The Libyan Journal of Science
University of Tripoli Vol. 27, No. 02
https://uot.ed.ly/journals/index.php/ljs

Metrizable Spaces with finitely many non–isolated points

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ARTICLE INFO

Article history:

Received 03/01/2024

Received in revised form 20/07/2024

Accepted 27/072024

ABSTRACT

A point x in a topological space X is said to be non-isolated if every open set containing x contains another point different from x. The aim of this paper is to give the topology and the metric induces that topology of a metrizable space X with exactly n non-isolated points .

Keywords: metrizable space; isolated point; free filter

1. Preliminaries

Let X be an infinite set, a space (X, τ) is said to be metrizable if there exists a metric induces the topology τ [1, 6]. A collection $\mathfrak{J}$ of non- empty subsets of X is called a filter if $F_1, F_2 \in \mathfrak{J}$ then $F_1 \cap F_2 \in \mathfrak{J}$ and if $G \in \mathfrak{J}$, $G \subseteq X$ with $F \subseteq G$, then $G \in \mathfrak{J}$. A filter $\mathfrak{J}$ in X is said to be free filter if $\bigcap_{F \in \mathfrak{J}} F = \varphi$. A subcollection $\wp$ of *a* filter $\mathfrak{J}$ is said to a filter base for $\mathfrak{J}$ if for any $F \in \mathfrak{J}$ there exists $C \in \wp$ with $C \subseteq F$[1 , 6] . A collection of subsets of a topological space X is said to be locally finite collection if every point in X has an open neighborhood which intersects only finitely many members of $\mathfrak{B}$. A collection $\mathfrak{B}$ of subsets of a topological space X is said to be $\sigma -$ locally finite if $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ where $\mathfrak{B}_n$ is locally finite collection for all [6]. A point x of a topological space X is called an isolated point if $\{x\}$ is open The following Theorems are respectively Theorem 2.2 and Theorem 2.1 of [5].

1.1. Theorem

A space (X, τ) is metrizable space with only one non- isolated point x_1 iff

$\tau = p(X \backslash \{x_1\}) \cup \{\{x_1\}) \cup F : F \in \mathfrak{J}\}$, where $\mathfrak{J}$ is a free filter in $X \backslash \{x_1\}$ with countable filter base $\{C_n\}_{n=1}^{\infty}$ with $C_{n+1} \subsetneq C_n$ for all n, $\bigcap_{n=1}^{\infty} C_n = \varphi$, and $P(X|\{x_1\})$ is the power set of $X \backslash \{x_1\}$.

1.2. Theorem

If X, τ, x_1 , $\{C\}_{n=1}^{\infty}$ and $\Im$ are as in the above Theorem , then the function $d: X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,\ y): \begin{cases} 1 & if\ x,y \in X\backslash\{x_1\}, x \neq y, x \notin C_1 or\ y \notin C_1 \\ 1 & if\big(\ x = x_1, y \notin C_1\big) or\ \big(y = x_1, x \notin C_1\big) \\ \frac{1}{n+1} & if\big(\ x = x_1, y \in C_1\big) \text{ or } \big(y = x_1, x \in C_1\big) \\ & where\ n\ is\ the\ least\ integer\ such\ that\ x \notin C_n \big(resp.\ y \notin C_n\big) \\ max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & if\ x \neq y;\ x,y \in C_1 \\ & and\ m, m\ are\ respectively\ the\ least \\ & intergers\ such\ that\ \ x \notin C_n , y \notin C_m \\ 0 & if\ x \neq y\ ,\ x,y \in X \end{cases}$$

is a metric and induces the topology τ.

1.3. Remark

Since the metric defined above is depending on the collection $\{C_n\}_{n=1}^{\infty}$and the point x_1 we call it the metric associated with the collection $\{C_n\}_{n=1}^{\infty}$and the pointx_1 . The aim of this paper is to give the topology and a matric induces that topology for any metrizable space with finitely many of non-isolated points.

2. The Main results

First, we start with metrizable spaces with only two non-isolated points.

