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The Auxiliary Equation Method and its Applications for a Nonlinear Dynamical System of a New Double-Chain Model of DNA

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Abstract

In this study, the auxiliary equation method was applied to find the exact solutions with parameters of the general nonlinear dynamical system of a new double-chain model of DNA. When the parameters are assigned special values, the solitary wave solutions were derived from the exact solutions. Comparison between our results and the well-known results is given.

Keywords: The auxiliary equation method; Nonlinear dynamical system of DNA; Double-chain model of DNA.

المستخلص

في هذا البحث تم تطبيق طريقة المعادلة المساعدة لإيجاد الحلول التامة لمنظومة ديناميكية غير خطية لنموذج ثنائي السلسلة لـ دي إن أي DNA. عندما أعطيت البارمترات قيماً خاصة, تم أستنتاج الحلول السلوتونيه للحلول التامة. وفي النهاية تمت مقارنة نتائجنا في هذا البحث بالنتائج المعروفة سابقاً.

Introduction

The investigation of the traveling wave solutions of nonlinear partial differential equations plays an important role in the study of a nonlinear physical phenomena.

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Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry.

Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations.In recent years, new exact solutions may help us to find new phenomena. A variety of powerful methods can be used, such as the inverse scattering method [2], the Hirota bilinear transformation [10], the exp-function method [23,33,5], the tanh-function method [1,7,34], the Jacobi elliptic function expansion method [6,14,15], the $(\frac{G'}{G})$-expansion method [19,25,29,32,36], the modified $(\frac{G'}{G})$-expansion method [36], the $(\frac{G'}{G},\frac{1}{G})$-expansion method [13,26,27,28], the multiple exp-function algorithm method [17], the transformed rational function method [16], and the modified simple equation method [11,24]. An attractive nonlinear model for the nonlinear science is the deoxyribonucleic acid (DNA). The dynamics of DNA molecules is one of the most fascinating problems of modern biophysics because it touches the basis of life. The DNA structure has been studied during last decades. The investigation of DNA dynamics has successfully predicted the appearance of important nonlinear structures. It has been shown that the nonlinearity is responsible for forming localized waves. These localized waves are interesting because they have the capability to transport energy without dissipation [3,4,8,9,12,18,20,21,22,30,31]. In Aka et al. [4,30] and Zayed and Arnous [31], it is given that a new double-chain model of DNA consists of two long elastic homogeneous strands which represent two polynucleotide chains of the DNA molecule, connected with each other by an elastic membrane representing the hydrogen bonds between the base pair of the two chains. Under some appropriate approximation, the new double-chain model of DNA can be described by the following two general nonlinear dynamical system:

$$u_{tt} - c_1^2 u_{xx} = \lambda_1 u + \gamma_1 uv + \mu_1 u^3 + \beta_1 uv^2, \tag{1}$$

$$v_{tt} - c_2^2 v_{xx} = \lambda_2 v + \gamma_2 u^2 + \mu_2 u^2 v + \beta_2 v^3 + c_0, \tag{2}$$

where

$$\begin{gathered} c_1 = \pm\sqrt{\frac{Y}{\rho}};\quad c_2 = \pm\sqrt{\frac{F}{\rho}};\quad \lambda_1 = \frac{-2\mu(h-l_0)}{\rho\sigma h};\quad \lambda_2 = \frac{-2\mu}{\rho\sigma};\quad \gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2}; \\ \mu_1 = \mu_2 = \frac{-2\mu l_0}{\rho\sigma h^3};\quad \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3};\quad c_0 = \frac{\sqrt{2}\mu(h-l_0)}{\rho\sigma} \end{gathered} \tag{3}$$

where ρ, σ, Y and F denote respectively the mass density, the area of transverse cross-section, the Young's modulus and the tension density of each strand; μ is the rigidity of the elastic membrane; h is the distance between the two strands, and l_0 is the height of the membrane in the equilibrium position. In Eqs. (1) and (2), u is the difference of the longitudinal displacements of the bottom and top strands, while v is the difference of the transverse displacements of the bottom and top strands.

