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Adomian Decomposition Method for Solving Linear Wave Equation

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Abstract

The adomian decomposition method (ADM) was used to solve various wave equations. We compared the obtained solution by ADM with the Reduced Differential Transform Method (RDTM) and the Variational Iteration Method (VIM). The results show that ADM is very effective, simple and easy compared with other methods.

Keywords: Adomian decomposition method; Wave equation; RTDM; VIM.

المستخلص

أستخدمت طريقة تحليل أدومين لحل معادلات الموجه المختلفة. وقارنا الحل المتحصل عليه بإستخدام طريقة تحليل أدومين مع طريقة تحويل تفاضلي مخفض والطريقة التكرارية المتناوبة، وقد اظهرت النتائج بأن طريقة أدومين فعالة جدا وبسيطة وسهلة مقارنة مع الطرق الأخرى.

Introduction

In this paper, we shall solve exactly the wave equation of second order in one dimension by adomain decomposition method in following forms:

1) $u_{tt} = c^2 u_{xx}\ , I = \{x \ \backslash 0 < x < a\}$. (hom. in finite dom.)
2) $u_{tt} = c^2 u_{xx}\ , I = \{x \ \backslash -\infty < x < \infty\}$. (hom. in infinite dom.)
3) $u_{tt} = c^2 u_{xx} + h(x,t)\ , I = \{x \ \backslash 0 < x < a\}$.(inhom. in finite dom.)
4) $u_{tt} = c^2 u_{xx} + h(x,t)\ , I = \{x \ \backslash -\infty < x < \infty\}$. (inhom. in infinite dom.)

With initial conditions corresponding to every form, over the interval I, where u=u(x,t) denotes the wave amplitude with dimensions of distance, c is the wave speed with dimensions of distance pertime, which is a positive Constant, and $h(x,t)$ a ccounts for any

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external forces acting on the system. In recent years, the Adomian decomposition method has received much attention in applied mathematics in general, and in the area of series solution, in particular. It was introduced and developed by George Adomian [1-4]. Some wave equations are solvable by many methods such the reduced differential transform method [5], introduced by Zhou in 1986 [6,7], the variational iteration method introduced by He [9], and a Finite Element method [13]. We aim in this paper to obtain exact solutions to the Wave Equation in four different cases. We also show that the ADM is a powerful, effective and fast convergence to the exact solution obtained by the previously mentioned methods.

Adomain Decomposition Method

The Adomian decomposition method [4] consists of decomposing the unknown function u(x,y) of any equation into a sum of an infinite number of components defined by the decomposition series

$$U(x,y)=\sum_{n=0}^{\infty} u_n(x,y) \text{ or briefly } u=\sum_{n=0}^{\infty} u_n \quad (1)$$

Where the components, $u_n(x,y)$, n≥ 0 are to be determined in a recursive Manner.
To give a clear overview of adomian decomposition method we first consider the linear differential equation written in an operator form by

$$Lu+Ru=g \quad (2)$$

Where L is mostly the lower order derivative, which is assumed invertible, R is other linear differential operator, and g is a source term. Then apply the inverse operator L^{-1} to both sides of equation (2) and using the given condition to obtain

$$u = f - L^{-1}(Ru) \quad (3)$$

Where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed.
Substituting (1) into both sides of (3) leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}(R(\sum_{n=0}^{\infty} u_n)). \quad (4)$$

For simplicity, equation (4) can be rewritten as,

$$u_0 + u_1 + u_2 + u_3 + \cdots = f - L^{-1}(R(u_0 + u_1 + u_2 + \cdots)) \quad (5)$$

The formal recursive relation is defined by,

$$u_0 = f$$
$$u_{k+1} = - L^{-1}(R(u_k)) , k \geq 0 \quad (6)$$

Or equivalently u_0 = f

$$u_1 = - L^{-1}(\mathrm{R}(u_0)) ,$$
$$u_2 = - L^{-1}(\mathrm{R}(u_1)) , \quad (7)$$
$$u_3 = - L^{-1}(\mathrm{R}(u_2)) ,$$
$$\vdots$$

Then substitute into (1) to obtain the solution in a series form.

