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A New Application of Sawi Transform for Solving Volterra Integral Equations and Volterra Integro-differential Equations

Mohammed E. H. Attaweel[1] and Haneen A. A. Almassry[2]
Mathematics Dept., Arts and Science Faculty-Gasr Khyar, Al Mergheb University, Libya.
[1]E-mail: meattaweel@elmergib.edu.ly; [2]haneenalmassry1995@gmail.com

Abstract

In this paper, a new application of Sawi transform is given for solving Volterra integral equations and Volterra integro-differential equations. In the application section of this paper, some applications are given to explain the importance of Sawi transform for solving the indicated equations. The results show that Sawi transform is a very useful integral transform for solving these equations.

Keywords: Volterra Integral Equations; Volterra Integro-differential Equations; Sawi Transform.

المستخلص

في هذه الورقة قدمنا تطبيقاً جديداً لتحويل ساوي لحل معادلات فولتيرا التكاملية ومعادلات فولتيرا التكاملية–التفاضلية وفي بند التطبيقات في هذه الورقة أعطيت بعض التطبيقات لتشرح أهمية تحويل ساوي لحل المعادلات المشار إليها. وقد وضحت النتائج أن تحويل ساوي تحويل تكاملي مفيد جداً لحل المعادلات المذكورة.

Introduction

In modern times, integral transforms, Laplace transform [12, 16,20,21,24,29,30,31,32], Fourier transform [24], Hankel transform [24], Mellin transform [24], Mahgoub transform [7,11,13,23,25], Kamal transform [1,2,5,8,10,22], Elzaki transform [6,15,17,18,19,30], Aboodh transform [3,4,14], Mohand transform [4,27],Sawi transform [26], Sadik transform [28] and Hermite transform [24] etc. have very useful role in mathematics, physics, chemistry, social science, biology, astronomy, nuclear science, electrical and mechanical engineering for solving the advanced problems of these fields. Many scholars use these transforms to solve problems of differential equations, partial differential equations, integral equations, integro-differential equations, partial integro-differential equations, delay differential equations and population growth and decay problems, and the mechanics and

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electrical circuit problems. In this paper, we concentrate mainly on an application of the Sawi transform to solve some linear Volterra integral equations and linear Volterra integro-differential equations.

Definitions and Standard Results

Sawi Transform

Definition: A new transform called the Sawi transform defined for function of exponential order, we consider functions in the set A defined by:

$$A=\left\{ f\left(t\right): \exists\ M, K_1, K_2>0,\ \left|f\left(t\right)\right|<M\,e^{\frac{|t|}{k_j}}, if\ \ t\in\left(-1\right)^j\times\left[0,\infty\right)\right\} \quad (1)$$

For a given function in the set A, the constant M must be a finite number. K_1, K_2 may be finite or infinite. Sawi Transform is denoted by the operator $S\left(\cdot\right)$; defined by the integral equations

$$S\left[f\left(t\right)\right]=R\left(v\right)=\frac{1}{v^2}\int_0^{\infty} f\left(t\right)e^{\frac{-t}{v}}\,dt\quad , \ t\geq 0,\ K_1\leq v\leq K_2 \qquad (2)$$

Some Properties of Sawi Transform

Linearity Property of Sawi Transform

If $S\left[f_1\left(t\right)\right]=R_1\left(v\right)$ *and* $S\left[f_2\left(t\right)\right]=R_2\left(v\right)$ then

$$S\left[a\,f_1\left(t\right)+b\,f_2\left(t\right)\right]=aS\left[f_1\left(t\right)\right]+b\,S\left[f_2\left(t\right)\right]=a\,R_1\left(v\right)+b\,R_2\left(v\right) \quad (3)$$

Where a, b are arbitrary constants.

