k}) \right| < \varepsilon$$

By compact imbedding theorem and (18) this shows that

$$g_j^{l_k}(x, u^{l_k}) \to g_j \ (x, u) \ in \ L^1(\Omega)$$

lemma 3 : The Operator

$T + S_\mu : X \to X'$ is coercive, i.e.

$$\lim_{\|u\|\to\infty} \frac{\langle (T + S_\mu)(u), u \rangle}{\|u\|} = +\infty$$

Proof : From f) we have

$$\int_\Omega g_j^\mu(x, u) u_j \ge 0$$

This implies that $\langle S_\mu(u), u \rangle \ \ge 0$

Thus by using conditions d) and g) we obtain

$$\frac{\langle (T+S_\mu)(u),u\rangle}{\|u\|} = \frac{\langle T(u),u\rangle}{\|u\|} + \frac{\langle S_\mu(u),u\rangle}{\|u\|} \geq \frac{\langle (T)(u),u\rangle}{\|u\|} \geq C_2\|u\|_X^\rho - C_3 - C_4\|u\|_X^{\rho+1} - C_5 \quad (20)$$

($C_2 - C_5$ are positive constants) . From this inequality and $\rho + 1 < p$ it follows that $T + S_\mu$is coercive.

Proof of the Theorem

By Lemmas 1 and 2 the operator $T + S_j$ is bounded pseudomonotone and coercive of all $j = 1,2,3,\ldots\ldots, by$ using the well known theory of pseudo monotone operator in reflexive Banach space we obtain that for any $F \in X'$ there exists $u^l \in X$ such that

$$(T + S_j)(u^l) = F \quad (21)$$

By (20) where the constants are independent of (μ) and (21) the sequence$(u^l)$is bounded in X' . T is a bounded operator and so the sequence $T(u^{l_k})$ is bounded in X'. Since X is a reflexive Banach space, thus there exists a subsequence (u^{l_k}) of (u^l) and $u \in X$ such that

$$lim(u^{l_k}) = u \text{ weakly in } X, \quad (22)$$

and $limT(u^{l_k}) = y$ weakly in X' For some $y = X'$. Combining the definition of S_j with (21) we find that

$$\sum_{j=1}^{M} \int_\Omega g_j^{l_k}(x,u^{l_k})\, u_j^{l_k} = \langle S_{l_k}(u^{l_k}),u^{l_k}\rangle = \langle F,u^{l_k}\rangle \leq \|F\|_{X'}\, \|u^{l_k}\|_X < C$$

Thus by lemma 2.

$$g_j\,(x,u)\,u_j \in L^1(\Omega),\ g_j\,(x,u) \in L^1(\Omega) \quad (23)$$

And there is a subseqence (u^{l_k}) of (u^l) such that

$$lim(u^{l_k}) = u \text{ a.e. in } \Omega \text{ and on } \partial\Omega \text{and also} \quad (24)$$

$$\lim_{k\to\infty} \left\| g_j^{l_k}(x,u^{l_k}) - g_j\,(x,u)\right\|_{L^1(\Omega)} = 0 \quad (25)$$

According to (21) for any $v \in X$

$$\langle (T + S_{l_k})(u^{l_k}),v\rangle = \langle F,v\rangle \quad (26)$$

Consider in (26) a fixed $v \in X$ such that $v \in L^\infty(\Omega) \times \ldots \times L^\infty(\Omega)$
And $v|_{\partial\Omega} \in L^\infty(\partial\Omega) \times \ldots \times L^\infty(\partial\Omega)$
By using (23)-(26) we obtain as $k \to \infty$

$$\langle y,v\rangle + \textstyle\sum_{j=1}^{M} \int_\Omega g_j(x,u)v_j = \langle F,v\rangle \quad (27)$$