2.1. Theorem

$if\ x_1, x_2 \in X,\ x_1 \neq x_2$

$\tau = P(X\backslash\{x_1, x_2\}) \cup \{\{x_1\}\cup F:\ F \in \Im_1\} \cup \{\{x_2\}\cup G: G \in \Im_2\} \cup \{\{x_1, x_2\}\cup F: F \in \Im_1\} \cup \{\{x_1, x_2\}\cup G: G \in \Im_2\}$.

Where $\Im_1, \Im_2$ are free filters in $X\backslash\{x_1, x_2\}$ with filter base $\{C_{1n}\}_{n=1}^{\infty}$, $\{C_{2n}\}_{n=1}^{\infty}$ Respectively such that $C_{1n+1} \subsetneq C_{1n}$, $C_{2n+1} \subsetneq C_{2n}$ for all n, $\bigcap_{n=1}^{\infty} C_{1n} = \bigcap_{n=1}^{\infty} C_{2n} = \varphi$ and $C_{11} \cap C_{21} = \varphi$,then τ is a topology on X and (X, τ) is a T_4 space .

Proof

It is routine to check that τ is a topology on X . To show that (X, τ) is Hausdorff, let $x, y \in X$ with $x \neq y$.If $x = x_1, y \in X\backslash\{x_1, x_2\}$ then since $\Im_1$ is a free filter so there exists $F \in \Im_1$ such that $y \notin F$and so $\{x_1\}\cup F$, $\{y\}$ are disjoint open sets containing x_1 and y respectively. Similarly if $x = x_2, y \in X\backslash\{x_1, x_2\}$. if $x = x_1$, $y = x_2$,then $\{x_1\} \cup C_{11}, \{x_2\} \cup C_{21}$ are disjoint open sets containing x_1 and x_2 respectively. So (X, τ) is Hausdorff. To show (X, τ) is normal space let A, B be any two disjoint closed sets , then we have the following cases :

i) If A, B are subsets of $X\backslash\{x_1, x_2\}$, then A, B are both open .

ii) If $x_1 \in A, x_2 \notin B$ then B is an open set and so B and B^C are two disjoint open sets containing B and A respectively . Similarly if $x_2 \in B, x_1 \notin A$

iii) If $x_1 \in A, x_2 \in B$ then A^C is an open set. $B \subset A^C$ So for any $b \in B$ there exists with $F_b \in \Im$. with $\{b\}\cup F_b \subset A^C$. Let $= \bigcup_{b\in B}\{b\} \cup F_b$, then K is a clopen (closed and open) set containing B and disjoint from A. Therefore (X, τ) is a normal space and hence (X, τ) is a T_4 space.

The following Theorem is Theorem 23.9 of [6] .

2.2. Theorem

A topological space X is metrizable if and only if it is T_3 and has a $\sigma -$ locally finite base.

2.3. Theorem

If τ as in Theorem 2.1, then (X, τ) is a metrizable space .

Proof:

Let $\{C_{1n}\}_{n=1}^{\infty}, \{C_{2n}\}_{n=1}^{\infty}$ be as in Theorem 2.1 and let

$\mathfrak{B}_1 = \{\{x\} : x \in X \backslash \{x_1, x_2\}, x \notin C_{11} \cup C_{21}\} \cup \{\{x_1\} \cup C_{11} \cup \{\{x_2\} \cup C_{21}\}$

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$\mathfrak{B}_n = \{\{x\} : x \in X \backslash \{x_1, x_2\}, x \notin C_{1n} \cup C_{2n}\} \cup \{\{x_1\} \cup C_{1n}\} \cup \{\{x_2\} \cup C_{2n}\}$.

Then $\mathfrak{B}_n$ is locally finite collection for all n and $\mathfrak{B} = \cup_{n=1}^{\infty} \mathfrak{B}_n$ is a $\sigma -$locally finite collection. Since $\mathfrak{B}$ contains the collection $\{\{x\}: x \in X \backslash \{x_1, x_2\} \cup \{\{x_1\} \cup C_{1n}\}_{n=1}^{\infty} \cup \{\{x_2\} \cup C_{2n}\}_{n=1}^{\infty}$. So $\mathfrak{B}$ is a base for the topology and since (X, τ) is T_4, so by the above Theorem (X, τ) is a metrizable space .