The objective of this paper is to apply the auxiliary equation method to find the exact traveling wave solutions of the dynamical system (1) and (2).

Description of the Auxiliary Equation Method

Consider the following nonlinear evolution equation:

$$E(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{4}$$

where E is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [37]

Step 1. We use the wave transformation:

$$u(x,t) = u(\xi), \quad \xi = kx + \omega t, \tag{5}$$

where k and ω are constants, to reduce Eq. (4) to the ODE:

$$P(u, u', u'', ...) = 0, \tag{6}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, such that $' = d/d\xi$, i.e. the prime notation in Eq. (6) denotes differentiation with respect to ξ.

Step 2. We assume that the solution of (6) has the form:

$$u(\xi) = \sum_{i=0}^{2M} a_i F^i(\xi), \tag{7}$$

where a_i (i = 0, 1, 2, ..., 2M) are constants to be determined, such that $a_{2M} \neq 0$. The function $F(\xi)$ satisfies the auxiliary equation

$$[F'(\xi)]^2 = pF^2(\xi) + qF^4(\xi) + sF^6(\xi), \tag{8}$$

where p, q, s are constants. Eq. (8) admits several types of solutions [37]:

Table1. Solutions of Eq. (8) with $\Delta = q^2 - 4ps$ and $\varepsilon = \pm 1$.

NO	$F(\xi)$		NO	$F(\xi)$	
1.	$\left(\dfrac{-pq\,\text{sech}^2(\xi\sqrt{p})}{q^2 - ps\left[1+\varepsilon\tanh(\xi\sqrt{p})\right]^2}\right)^{\frac{1}{2}}$	$p>0$	8.	$\left(\dfrac{-p\sec^2(\xi\sqrt{-p})}{q+2\varepsilon\sqrt{-ps}\tan(\xi\sqrt{-p})}\right)^{\frac{1}{2}}$	$p<0,\ s>0$
2.	$\left(\dfrac{pq\,\text{csch}^2(\xi\sqrt{p})}{q^2 - ps\left[1+\varepsilon\coth(\xi\sqrt{p})\right]^2}\right)^{\frac{1}{2}}$	$p>0$	9.	$\left(\dfrac{p\,\text{csch}^2(\xi\sqrt{p})}{q+2\varepsilon\sqrt{ps}\coth(\xi\sqrt{p})}\right)^{\frac{1}{2}}$	$p>0,\ s>0$
3.	$\left(\dfrac{2p}{-q+\varepsilon\sqrt{\Delta}\cosh(2\xi\sqrt{p})}\right)^{\frac{1}{2}}$	$p>0,\ \Delta>0$	10.	$\left(\dfrac{-p\csc^2(\xi\sqrt{-p})}{q+2\varepsilon\sqrt{-ps}\cot(\xi\sqrt{-p})}\right)^{\frac{1}{2}}$	$p<0,\ s>0$
4.	$\left(\dfrac{2p}{-q+\varepsilon\sqrt{\Delta}\cos(2\xi\sqrt{-p})}\right)^{\frac{1}{2}}$	$p<0,\ \Delta>0$	11.	$\left(\dfrac{-p}{q}\left[1+\varepsilon\tanh(\xi\sqrt{p})\right]\right)^{\frac{1}{2}}$	$p>0,\ \Delta=0$
5.	$\left(\dfrac{2p}{-q+\varepsilon\sqrt{-\Delta}\sinh(2\xi\sqrt{p})}\right)^{\frac{1}{2}}$	$p>0,\ \Delta<0$	12.	$\left(\dfrac{-p}{q}\left[1+\varepsilon\coth(\xi\sqrt{p})\right]\right)^{\frac{1}{2}}$	$p>0,\ \Delta=0$
6.	$\left(\dfrac{2p}{-q+\varepsilon\sqrt{\Delta}\sin(2\xi\sqrt{-p})}\right)^{\frac{1}{2}}$	$p<0,\ \Delta>0$	13.	$4\left(\dfrac{pe^{2\varepsilon\xi\sqrt{p}}}{\left(e^{2\varepsilon\xi\sqrt{p}}-4q\right)^2-64ps}\right)^{\frac{1}{2}}$	$p>0$
7.	$\left(\dfrac{-p\,\text{sech}^2(\xi\sqrt{p})}{q+2\varepsilon\sqrt{ps}\tanh(\xi\sqrt{p})}\right)^{\frac{1}{2}}$	$p>0,\ s>0$	14.	$4\left(\dfrac{\pm pe^{2\varepsilon\xi\sqrt{p}}}{1-64pse^{4\varepsilon\xi\sqrt{p}}}\right)^{\frac{1}{2}}$	$p>0,\ q=0$