Application

Now, we use the method to solve the following different forms of wave Equation.
Example (1): we consider the homogeneous wave equation in finite domain [5] defined as:

$$u_{tt} = u_{xx} - 3u \, , 0 < x < \pi \, , t > 0 \quad (8)$$

With the initial conditions:

$$u(x,0) = 0 \, , u_t(x,0) = 2\cos(x) \quad (9)$$

Solution: equation (8) in an operator form becomes

$$L_t u = L_x u - 3u \quad (10)$$

Where; $L_t = \frac{\partial^2}{\partial t^2}$ (11)

$$L_x = \frac{\partial^2}{\partial x^2} \quad (12)$$

Applying L^{-1} to both sides of (10) and using the initial condition we obtain:

$L_t^{-1}(L_t u) = L_t^{-1}(L_x u - 3u)$ (13) , where $L_t^{-1}(.) = \int_0^t \int_0^t (.) dt dt.$ (14)

So that

$$u(x,t) - 2t cos(x) = L_t^{-1}\left(\frac{\partial^2}{\partial x^2} u(x,t) - 3u(x,t)\right) \quad (15)$$

Substituting the series assumption (1) into both sides of (15) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = 2t cos(x) + L_t^{-1}\left(\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} u_n(x,t) - 3 \sum_{n=0}^{\infty} u_n(x,t)\right) \quad (16)$$

The recursive relation of (16) is

$$u_0(x,t) = 2t cos(x) \quad (17)$$

$$u_{k+1}(x,t) = L_t^{-1}\big(L_x u_k(x,t) - 3u_k(x,t)\big), k \geq 0 \tag{18}$$

Consequently, we obtain

$$u_0(x,t) = 2tcos(x)$$

$$u_1(x,t) = L_t^{-1}\big(L_x u_0(x,t) - 3u_0(x,t)\big) = -\frac{(2t)^3}{3!}\cos(x) \tag{19}$$

$$u_2(x,t) = L_t^{-1}\big(L_x u_1(x,t) - 3u_1(x,t)\big) = \frac{(2t)^5}{5!}\cos(x)$$

$$u_3(x,t) = L_t^{-1}\big(L_x u_2(x,t) - 3u_2(x,t)\big) = -\frac{(2t)^7}{7!}\cos(x)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

The solution $u(x,t)$ is given by

$$= \cos(x)\,[\,2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \dots\dots \quad] \text{ in a series form} \tag{20}$$

$$u(x,t) = \cos(x)\sin(2t) \tag{21}$$

Is in a closed form, which is the exact Solution as in [5].

Example (2): we consider the homogeneous wave equation in infinite Domain [10] defined as:

$$u_{tt} = u_{xx}, -\infty < x < \infty, t > 0 \tag{22}$$

With the initial conditions:

$$u(x,0) = \sin(x),\ u_t(x,0) = 0 \tag{23}$$

Solution: in an operator form eq (17) becomes

$$L_t u = L_x u \tag{24}$$

Where L_t, L_x as (11),(12)
Operating with L_t^{-1} on both sides (19) leads

$$L_t^{-1}(L_t u) = L_t^{-1}(L_x u) \tag{25}$$

Where $L_t^{-1}(.)$ as (14)

So that

$$u(x,t) = \sin(x) + L_t^{-1}(L_x u) \quad (26)$$

Consequently, we obtain,

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin(x) + L_t^{-1}(L_x \sum_{n=0}^{\infty} u_n(x,t)) \quad (27)$$

The following recursive relation

$$u_0(x,t) = sin(x) \quad (28)$$

$$u_{k+1}(x,t) = L_t^{-1}(L_x u_k(x,t)) \quad (29)$$

Proceeding as before, we set

$$u_1(x,t) = L_t^{-1}(L_x u_0(x,t)) = -\frac{t^2}{2!}\sin(x)$$

$$u_2(x,t) = L_t^{-1}(L_x u_1(x,t)) = \frac{t^4}{4!}\sin(x) \quad (30)$$

$$u_3(x,t) = L_t^{-1}(L_x u_2(x,t)) = -\frac{t^6}{6!}\sin(x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$u(x,t) = \sin(x)(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \quad) \quad (31)$$

Is solution in a series form, but in a closed form is $u(x,t) = \sin(x)cos(t)$ (32) which as given in [10].

