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Comparison of Three Methods of Estimation for the Two-Parameter Weibull Distribution

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Abstract

Three methods of estimation of the two-parameter Weibull distribution were compared. A computer program was used to generate random data which was then used to get the desired estimates. The methods are: the graphical method, the maximum likelihood method and Menon's method. Three sample sizes and two sets of parameter values were used. The estimates were then compared on the basis of their mean square errors.

Keywords: Weibull distribution; Graphical method; maximum likelihood method; Menon's method

المستخلص

تمت مقارنة ثلاث طرق لتقدير توزيع وأيبول Weibull الثنائي البارمتر. وقد تم استعمال برنامج حاسوب لإنتاج البيانات العشوائية التي استعملت بعد ذلك في الحصول على التقديرات المطلوبة. والطرق المستعملة هي: الطريقة البيانية وطريقة أقصى امكانية وطريقة منون Menon. استعمالت ثلاثة أحجام عينات ومجموعتي بارمترات حيث أعقب ذلك المقارنة بينها باستعمال متوسط مربع الاخطاء.

Introduction

Weibull distribution has been used widely in analysing life-tests using time to failure distributions. The family of Weibull distribution is considered as a sub-family within the group of ''extreme value'' distributions. The Weibull distribution is closely related to the exponential distribution with more parameters.

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However, in the exponential distribution we get only a single constant failure rate λ where in the Weibull model, a variety of hazard situations can be treated. So, the reliability function $R(t)$ of the exponential distribution, the probability that failure occurs after time t, is given by:

$$R(t) = \exp(-\lambda t), \quad t \geq 0 \qquad (1)$$

Put $\lambda = \frac{1}{\delta}$ and replace t by $t - \mu$ gives $\lambda t = \frac{t-\mu}{\delta}$, $t \geq \mu$, and by inserting β as a power, the reliability function of the three-parameter Weibull distribution is:

$$R(t) = exp\left[-\left\{\frac{t-\mu}{\delta}\right\}^{\beta}\right], \quad t \geq \mu. \qquad (2)$$

Where the parameter μ is the location parameter, δ is the scale parameter, and β is the shape parameter. Clearly μ is the smallest value that the observations can assume and the time of failure cannot occur before time μ.
The Weibull distribution function is given by:

$$F(t) = 1 - exp\left[-\left\{\frac{t-\mu}{\delta}\right\}^{\beta}\right], \quad \beta > 0, \; t \geq 0, \; \mu \geq 0, \; \delta > 0. \qquad (3)$$

The Weibull probability density function is:

$$f(t) = \frac{\beta}{\delta}\left\{\frac{t-\mu}{\delta}\right\}^{\beta-1} exp\left[-\left\{\frac{t-\mu}{\delta}\right\}^{\beta}\right] \qquad (4)$$

However, the probability density function of the two-parameter (β and δ)is:

$$f(t) = \frac{\beta\, t^{\beta-1}}{\delta^{\beta}} exp\left[-\left\{\frac{t}{\delta}\right\}^{\beta}\right] \qquad (5)$$

If $\beta = 1$, this gives the exponential probability density function (PDF). For different values of β, the Weibull distribution has many different shapes. The failure rate, $h(t)$, is defined as the ratio of the PDF to the reliability function. So,

$$h(t) = \frac{\beta\, t^{\beta-1}}{\delta^{\beta}}, \qquad (6)$$

The graph of the hazard function has many different shapes. The hazard decreases with time for $\beta < 1$, is constant for $\beta = 1$, increasing linearly for $\beta = 2$, and for $\beta > 2$, the failure rate increases faster as the time increases.

Methods of Parameter Estimation

Many authors have studied the estimates of the Weibull parameters following many different methods where the maximum likelihood method was one of the most important methods. Cohen (1965) used it for complete, singly censored, and progressively censored samples. Lemon (1975) used it with left and right progressive censoring. Dubey (1960, 1967) used some percentile estimators and compared them to the maximum likelihood estimators. In the percentile estimation technique, an explicit estimator for β can be found. Moments is also another estimation method, where the first 3 moments are required to construct three equations to be solved for β, δ, and μ to obtain their estimates.