Step 3. We determine the positive integer M in (7) by balancing the highest order derivatives and the nonlinear terms in (6).

Step 4. We substitute (7) along with Eq. (8) into Eq. (6) and collecting all the terms of the same power $F(\xi)$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i , k and ω.

Step 5. Substituting these values and the solutions of Eq. (8) into (7) we have the exact solutions of Eq. (4).

Application

In this section, we will apply the proposed method described above, to find the exact solutions of the dynamical system (1) and (2). To this end, we first introduce the transformation:

$$v = au + b \ , \tag{9}$$

where a and b are constants, to reduce Eqs. (1) and (2) to the following system of equations:

$$u_{tt} - c_1^2 u_{xx} = u^3(\mu_1 + \beta_1 a^2) + u^2(2\beta_1 ab + a\gamma_1) + u(\lambda_1 + b\gamma_1 + \beta_1 b^2), \tag{10}$$

and

$$u_{tt} - c_2^2 u_{xx} = u^3(\mu_2 + \beta_2 a^2) + u^2(\frac{\gamma_2}{a} + \frac{\mu_2 b}{a} + 3\beta_2 ab) + u(\lambda_2 + 3\beta_2 b^2) + \frac{\lambda_2 b}{a} + \frac{\beta_2 b^3}{a} + \frac{c_0}{a}, \tag{11}$$

Comparing Eqs. (9) and (10) and using (3) we deduce that $b = \frac{h}{\sqrt{2}}$ and $F = Y$. Now Eqs. (9) and (10) can be re-written as

$$u_{tt} - c_1^2 u_{xx} - Au^3 - Bu^2 - Cu = 0, \tag{12}$$

where

$$A = \left(\frac{-2\alpha}{h^3} + \frac{4a^2\alpha}{h^3}\right); \ B = \frac{6\sqrt{2}a\alpha}{h^2}; \ C = \left(\frac{-2\alpha}{l_0} + \frac{6\alpha}{h}\right); \ \alpha = \frac{\mu l_0}{\rho\sigma}; \ c_1^2 = \frac{Y}{\rho}.$$

(13)

The wave transformation (5) of Sec. 2, reduces Eq. (12) to the following ODE:

$$(\omega^2 - k^2 c_1^2)u'' - Au^3 - Bu^2 - Cu = 0, \tag{14}$$

where $\omega^2 - k^2 c_1^2 \neq 0$. Balancing u'' with u^3 yields M. where M is the balance indicated in step 3. Consequently, we have the formal solution:

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi) \quad , \tag{15}$$

where a_0, a_1 and a_2 are constants to be determined. Substituting (15) along with equation (8) into (14) and setting the coefficients of $F^i(\xi)$ $(i = 0, 1, 2, ..., 2M)$ to zero yields a set of algebraic equations for a_0, a_1, a_2, k and ω as follows:

$$\begin{aligned}
&F^6 : 8sa_2\left(\omega^2-k^2c_1^2\right)-Aa_2^3=0,\\
&F^5 : 3sa_1\left(\omega^2-k^2c_1^2\right)-3Aa_1a_2^2=0,\\
&F^4 : 6qa_2\left(\omega^2-k^2c_1^2\right)-Ba_2^2-A\left(a_0a_2^2+2a_1^2a_2+a_2\left(a_1^2+2a_0a_2\right)\right)=0,\\
&F^3 : 2qa_1\left(\omega^2-k^2c_1^2\right)-A\left(a_1\left(a_1^2+2a_0a_2\right)+4a_0a_1a_2\right)-2Ba_1a_2=0,\\
&F^2 : 4pa_2\left(\omega^2-k^2c_1^2\right)-Ca_2-A\left(2a_0a_1^2+a_0^2a_2+a_0\left(a_1^2+2a_0a_2\right)\right)-B\left(a_1^2+2a_0a_2\right)=0,\\
&F : pa_1\left(\omega^2-k^2c_1^2\right)-2Ba_1a_0-Ca_1-3Aa_1a_0^2=0,\\
&F^0 : -Aa_0^3-Ba_0^2-Ca_0=0.
\end{aligned}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following result:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = \frac{3Cq}{2Bp}, \quad k = k, \quad \omega = \pm\sqrt{\frac{C+4pk^2c_1^2}{4p}}, \quad A = \frac{8psB^2}{9q^2C}. \tag{16}$$

Form (15), (16) and Table.1, we deduce the traveling wave solutions of Eq. (14) as follows:

$$u_1(\xi) = -\frac{3Cq^2}{2B}\left(\frac{\operatorname{sech}^2(\xi\sqrt{p})}{q^2-ps\left[1+\varepsilon\tanh(\xi\sqrt{p})\right]^2}\right) \quad , \quad p > 0 \tag{17}$$

$$u_2(\xi) = \frac{3Cq^2}{2B}\left(\frac{\operatorname{csch}^2(\xi\sqrt{p})}{q^2-ps\left[1+\varepsilon\coth(\xi\sqrt{p})\right]^2}\right) \quad , \quad p > 0 \tag{18}$$

$$u_3(\xi) = \frac{3Cq}{B}\left(\frac{1}{-q+\varepsilon\sqrt{\Delta}\cosh(2\xi\sqrt{p})}\right) \quad , \; p>0\,,\; \Delta>0 \tag{19}$$

$$u_4(\xi) = \frac{3Cq}{B}\left(\frac{1}{-q+\varepsilon\sqrt{\Delta}\cos(2\xi\sqrt{-p})}\right) \quad , \; p<0\,,\; \Delta>0 \tag{20}$$

$$u_5(\xi) = \frac{3Cq}{B}\left(\frac{1}{-q+\varepsilon\sqrt{-\Delta}\sinh(2\xi\sqrt{p})}\right) \quad , \; p>0\,,\; \Delta<0 \tag{21}$$

$$u_6(\xi) = \frac{3Cq}{B}\left(\frac{1}{-q+\varepsilon\sqrt{\Delta}\sin(2\xi\sqrt{-p})}\right) \quad , \; p<0\,,\; \Delta>0 \tag{22}$$

$$u_7(\xi) = -\frac{3Cq}{2B}\left(\frac{\operatorname{sech}^2(\xi\sqrt{p})}{q+2\varepsilon\sqrt{ps}\tanh(\xi\sqrt{p})}\right) \quad , \; p>0\;,\; s>0 \tag{23}$$

$$u_8(\xi) = -\frac{3Cq}{2B}\left(\frac{\sec^2(\xi\sqrt{-p})}{q+2\varepsilon\sqrt{-ps}\tan(\xi\sqrt{-p})}\right) \quad , \; p<0\;,\; s>0 \tag{24}$$

$$u_9(\xi) = \frac{3Cq}{2B}\left(\frac{\operatorname{csch}^2(\xi\sqrt{p})}{q+2\varepsilon\sqrt{ps}\coth(\xi\sqrt{p})}\right) \quad , \; p>0\;,\; s>0 \tag{25}$$

$$u_{10}(\xi) = -\frac{3Cq}{2B}\left(\frac{\csc^2(\xi\sqrt{-p})}{q+2\varepsilon\sqrt{-ps}\cot(\xi\sqrt{-p})}\right) \quad , \; p<0\;,\; s>0 \tag{26}$$

$$u_{11}(\xi) = -\frac{3C}{2B}\left(1+\varepsilon\tanh(\xi\sqrt{p})\right) \quad , \; p>0\,,\Delta=0 \tag{27}$$

$$u_{12}(\xi) = -\frac{3C}{2B}\left(1+\varepsilon\coth(\xi\sqrt{p})\right) \quad , \; p>0\,,\Delta=0 \tag{28}$$

$$u_{13}(\xi) = \frac{24Cq}{B}\left(\frac{e^{2\varepsilon\xi\sqrt{p}}}{\left(e^{2\varepsilon\xi\sqrt{p}}-4q\right)^2-64ps}\right) \quad , \; p>0\,. \tag{29}$$