2.2. Lemma

If d_1, d_2 are two metrics on X and A, B are two non-empty subsets of X such that

$X = A \cup B$, $A \cap B = \varphi$ and $d_1(x,) = d_2(x,y)$ if $x \in A$ or $y \in A$ then the function $d: X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} d_1(x,y) & if\ x \in A\ \ or\ \ y \in A \\ d_2(x,y) & if\ \ x \in B,\ \ y \in B \end{cases}$$

is a metric on X.

Proof

The proof follows since

$X \times X = (A \times A) \cup (B \times A) \cup (A \times B) \cup (B \times B)$ and d_1, d_2 are two metrics on X.

2.3 Theorem

If τ, $\{C_{1n}\}_{n=1}^{\infty}$ and $\{C_{2n}\}_{n=1}^{\infty}$ as in Theorem 2.1 and $X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & if\quad x \neq y, x, y \in X \backslash \{x_1, x_2\}, (x,y) \notin C_{11} \times C_{11} and (x,y) \notin C_{21} \times C_{21} \\ 0 & if\quad x = y \\ 1 & if\quad (x = x_1, y \notin C_{11}) or (y = x_1, x \notin C_{11}) \\ 1 & if\quad (x = x_2, y \notin C_{21}) or (y = x_2, x \notin C_{21}) \\ \frac{1}{n+1} & if\quad x = x_1, y \in C_{11}\ and\ n\ is\ the\ least\ integer\ such\ that\ y \notin C_{1n} \\ & or \\ & y = x_1, x \in C_{11}\ and\ n\ is\ the\ least\ integer\ such\ that\ x \notin C_{1n} \\ \frac{1}{m+1} & if\quad x = x_2, y \in C_{21}\ and\ m\ is\ the\ least\ integer\ such\ that\ y \notin C_{2m} \\ & or \\ & y = x_2, x \in C_{21}\ and\ m\ is\ the\ least\ integer\ such\ that\ x \notin C_{2m} \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & if\quad x, y \in C_{11} and\ n, m\ are\ respectively\ the\ least\ intergers \\ & such\ that\ x \notin C_{1n}, y \notin C_{1m} \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & if\quad x, y \in C_{21} and\ n, m\ are\ respectively\ the\ least\ intergers \\ & such\ that\ x \notin C_{2n}, y \notin C_{2m} \end{cases}$$

Then is a metric and induces the topology τ.

Proof

Let d_1 be the metric corresponding to the collection $\{C_{1n}\}_{n=1}^{\infty}$ and the point x_1 and d_2 be the metric corresponding to the collection $\{C_{2n}\}_{n=1}^{\infty}$ and the point x_2,

(See Remark 1.3).

Let $A = X\backslash(C_{21} \cup \{x_2\})$, $B = C_{21} \cup \{x_2\}$, since A, B are two disjoint non empty subsets with $X = A \cup B$, $d_1(x,y) = d_2(x,y)$ if $x \in A$ or $y \in A$ and

$$d(x,y)=\begin{cases} d_1(x,y) & if\ x \in A \quad or \quad y \in A \\ d_2(x,y) & if \quad x \in B, \quad y \in B \end{cases}$$

So by Lemma 2.2 d is a matric on X and it is routine to show that induces the topology τ.

2.4. Remarks:

i) The above metric can be written in the following way

$$d(x,y) = \begin{cases} 1 & if\ x \neq y, x, y \in X \backslash\{x_1, x_2\}, (x,y) \notin C_{i1} \times C_{i1}\ for\ i = 1,2 \\ 0 & if\ x = y \\ 1 & if\big(x = x_i, y \notin C_{i1}\big) or \big(y = x_i, x \notin C_{i1}\big) for\ i = 1,2 \\ \dfrac{1}{n+1} & if\ x = x_i, y \in C_{i1}\ and\ n\ is\ the\ least\ integer\ such\ that\ y \notin C_{in} \\ & or \\ & y = x_i, x \in C_{i1}\ and\ n\ is\ the\ least\ integer\ such\ that\ x \notin C_{in} for\ i = 1,2 \\ \max\left\{\dfrac{1}{n+1}, \dfrac{1}{m+1}\right\} & if\ x, y \in C_{i1} and\ n, m\ are\ respectively\ the\ least\ intergers \\ & such\ that\ x \notin C_{in}, y \notin C_{im}\ for\ i = 1,2 \end{cases}$$