The best linear unbiased estimator (BLUE) and the best linear invariant estimator (BLIE) were also used to estimate Weibull parameters (Mann, 1967, 1971). The BLIE method involves weighting the ordered observations and using censored samples. For large samples, this approach needs to divide the unordered samples randomly into sub-samples of small size and to obtain the BLUE for each sub-sample. Then one calculates an optimum weighted average which approximates a BLUE for the entire sample.

These methods are considered as traditional estimation techniques. Recently, two new methods were developed for the same problem, namely, the entropy (Singh, 1987), and the use of the probability weighted moments (Greenwood et al., 1979). Singh et al. (1990), evaluated these techniques on the basis of their relative performance in terms of variability, robustness and bias. However, more recently, Weibull distribution became one of the most commonly used distribution to determine wind energy potential. Seguro and Lamber (2000) used Weibull wind speed distribution to estimate the parameters. They presented a modified maximum likelihood method which was recommended if the wind data is in a frequency format. If the data is in a time series format they recommended the maximum likelihood method. Akdag and Dinler (2009) used the mean of wind speed to formulate the energy pattern factor. Weibull parameters were estimated by solving the energy pattern factor numerically or using a simple formula for power density (PD) method. In this method the parameters can be estimated provided that the power density and mean wind speed are available.

In this work, the three considered methods were: the graphical method, the maximum likelihood method and the Menon's method.

The Graphical Method

This method used the idea of linear regression. It was studied by Kao (1959, 1960). Thus,

$$R(t) = exp\left[-\left\{\frac{t}{\delta}\right\}^{\beta}\right],$$

Therefore,

$$\frac{1}{R(t)} = exp\left[\left\{\frac{t}{\delta}\right\}^{\beta}\right], \text{ and}$$

$$\ln ln\frac{1}{R(t)} = \beta \ln\ t - \beta \ln\ \delta \qquad (7)$$

This is a linear equation which can be plotted as a straight line on a ''log-log versus log'' graph paper. The linear equation is of the form: $y = ax + b$, where

$$y = \ln\ln\frac{1}{R(t)}, \quad x = \ln t, \quad b = -\beta \ln \delta, \quad \text{and} \quad a = \beta.$$

The estimates of the two parameters β and δ can be obtained by fitting a straight line to the data. The slope of the line provides an estimate of β and the intercept can then be used to estimate δ. The observations are arranged in order so that $t_1 \leq t_2 \leq \cdots \leq t_n$. Doris (1989) uses the result $E\big(R(t_i)\big) = \frac{n+1-i}{n+1}$. Then $R(t_i)$ can be estimated as $\hat{R}(t_i) = \frac{n+1-i}{n+1}$. This gives $y_i = lnln\frac{n+1}{n+1-i}$, as the vertical coordinate, and $x_i = \ln(t_i)$ as the horizontal coordinate. The n points (x_i, y_i) were plotted and a straight line was fitted to them. The parameters were obtained from the slope and the intercept of the fitted line. Thus, the required formulas:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}[\ln(t_i) - \overline{\ln(t)}]y_i}{\sum_{i=1}^{n}[\ln(t_i) - \overline{\ln(t)}]^2}, \qquad (8a)$$

$$\hat{\delta}_1 = exp\left[\overline{\ln(t)} - \frac{\bar{y}}{\hat{\beta}_1}\right] \qquad (8b)$$

In getting equations (8a) and (8b), $y_i = \ln\ln\frac{1}{R(t_i)}$ was used as the dependent variable and $x_i = \ln(t_i)$ as the independent variable. However, the y_i were fixed and the x_i were random variables. Therefore, switch the variables to get an alternative linear equation

$$\ln(t) = \frac{1}{\beta}\ln\ln\frac{1}{R(t)} + \ln\delta. \qquad (9)$$

Again, the fitting of this equation will provide estimates of β and δ. The required formulas are:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{\sum_{i=1}^{n}(y_i-\bar{y})\ln(t_i)}, \qquad (10a)$$

$$\hat{\delta}_2 = exp\left[\overline{\ln(t)} - \frac{\bar{y}}{\hat{\beta}_2}\right]. \qquad (10b)$$

In place of $E\big(R(t_i)\big)$, one can also use the median of $R(t_i)$. While an exact calculation of the median of $R(t_i)$ is difficult, an approximation by Kenneth (2011) is available, namely, $\hat{R}(t_i) = \frac{n+0.7-i}{n+0.4}$. This gives $y_i = \ln\ln\frac{n+0.4}{n+0.7-i}$. With this change, one can obtain two more sets of estimates by using (8a), (8b), (10a), and (10b).