Sawi Transform of the Derivative of the Function $f\left(t\right)$

Theorem: let $S\left[f\left(t\right)\right]=R\left(v\right)$ then

$$(i)\quad S\left[f'\left(t\right)\right]=\frac{1}{v}R\left(v\right)-\frac{1}{v^2}f\left(0\right) \qquad (4)$$

$$(ii) \quad S\left[f''(t)\right] = \frac{1}{v^2} R(v) - \frac{1}{v^2} f'(0) - \frac{1}{v^3} f(0) \tag{5}$$

$$(iii) \quad S\left[f^{(n)}(t)\right] = \frac{1}{v^n} R(v) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-k+1}} \tag{6}$$

Sawi Transform of Integral of a Function $f(t)$

Theorem: let $S\left[f(t)\right] = R(v)$ then

$$S\left[\int_0^t f(t)dt\right] = v\,R(v) \tag{7}$$

Proof: let $H(t) = \int_0^t f(t)dt \quad \Rightarrow H'(t) = f(t), \quad H(0) = 0$

Now using Sawi transform of the derivative of a function, we get

$$S\left[H'(t)\right] = \frac{1}{v} S\left[H(t)\right] - \frac{1}{v^2} H(0) \Rightarrow \quad S\left[H'(t)\right] = \frac{S\left[H(t)\right]}{v} \tag{8}$$

$$\Rightarrow \quad S\left[H(t)\right] = v\,S\left[H'(t)\right] = v\,S\left[f(t)\right] = v\,R(v) \tag{9}$$

$$\Rightarrow \quad S\left[\int_0^t f(t)dt\right] = v\,R(v) \tag{10}$$

Change of Scale Property

Theorem: let $S\left[f(t)\right] = R(v)$ then $S\left[f(at)\right] = a\,R(av)$

Proof:

$$\because S\left[f(t)\right] = \frac{1}{v^2}\int_0^\infty f(t)\, e^{\frac{-t}{v}}\, dt \;\Rightarrow\; S\left[f(at)\right] = \frac{1}{v^2}\int_0^\infty f(at)\, e^{\frac{-t}{v}}\, dt \tag{11}$$

Let $at = u \quad \Rightarrow \quad t = \frac{u}{a} \Rightarrow \quad dt = \frac{1}{a} du$

$$\Rightarrow \; S\left[f(u)\right]=\frac{1}{v^2}\int_0^\infty f(u)\,e^{\frac{-u}{av}}\;\frac{1}{a}du=\frac{1}{a}\frac{1}{v^2}\int_0^\infty f(u)\,e^{\frac{-u}{(av)}}\;du \qquad (12)$$

$$\Rightarrow \; S\left[f(u)\right]=a\;\frac{1}{(av)^2}\int_0^\infty f(u)\,e^{\frac{-u}{(av)}}\;du=a\,R(av) \qquad (13)$$

Convolution Property

Theorem: let $f(t),\; g(t)\in A$ *and* $S\left[f(t)\right]=F(v)$, $S\left[g(t)\right]=G(v)$ then

$$S\left[f(t)*g(t)\right]=v^2F(v)G(v) \qquad (14)$$

where $f(t)*g(t)=\int_0^t f(t-x)g(x)dx=\int_0^t f(x)g(t-x)dx$

Proof: Consider $F(v)=\frac{1}{v^2}\int_0^\infty e^{\frac{-r}{v}}f(r)\;dr$, $G(v)=\frac{1}{v^2}\int_0^\infty e^{\frac{-s}{v}}g(s)\;ds$

$$\Rightarrow F(v)G(v)=\frac{1}{v^4}\int_0^\infty e^{\frac{-r}{v}}f(r)\;dr\;\int_0^\infty e^{\frac{-s}{v}}g(s)\;ds=\frac{1}{v^4}\int_0^\infty\int_0^\infty e^{\frac{-1}{v}(r+s)}f(r)g(s)ds\,dr \quad (15)$$

put $r+s=t \quad \Rightarrow \quad ds=dt$ and by changing the order of integration and solving we obtain

$$F(v)G(v)=\frac{1}{v^2}\left[\frac{1}{v^2}\int_0^\infty\int_0^t e^{\frac{-t}{v}}f(r)g(t-r)dr\,dt\right]=\frac{1}{v^2}S\left[f(t)*g(t)\right] \qquad (16)$$

$$\Rightarrow \quad S\left[f(t)*g(t)\right]=v^2F(v)G(v) \qquad (17)$$

Sawi Transform for Some Functions

Table 1. Sawi Transform functions.