Example (3): we next consider the inhomogeneous wave equation in finite Domain [8] defined as:

$$u_{tt} = u_{xx} + \sin(x), 0 < x < \pi, t > 0 \quad (33)$$

With the initial conditions:

$$u(x,0) = sin(x) \;, u_t(x,0) = sin(x) \quad (34)$$

Solution: operating with L_t^{-1}on both sides of (33) yields

$$u(x,t) = sin(x) + tsin(x) + \frac{t^2}{2}\sin(x) + L_t^{-1}(L_x u) \quad (35)$$

Where L_t, L_x and $L_t^{-1}(.)$ as (11),(12) and (14)

So that $\sum_{n=0}^{\infty} u_n(x,t) = \sin(x) + t sin(x) + \frac{t^2}{2} sin(x) + L_t^{-1}(L_x \sum_{n=0}^{\infty} u_n(x,t))$(36)

For proceeding discussion, we set

$u_0(x,t) = sin(x) + t sin(x) + \frac{t^2}{2} sin(x)$

$$u_1(x,t) = L_t^{-1}(L_x u_0(x,t)) = -\frac{t^2}{2!}\sin(x) - \frac{t^3}{3!} sin(x) - \frac{t^4}{4!} sin(x) \qquad (37)$$

$$u_2(x,t) = L_t^{-1}(L_x u_1(x,t)) = \frac{t^4}{4!} sin(x) + \frac{t^5}{5!} sin(x) + \frac{t^6}{6!} sin(x)$$

$$u_3(x,t) = L_t^{-1}(L_x u_2(x,t)) = -\frac{t^6}{6!}\sin(x) - \frac{t^7}{7!} sin(x) - \frac{t^8}{8!} sin(x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$u(x,t) = sin(x) + \sin(x)[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots] \qquad (38)$$

is solution in a series form, and in a closed form is

$$u(x,t) = \sin(x)[1 + \sin(t)] \qquad (39)$$

Which is the same solution in [8].

Example (4): Finally, we consider the inhomogeneous wave equation in infinite Domain [4] defined as:

$$u_{tt} = u_{xx} + 2x + 6t, -\infty < x < \infty, t > 0 \qquad (40)$$

With the initial conditions:

$$u(x,0) = 0, u_t(x,0) = sin(x) \qquad (41)$$

Solution: from previous discussion and using (11), (12) and (14), we obtain

$$u(x,t) = t\sin(x) + xt^2 + t^3 + L_t^{-1}(L_x u) \qquad (42)$$

So that

$$\sum_{n=0}^{\infty} u_n(x,t) = t^3 + xt^2 + t sin(x) + L_t^{-1}(L_x \sum_{n=0}^{\infty} u_n(x,t)) \qquad (43)$$

Proceeding as before, we find

$u_0(x,t) = t^3 + xt^2 + t sin(x)$

$$u_1(x,t) = L_t^{-1}(L_x u_0(x,t)) = -\frac{t^3}{3!}\sin(x)$$

$$u_2(x,t) = L_t^{-1}(L_x u_1(x,t)) = \frac{t^5}{5!} sin(x) \quad (44)$$

$$u_3(x,t) = L_t^{-1}(L_x u_2(x,t)) = -\frac{t^7}{7!} sin(x)$$

And so on.

The solution is $u(x,t) = xt^2 + t^3 + sin(x)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]$ (45)

In a series form, and in a closed form is

$$u(x,t) = xt^2 + t^3 + sin(x)\sin(t) \quad (46)$$

Which is the same exact solution for example solved by variational iteration method (VIM) in [4].

Conclusion

In this paper, the adomain decomposition method has been successfullya for finding the solution of the wave equations in four different cases. The solution obtained by the adomain decomposition, in turn is expressed in a closed form, which is convergent to the exact solution. The Simplicity of the method and the obtained results show that the Adomain Decomposition method is an effective mathematical tool for solving the Wave Equation.

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