The above discussion assumes that the value of μ for the three-parameter distribution is known. If it is unknown, then the graphical method can still be used. Clearly, one has to use $x_i = \ln(t_i - \hat{\mu})$ where $\hat{\mu}$ is a preliminary estimate. The points (x_i, y_i) were plotted as shown above. The trial and error method of selecting $\hat{\mu}$ requires increasing the estimate if the curve plotted is concave upward, and decreasing it if the curve is concave downward, until one gets a straight line. Once $\hat{\mu}$ is obtained, the above method can be used to estimate β and δ.

The Maximum Likelihood Method

The density of the two-parameter Weibull distribution is written by Cohen (1965) as follows:

$$f(t) = \frac{\beta}{\theta}\, t^{\beta-1} exp\left[-\frac{t^{\beta}}{\theta}\right], \quad t > 0,\ \theta > 0,\ \beta > 0, \qquad (11)$$

Where the scale parameter δ is written as $\theta^{\frac{1}{\beta}}$. This particular form of the density was chosen for the purpose of simplifying derivation of the maximum likelihood estimating equations. Consider a random sample of n observations from a distribution whose density function is given by (11). The likelihood function is:

$$L(\beta,\theta,t_1,t_2,\cdots,t_n)=\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\theta}\right)t_i^{\beta-1}exp\left[-\frac{t_i^{\beta}}{\theta}\right]\right\}. \qquad (12)$$

Taking logarithms in (12), differentiating partially with respect to β and θ and equating the result to zero, we get

$$\frac{\partial \ln L}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln t_i-\frac{1}{\theta}\sum_{i=1}^{n}t_i^{\beta}\ln t_i=0, \qquad (13)$$

$$\frac{\partial \ln L}{\partial\theta}=\frac{-n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}t_i^{\beta}=0. \qquad (14)$$

After eliminating θ from these two equations and simplifying, we get

$$\frac{\sum_{i=1}^{n}t_i^{\beta}lnt_i}{\sum_{i=1}^{n}t_i^{\beta}}-\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^{n}\ln t_i=0, \qquad (15)$$

To solve equation (15), use the Newton-Raphson Procedure. So, let

$$g(\beta)=\sum_{i=1}^{n}t_i^{\beta}\ln t_i-\frac{1}{\beta}\sum_{i=1}^{n}t_i^{\beta}-\frac{1}{n}\sum_{i=1}^{n}\ln t_i\sum_{i=1}^{n}t_i^{\beta} \qquad (16)$$

Equation (15) is then equivalent to the equation $g(\beta)=0$. This equation has a unique solution; see Lehmann (1983, p. 437). It is easy to see that

$$g'(\beta)=\sum_{i=1}^{n}t_i^{\beta}(\ln t_i)^2+\frac{1}{\beta^2}\sum_{i=1}^{n}t_i^{\beta}$$
$$-\frac{1}{\beta}\sum_{i=1}^{n}t_i^{\beta}\ln t_i-\frac{1}{n}\sum_{i=1}^{n}\ln t_i\sum_{i=1}^{n}t_i^{\beta}\ln t_i \qquad (17)$$

We started with a good guess β_0 for β which might, for instance, be an estimate obtained by the graphical method. Then, calculate the correction $\Delta=-\frac{g(\beta_0)}{g'(\beta_0)}$, which yields a revised estimate $\beta_1=\beta_0+\Delta$. The next iteration used β_1 to calculate another correction Δ. This process is repeated until Δ became very small. The last value of β is the required estimate $\hat{\beta}$ which allows estimating θ from (14) as:

$$\hat{\theta}=\sum_{i=1}^{n}\frac{t_i^{\hat{\beta}}}{n}, \text{ and finally, } \quad \hat{\delta}=\left(\hat{\theta}\right)^{\frac{1}{\hat{\beta}}}.$$