Physical Explanations of Some Obtained Solutions

Here, we present some graphs of the obtained solutions constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equation. Using the mathematical software Maple, three-dimensional plots of some obtained exact solutions have been shown (Figs. 1-4). The obtained solutions for the nonlinear PDE (12) incorporate three types of explicit solutions namely, hyperbolic, trigonometric and rational. From these explicit results, it is easy to say that the solution (27) is a kink shaped soliton solution; the solution (19) is a bell shaped soliton solution; the solution (21) is a

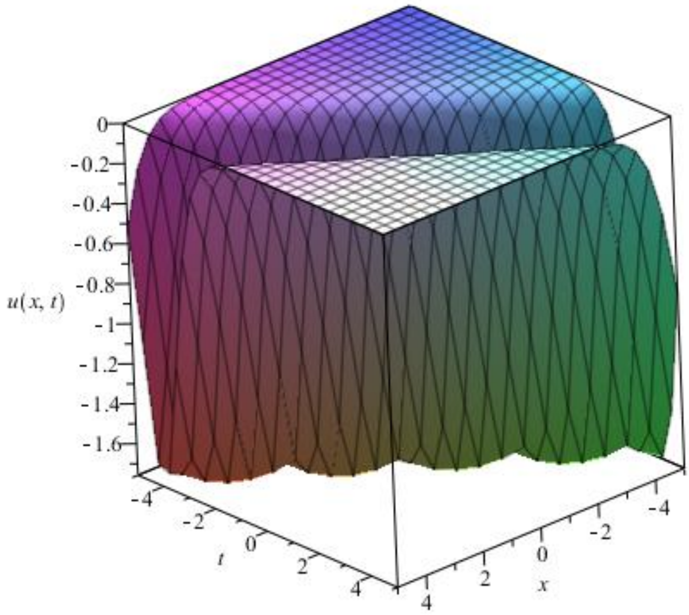

Fig.1. Plot of solution (17) when $p = 1, q = 2, s = \frac{1}{2}, C = 1, B = 1, c_1 = 1, k = 1$.

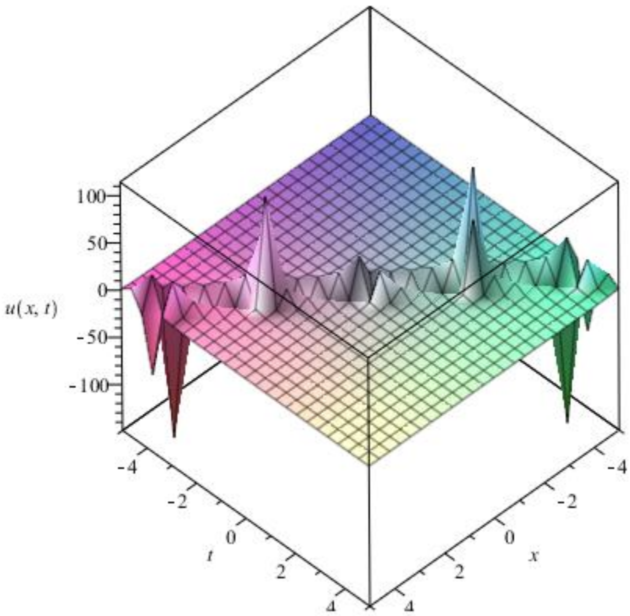

Fig.2. Plot of solution (19) when $p = 1, q = 2, s = \frac{1}{2}, C = 1, B = 1, c_1 = 1, k = 1$.