$f(t)$	$S\left[f(t)\right]$
1	$\frac{1}{v}$
t	1
t^2	$2v$
t^n	$n!v^{n-1}$
e^{at}	$\frac{1}{v(1-av)}$

$\sin at$	$\frac{a}{1+a^2v^2}$
$\cos at$	$\frac{1}{v\left(1+a^2v^2\right)}$
$\sinh at$	$\frac{a}{1-a^2v^2}$
$\cosh at$	$\frac{1}{v\left(1-a^2v^2\right)}$

Applications

Example 1.

Consider linear Volterra integral equation of second kind

$$y(x)=1-x+\int_0^x (x-t)\,y(t)\,dt \tag{18}$$

Solution: Applying the Sawi transform to both sides we have

$$S\left[y(x)\right]=S[1]-S[x]+S\left\{\int_0^x (x-t)\,y(t)\,dt\right\} \tag{19}$$

$$R(v)=\frac{1}{v}-1+v^2R(v) \tag{20}$$

$$\Rightarrow \quad R(v)=\frac{1-v}{v\left(1-v^2\right)}=\frac{1}{v(1+v)} \tag{21}$$

Now using inverse Sawi transform, we get

$$y(x)=e^{-x} \tag{22}$$

Example 2.

Consider linear Volterra integral equation of second kind

$$y(x)=3x^2+\int_0^x \sin(x-t)\,y(t)\,dt \tag{23}$$

Solution: Applying the Sawi transform to both sides we have

$$S[y(x)]=3S[x^2]+v^2 S[\sin x]\,S[y(x)] \tag{24}$$

$$R(v)=6v+v^2\frac{1}{1+v^2}R(v) \quad \Rightarrow R(v)\left[1-\frac{v^2}{1+v^2}\right]=6v \tag{25}$$

$$\Rightarrow R(v)\left[\frac{1}{1+v^2}\right]=6v \quad \Rightarrow \quad R(v)=6v+6v^3 \tag{26}$$

$$\Rightarrow R(v)=3[2v]+\frac{1}{4}[4!v^3] \tag{27}$$

Now using inverse Sawi transform, we get

$$y(x)=3x^2+\frac{1}{4}x^4 \tag{28}$$

Example 3.

Consider linear Volterra integral equation of second kind

$$y(x)=x+\frac{1}{6}\int_0^x (x-t)^3\,y(t)\,dt \tag{29}$$

Solution: Applying the Sawi transform to both sides we have

$$S[y(x)]=S[x]+\frac{1}{6}v^2 S[x^3]S[y(x)] \tag{30}$$

$$R(v)=1+v^4 R(v) \tag{31}$$

$$\Rightarrow R(v)=\frac{1}{1-v^4}=\frac{1}{2}\frac{1}{1-v^2}+\frac{1}{2}\frac{1}{1+v^2} \tag{32}$$

Now using inverse Sawi transform, we get

$$y(x)=\frac{1}{2}\left[\sinh x+\sin x\right] \qquad (33)$$

Example 4.

Consider linear Volterra integral equation

$$y(x)=1-\frac{1}{2}x^{2}+\int_{0}^{x}y(t)\,dt \qquad (34)$$

Solution: Applying the Sawi transform to both sides we have

$$R(v)=\frac{1}{v}-v+v\,R(v) \qquad (35)$$

$$\Rightarrow \quad R(v)=\frac{1-v^{2}}{v(1-v)}=\frac{1}{v}+1 \qquad (36)$$

Now using inverse Sawi transform, we get

$$y(x)=1+x \qquad (37)$$

Example 5.