ii) For any $x \in X$, $\varepsilon > 0$ we have

$$B_\varepsilon(x) = \begin{cases} \{x\} & if\ x \notin C_{11} \cup C_{21} \quad x \neq x_1, x \neq x_2, \varepsilon = 1 \\ \{x\} & if\ , x \in C_{11}, \varepsilon = \dfrac{1}{n+2}\ where\ n\ is\ the\ least\ integer\ such\ that\ x \in C_{1n} \\ \{x\} & if\ , x \in C_{21}, \varepsilon = \dfrac{1}{m+2}\ where\ m\ is\ the\ least\ integer\ such\ that\ x \in C_{1m} \\ \{x_1\} \cup C_{1n} & if\ x = x_1, \varepsilon = \dfrac{1}{n} \\ \{x_2\} \cup C_{2m} & if\ x = x_2, \varepsilon = \dfrac{1}{m} \end{cases}$$

so as base for the metric topology we have the collection

$\{\{x\} : x \neq x_1, x \neq x_2\} \cup \{\{x_1\} \cup C_{1n}\}_{n=1}^{\infty} \cup \{\{x_2\} \cup C_{2n}\}_{n=1}^{\infty}$

2.5. Notation

Let **Ω** denotes the collection of all non-empty subsets of the set $\{x_1, x_2, \ldots, x_n\}$.

The proof of the following Theorem is similar to the proof of Theorem 2.1

2.6. Theorem

If $x_1, x_2, \ldots, x_n$ are elements of X,

$\tau = p(X\backslash\{x_1, x_2, \ldots, x_n\}) \cup [\cup_{i=1}^{n}[\cup_{A\in\Omega}\{A \cup F : F \in \mathfrak{I}_i\}]]$ where $\mathfrak{I}_i$ is a free filter in $X\backslash\{x_1, x_2, \ldots, x_n\}$ with countable filter base , then τ is a topology on X and is (X, τ) a T_4 space .

2.7. Lemma

If $x_1, x_2, \ldots, x_n$ are distinct points in a matric space X ,then there exists $\varepsilon > 0$ such that $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varphi$ *for* $i \neq j$.

2.8. Theorem

A space (X, τ) with exactly n non-isolated points $x_1, x_2, \ldots, x_n$ is metrizable iff $\tau = P(X\backslash\{x_1, x_2, \ldots, x_n\}) \cup [\cup_{i=1}^{n}[\cup_{A\in\Omega}\{A \cup F : F \in \mathfrak{I}_i\}]]$ where $\mathfrak{I}_i$ is a free filter in $X\backslash\{x_1, x_2, \ldots, x_n\}$ with countable filter base.

Proof

$\Rightarrow$ Let X be a metrizable space with $x_1, x_2, \ldots, x_n$ as the only non-isolated points .UsingLemma 2.7 we

choose $\varepsilon > 0$ such that $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varphi$ *for* $i \neq j$

and let $C_{ik} = B_{\varepsilon/k}(x_i) \setminus \{x_1, x_2, \ldots, x_n\}$for all k=1,2,…,n then C_{ik} ⊆ $X \setminus \{x_1, x_2, \ldots, x_n\}$,

$C_{ik+1} \subsetneq C_{ik}$ for all k and $\bigcap_{k=1}^{\infty} C_{ik} = \varphi$.