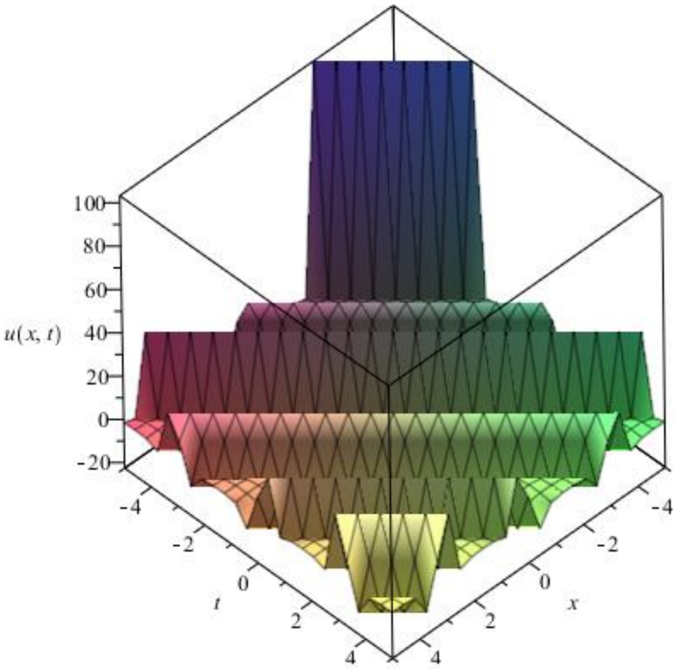

Fig. 3. Plot of solution (24) when $p = -1, q = 2, s = 1, C = 2, B = 2, c_1 = \frac{\sqrt{6}}{2}, k = 1$.

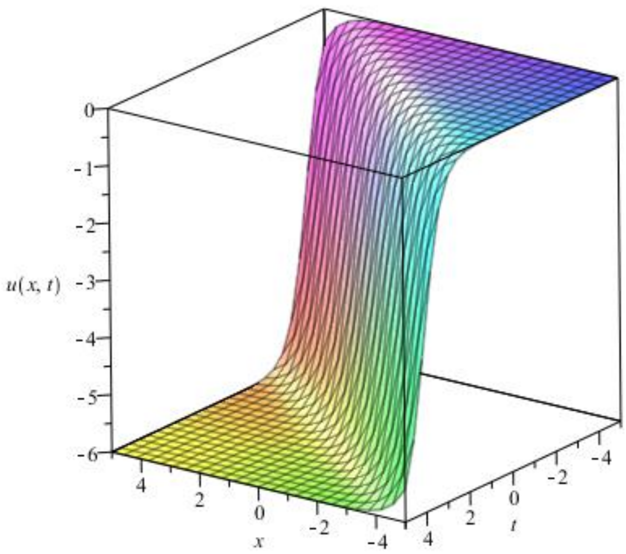

Fig. 4. Plot of solution (27) when $p = 1, C = 2, B = 1, c_1 = 1, k = 1$.

singular bell shaped soliton solution; the solution (28) is a singular kink shaped soliton solution; the solutions (17) and (23) are bell-kink shaped soliton solutions; the solution (18) and (25) are singular bell-kink shaped soliton solutions, the solutions (20), (22), (24), (26) are periodic solutions and the solution (29) is rational solution. The graphical representation of the solutions (17), (19), (24) and (27) can be plotted as shown (Figs. 1 to 4).

Conclusions

The auxiliary equation method described earlier in this study has been applied to construct many new exact solutions of the general nonlinear dynamical system (1) and (2) which describes the new double-chain model of DNA with the aid of Maple. On comparing our results obtained in the Application section above, with the results obtained in [4,30] using the improved generalized Riccati equation mapping method and the Riccati parameterized factorization method respectively, we conclude that our results are new and not published elsewhere. In last section, above, we presented some graphs of some exact solutions discussed in the Application section by choosing suitable values of parameters. Also we deduce that the auxiliary equation method used in this study is direct, effective and can be applied to many other nonlinear PDEs. Finally, all solutions obtained here have been checked with the Maple 14 by putting them back into the original equation.

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