Consider linear Volterra integral equation of first kind

$$\sin x=\int_{0}^{x}e^{(x-t)}\,y(t)\,dt \qquad (38)$$

Solution: Applying the Sawi transform to both sides we have

$$\frac{1}{1+v^{2}}=\frac{v^{2}}{v(1-v)}R(v) \qquad (39)$$

$$\Rightarrow \quad R(v)=\frac{1-v}{v\left(1+v^{2}\right)}=\frac{1}{v\left(1+v^{2}\right)}-\frac{1}{1+v^{2}} \qquad (40)$$

Now using inverse Sawi transform, we get

$$y(x) = \cos x - \sin x \qquad (41)$$

Example 6.

Consider linear Volterra integral equation of first kind

$$x^2 = \frac{1}{2}\int_0^x (x-t)\, y(t)\, dt \qquad (42)$$

Solution: Applying the Sawi transform to both sides we have

$$2v = \frac{1}{2} v^2 R(v) \qquad (43)$$

$$\Rightarrow \quad R(v) = \frac{4}{v} = 4\frac{1}{v} \quad \Rightarrow \qquad y(x) = 4 \qquad (44)$$

Example 7.

Consider linear Volterra integral equation of first kind

$$x = \int_0^x e^{(x-t)}\, y(t)\, dt \qquad (45)$$

Solution: Applying the Sawi transform to both sides we have

$$1 = \frac{v}{1-v} R(v) \ \Rightarrow \quad R(v) = \frac{1}{v} - 1 \qquad (46)$$

Now using inverse Sawi transform, we get

$$y(x) = 1 - x \qquad (47)$$

Example 8.

Consider linear Volterra integro - differential equation of second kind

$$y'(x) = 2 + \int_0^x y(t)\, dt \quad with \quad y(0) = 2 \qquad (48)$$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v}-\frac{2}{v^2}=\frac{2}{v}+v\,R(v)\quad\Rightarrow R(v)\left(\frac{1}{v}-v\right)=\frac{2(v+1)}{v^2}\qquad(49)$$

$$\Rightarrow R(v)=\frac{2(v+1)}{v\left(1-v^2\right)}=\frac{2}{v\,(1-v)}\qquad(50)$$

Now Using inverse Sawi transform, we get

$$y(x)=2e^x\qquad(51)$$

Example 9.

Consider linear Volterra integro-differential equation of second kind

$$y''(x)=-1-x+\int_0^x (x-t)\,y(t)\,dt\qquad(52)$$

With $y(0)=1$, $y'(0)=1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^2}-\frac{1}{v^2}-\frac{1}{v^3}=-\frac{1}{v}-1+v^2R(v)\qquad(53)$$

$$\left[\frac{1-v^4}{v^2}\right]R(v)=\frac{v+1-v^2-v^3}{v^3}\Rightarrow R(v)=\frac{\left(1-v^2\right)+v\left(1-v^2\right)}{v\left(1-v^4\right)}\qquad(54)$$

$$\Rightarrow R(v)=\frac{1}{v\left(1+v^2\right)}+\frac{1}{1+v^2}\qquad(55)$$

Now using inverse Sawi transform, we get

$$y(x)=\cos x+\sin x\qquad(56)$$

Example 10.

Consider linear Volterra integro-differential equation of second kind

$$y''(x) = \cosh x + \int_0^x e^{(x-t)} y(t)\, dt \qquad (57)$$

With $y(0)=1,\ y'(0)=1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^2} - \frac{1}{v^2} - \frac{1}{v^3} = \frac{1}{v(1-v^2)} + \frac{v}{1+v} R(v) \qquad (58)$$

$$\left[\frac{1+v-v^3}{v^2(1+v)}\right] R(v) = \frac{v^2 + v(1-v^2) + (1-v^2)}{v^3(1-v^2)} = \frac{1+v-v^3}{v^3(1+v)(1-v)} \quad (59)$$

$$\Rightarrow\ R(v) = \frac{1}{v(1-v)} \qquad (60)$$

Now using Inverse Sawi transform, we get

$$y(x) = e^x \qquad (61)$$

Example 11.