Let $\Im_i$ be a free filter with filter base $\{C_{ik}\}_{k=1}^{\infty}$and let τ^* be the topology whose base is the collection $\mathfrak{B} = \{\{x\}: x \in X \setminus \{x_1, x_2, \ldots, x_n\}\} \cup [\bigcup_{i=1}^{n} \{\{x_i\} \cup C_{ik}\}_{k=1}^{\infty}]$

It is routine to check that $\tau = \tau^*$.

$\Leftarrow$ Let$\{C_{ik}\}_{k=1}^{\infty}$ be the filter base for $\Im_i$ for all$i$ with $C_{i1} \cap C_{j1} = \varphi$ for $i \neq j$

Let .

$\mathfrak{B}_1 = \{\{x\}: \ if \ \ x \notin \bigcup_{i=1}^{n} C_{i1}\} \cup \{\{\bigcup_{i=1}^{n}\{x_i\} \cup C_{i1}\}\}$

$\mathfrak{B}_2 = \{\{x\}: \ if \ \ x \notin \bigcup_{i=1}^{n} C_{i2}\} \cup \{\{\bigcup_{i=1}^{n}\{x_i\} \cup C_{i2}\}\}$

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$\mathfrak{B}_n = \{\{x\}: \ if \ \ x \notin \bigcup_{i=1}^{n} C_{in}\} \cup \{\{\bigcup_{i=1}^{n}\{x_i\} \cup C_{in}\}\}$

Then $\mathfrak{B}_n$ is locally finite collection for all n. Let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$, then $\mathfrak{B}$ is a σ −locally finite collection and since $\mathfrak{B}$ is a base for τ by Theorem 2.2 it follows that X is a metrizable space with $x_1, x_2, \ldots, x_n$ as the non-isolated points .

2.9. Theorem

If τ, $\{C_{ik}\}_{k=1}^{\infty}$ as in last Theorem and d : $X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} 1 & if \ x \neq y, x, y \in X \setminus \{x_1, x_2, \ldots, x_n\}, (x,y) \notin C_{i1} \times C_{i1} \ for \ i = 1,2,\ldots,n \\ 0 & if \ x = y \\ 1 & if (\ x = x_i, y \notin C_{i1}) or (y = x_i, x \notin C_{i1}) for \ i = 1,2,\ldots,n \\ \dfrac{1}{m+1} & if \ x = x_i, y \in C_{i1} \ and \ m \ is \ the \ least \ integer \ such \ that \ y \notin C_{im} \\ & \quad or \\ & y = x_i, x \in C_{i1} \ and \ m \ is \ the \ least \ integer \ such \ that \ x \notin C_{im} \\ \max\left\{\dfrac{1}{n+1}, \dfrac{1}{m+1}\right\} & if \ x, y \in C_{i1} and \ n, m \ are \ respectively \ the \ least \ intergers \\ & such \ that \ x \notin C_{in}, y \notin C_{im} \end{cases}$$

Then is a metric and inducs the topology τ.

Proof

By induction on the number of the non-isolated points . If n= 1, then this is Theorem 2.2 of [5] . If n =2, then this is Theorem 2.1. So suppose $n \geq 3$ and the Theorem is true for all i with $1 \leq i < n$. Let d_i be the metric corresponding to the collection $\{C_{ik}\}_{k=1}^{\infty}$ and the point x_i for all i .

Let d_* be the metric produced by $d_1, d_2, \ldots, d_{n-1}$ using Lemma 2.2 , d_n the metric corresponding to the collection $\{C_{nk}\}_{k=1}^{\infty}$ and the point x_n and define,

$$d(x,y) = \begin{cases} d_*(x,y) & if \ x \in X \setminus (C_{n1} \cup \{x_n\} \ or \ y \in X \setminus C_{n1} \cup \{x_n\}) \\ d_n(x,y) & if \ \ x, y \in C_{n1} \cup \{x_n\} \end{cases}$$

Then by Lemma 2.2 d is a matric on X with metric base the collection

$\mathfrak{B} = \{\{x\}: x \neq x_i \ for \ all \ i\}) \cup [\bigcup_{i=1}^{n} \{\{x_i\} \cup C_{ik}\}_{k=1}^{\infty}]$ and clearly τ is generated by $\mathfrak{B}$.

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