Now, we shall prove that $y = T(u)$.
Since T is pseudomonotone, thus, it is sufficient to prove the following inequalities

$$\lim_{k\to\infty} sup \langle T(u^{l_k}), u^{l_k} - u\rangle \leq 0$$

We have,

$$\langle T(u^{l_k}), u^{l_k} - u\rangle = \langle T(u^{l_k}), u^{l_k}\rangle - \langle T(u^{l_k}), u\rangle$$

And so by (22) and (26) and lemma 2.

$$\lim_{k\to\infty} sup \langle T(u^{l_k}), u^{l_k} - u\rangle = \lim_{k\to\infty} sup \langle F - S^{l_k}(u^{l_k}), u^{l_k}\rangle - \langle y, v\rangle -$$

$$- \leq \langle F - y, v\rangle - \lim_{k\to\infty} inf \left\{ \sum_{j=1}^{M} \int_{\Omega} \mathrm{g}^{l_k}{}_{\mathrm{j}} \left(\mathrm{x}, \mathrm{u}^{l_k}\right) \mathrm{v_j}^{l_k} \right\} \leq \langle F - y, v\rangle - \sum_{j=1}^{M} \int_{\Omega} \mathrm{g_j}(\mathrm{x}, \mathrm{u}) \mathrm{v_j}$$

Thus for any $w \in X \cap L^{\infty}(\Omega) \times \ldots \times L^{\infty}(\Omega)$by using (27).

$$\lim_{k\to\infty} sup \langle T(u^{l_k}), u^{l_k} - u\rangle \leq \langle F - y, u - w\rangle + \sum_{j=1}^{M} \int_{\Omega} [\mathrm{g_j}(\mathrm{x}, \mathrm{u})] \left(\mathrm{u_j} - \mathrm{u}\right) \quad (28)$$

Since $\partial\Omega$ is bounded and continuously differentiable , thus $u \in X$ can be extended to IR^nsuch that we obtain

$$u \in w_p^1(IR^n) \times \ldots\ldots \times w_p^1(IR^n)$$

We know that there is a subsequence $\left(w_j^l\right)$and $w_p^1(IR^n) \cap L^{\infty}(IR^n)$
such that$\left(w_j^l\right)$in $w_p^1(IR^n) \cap L^{\infty}(IR^n)$ and such that $\left(w_j^l\right)$ converges to $\left(\mathrm{u_j}\right)$
in $w_p^1(IR^n)$ and a.e. in IR^n, further

$$\left|w_j^l(x)\right| \leq \left|\mathrm{u_j}(x)\right| \text{a. e. in IR, j} = 1,2, \ldots, \mathrm{M} \quad (29)$$

consequently for the trace of $\left(w_j^l\right)$and $\mathrm{u_j}$. (29)

$$\left|w_j^l|_{\partial\Omega}(x)\right| \leq |\mathrm{u_j}|_{\partial\Omega}(x) \text{ in a.e. } X \in \partial\Omega \ . \quad (30)$$

By (29) and (30) we have

$$\langle F - y, u - w^l\rangle \to 0 \text{ and } \int_{\Omega} [\mathrm{g_j}(\mathrm{x}, \mathrm{u})] w_j^l \mathrm{dx} \to \int_{\Omega} [\mathrm{g_j}(\mathrm{x}, \mathrm{u})] u_j \mathrm{dx}$$

Since $\mathrm{g}(x, u)u \in L^1(\Omega)$. Thus from (28) it follows that

$$\lim_{k\to\infty} sup \langle T(u^{l_k}), u^{l_k} - u\rangle \leq 0$$

Consequently, $y = T(u)$, and

$$\langle T(u^{l_k}), u^{l_k} - u\rangle \to 0$$

Therefore from (27) we obtain (3) for all $v \in w_p^1(\Omega) \times \ldots\ldots \times w_p^1(\Omega)$
with $v \in L^{\infty}(\Omega) \times \ldots \times L^{\infty}(\Omega)$
$v|_{\partial\Omega} \in L^{\infty}(\partial\Omega) \times \ldots \times L^{\infty}(\partial\Omega)$.

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