Consider linear Volterra integro-differential equation of second kind

$$y'''(x) = -1 + \int_0^x y(t)\, dt \qquad (62)$$

With $y(0) = y'(0) = 1 \quad , \quad y''(t) = -1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^3} - \frac{1}{v^4} - \frac{1}{v^3} + \frac{1}{v^2} = \frac{-1}{v} + v\, R(v) \qquad (63)$$

$$R(v)\left[\frac{1-v^4}{v^3}\right] = \frac{1+v-v^2-v^3}{v^4} = \frac{(1-v^2)+v(1-v^2)}{v^4} \qquad (64)$$

$$\Rightarrow \quad R(v)=\frac{\left(1-v^{2}\right)+v\left(1-v^{2}\right)}{v\left(1-v^{2}\right)\left(1+v^{2}\right)}=\frac{1}{v\left(1+v^{2}\right)}+\frac{1}{1+v^{2}} \tag{65}$$

Now using inverse Sawi transform, we get

$$y(x)=\cos x+\sin x \tag{66}$$

Conclusion

In this present work, the Sawi transform is used to solve Volterra integral equations and Volterra Integro-differential Equations. Furthermore, we present several properties and theorems of Sawi transform. To see the efficiency of Sawi transform, we applied this transform on eleven different examples. The results show that the Sawi transform method is an appropriate method for solving Volterra integral equations and Volterra Integro-differential Equations.

REFERENCES

[1] Abdelilah, K. and Hassan, S. (2016). The new integral transform "Kamal Transform". Advances in Theoretical and Applied Mathematics, **11**(4), 451-458.

[2] Abdelilah, K. and Hassan, S. (2017). The use of Kamal transform for solving partial differential equations. Advances in Theoretical and Applied Mathematics, **12**(1), 7-13.

[3] Aboodh, K. S. (2013). The new integral transform "Aboodh Transform". Global Journal of Pure and Applied Mathematics, **9**(1), 35-43.

[4] Aggarwal, S. and Chauhan, R. (2018). A comparative study of Mohand and Aboodh transforms. International Journal of Research in Advent Technology, **7**(1), 520-529.

[5] Aggarwal, S., Chauhan, R. and Sharma, N. (2018). A new application of Kamal transform for solving linear Volterra integral equations. International Journal of Latest Technology in Engineering, Management & Applied Science, **7**(4), 138-140.

[6] Aggarwal, S., Chauhan, R. and Sharma, N. (2018). Application of Elzaki transform for solving linear Volterra integral equations of first kind. International Journal of Research in Advent Technology, **6**(12), 3687-3692.

[7] Aggarwal, S., Chauhan, R. and Sharma, N. (2018). A new application of Mahgoub transform for solving linear Volterra integral equations. Asian Resonance, **7**(2), 46-48.

[8] Aggarwal, S., Gupta, A. R., Asthana, N. and Singh, D. P. (2018). Application of Kamal transform for solving population growth and decay problems. Global Journal of Engineering Science and Researches, **5**(9), 254-260.

[9] Aggarwal, S., Gupta, A. R., and Sharma, S. D. (2019). A new application of Shehu transform for handling Volterra integral equations of first kind. International Journal of Research in Advent Technology, **7**(4), 438-445.

[10] Aggarwal, S., Sharma, N. and Chauhan, R. (2018). Application of Kamal transform for solving linear Volterra integral equations of first kind. International Journal of Research in Advent Technology, **6**(8), 2081-2088.
[11] Aggarwal, S., Sharma, N., Chauhan, R., Gupta, A. R. and Khandelwal, A. A. (2018). New application of Mahgoub transform for solving linear ordinary differential equations with variable coefficients. Journal of Computer and Mathematical Sciences, **9**(6), 520-525.
[12] Aggarwal, S., Gupta, A. R., Singh, D. P., Asthana, N. and Kumar, N. (2018). Application of Laplace transform for solving population growth and decay problems. International Journal of Latest Technology in Engineering, Management & Applied Science, **7**(9), 141-145.
[13] Aggarwal, S., Sharma, N. and Chauhan, R. (2018). Solution of linear Volterra integro-differential equations of second kind using Mahgoub transform. International Journal of Latest Technology in Engineering, Management & Applied Science, **7**(5), 173-176.
[14] Aggarwal, S., Sharma, N. and Chauhan, R. (2018), Application of Aboodh transform for solving Volterra integro-differential equations of second kind. International Journal of Research in Advent Technology, **6**(6), 1186-1190.
[15] Aggarwal, S., Singh, D. P., Asthana, N. and Gupta, A. R. (2018). Application of Elzaki transform for solving population growth and decay problems. Journal of Emerging Technologies and Innovative Research, **5**(9), 281-284.
[16] Dass, H. K. (2007). Advanced Engineering Mathematics. S. Chand & Co. Ltd.
[17] Elzaki, T. M. (2011). The new integral transform "Elzaki Transform". Global Journal of Pure and Applied Mathematics, **1**, 57-64.
[18] Elzaki, T. M. and Ezaki, S. M. (2011). On the Elzaki transform and ordinary differential equation with variable coefficients. Advances in Theoretical and Applied Mathematics, **6**(1), 41-46.
[19] Elzaki, T.M. and Ezaki, S.M. (2011). Applications of new transform "Elzaki transform" to partial differential equations. Global Journal of Pure and Applied Mathematics, **7**(1), 65-70.
[20] Greenberg, M.D. (1998). Advanced Engineering Mathematics, Prentice Hall.
[21] Jeffrey, A. (2002). Advanced Engineering Mathematics, Harcourt Academic Press.
[22] Gupta, A. R., Aggarwal, S. and Agrawal, D. (2018). Solution of linear partial integro-differential equations using Kamal transform. International Journal of Latest Technology in Engineering, Management & Applied Science, **7**(7), 88-91.
[23] Kiwne, S. B., and Sonawane, S. M. (2018). Mahgoub transform fundamental properties and applications. International Journal of Computer & Mathematical Sciences, **7**(2), 500-511.
[24] Lokenath, D. and Bhatta, D. (2006). Integral Transforms and Their Applications, Second edition, Chapman & Hall/CRC.
[25] Mahgoub, M. A. M. (2019). The new integral transform "Mahgoub Transform". Advances in Theoretical and Applied Mathematics, **1**(14), 81-87.
[26] Mahgoub, M. A. M. (2016). The new integral transform "Sawi Transform". Advances in Theoretical and Applied Mathematics, **14**(1), 81-87.

[27] Mohand, M. and Mahgoub, A. (2017). The new integral transform "Mohand Transform". Advances in Theoretical and Applied Mathematics, **12**(2), 113–20.
[28] Pawar, S. S. and Abhale, N. S. (2019). Application of Sadik transform for solving Bessel's functions and linear volterra integral equation of convolution type. Cikitusi Journal for Multidisciplinary Research, **6**(3), 85-91.
[29] Raisinghania, M. D. (2015). Advanced Differential Equations, S. Chand & Co. Ltd.
[30] Shendkar, A. M. and Jadhav, P. V. (2015). Elzaki transform: A solution of differential equations. International Journal of Science, Engineering and Technology Research, **4**(4), 1006-1008.
[31] Stroud, K. A. and Booth, D. J. (2001). Engineering Mathematics, Industrial Press, Inc.
[32] Wazwaz, A. M. (2011). Linear and Nonlinear Integral Equations, Higher Education Press, Beijing.