Menon's Method

Menon (1963) gave the following formulas for estimating the shape and the scale parameters of the Weibull distribution.

$$\hat{\beta} = \left[\left(\frac{6}{\pi^2}\right)\left\{\sum_{i=1}^{n}(\ln \mathrm{t_i})^2 - \frac{1}{n}\left(\sum_{i=1}^{n}\ln t_i\right)^2\right\}/(n-1)\right]^{\frac{-1}{2}}, \quad (18)$$

$$\hat{\delta} = exp\left[\frac{1}{n}\sum_{i=1}^{n}\ln t_i + \frac{\gamma}{\hat{\beta}}\right], \quad (19)$$

Where γ is Euler's constant. The rationale behind these formulae was as follows. Suppose that T has the Weibull probability density function given by (5). Using the fact that the distribution of $W = (T/\delta)^{\beta}$ is independent of both β and δ and has the PDF

$$\begin{cases} e^{-t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

Then, $var(\ln W) = E[(\ln W)^2] - [E(\ln W)]^2$

$$= \int_0^{\infty} e^{-t}(\ln t)^2 dt - \left[\int_0^{\infty} e^{-t}\ln t dt\right]^2$$

It is known that $\Gamma(p) = \int_0^{\infty} t^{p-1}\, e^{-t} dt$,

So, $\Gamma'(p) = \int_0^{\infty} t^{p-1}\, e^{-t}\ln t\, dt$,

And the higher derivations have similar expressions. Thus,

$$\Gamma'(1) = \int_0^{\infty} e^{-t}\ln t\, dt,$$

$$\Gamma''(1) = \int_0^{\infty} e^{-t}\,(\ln t)^2 dt$$

Therefore, $var(\ln W) = \Gamma''(1) - [\Gamma'(1)]^2$

Define $\Psi(p) = \frac{\Gamma'(p)}{\Gamma(p)}$. Then it is clear that

$$\Psi'(1) = \Gamma''(1) - [\Gamma'(1)]^2$$

Therefore, $var(ln\ W) = \Psi'(1)$.

It can be shown that, $\Psi'(p) = \sum_{k=0}^{\infty} \frac{1}{(p+k)^2}$.

Therefore, $\Psi'(1) = \sum_{k=0}^{\infty} \frac{1}{(1+k)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Thus, $var(\ln W) = \frac{\pi^2}{6}$.

Now, $var(\ln W) = \beta^2 var(\ln T)$, and so, $\frac{1}{\beta^2} = \frac{var(\ln T)}{var(\ln W)} = \frac{6}{\pi^2} var(\ln T)$.

Then, $var(\ln T)$ can be estimated by

$$\frac{1}{n-1}\left[\sum_{i=1}^{n}(\ln t_i)^2 - \frac{1}{n}\left(\sum_{i=1}^{n}\ln t_i\right)^2\right]$$

This gives $\hat{\beta}$ as in (18). To justify (19), write $\lambda_1 = \Gamma'(1)$. Then

$$\lambda_1 = \int_0^{\infty} e^{-t} \ln t\, dt = E(\ln W).$$

Further, $\ln(W) = \beta(\ln T - \ln \delta)$.
Taking expectations, then $E(\ln W) = \beta[E(\ln T) - \ln \delta]$.
So, $\lambda_1 = \beta[E(\ln T) - \ln \delta]$, and $\ln \delta = E(ln\ T) - \frac{\lambda_1}{\beta}$.
Now, $E(\ln T)$ can be estimated by $\frac{1}{n}\sum_{i=1}^{n} \ln t_i$ and $\hat{\beta}$ is given by (18). Therefore, we get

$$\hat{\delta} = exp\left[\frac{1}{n}\sum_{i=1}^{n}\ln t_i - \frac{\lambda_1}{\hat{\beta}}\right].$$

Since it is well known that $\lambda_1 = -\gamma$, formula (19) for $\hat{\delta}$ is also justified. In essence, Menon's method is just the method of moments applied to $\ln T$.

Simulation Study

A large number of samples of observations were used to study the properties of the various estimates for the parameters of the Weibull distribution. These observations were created by using a random number generator and converted from uniform (0,1) random variables X to Weibull random variables T.

The process starts by setting the values of the parameters at β = 2.5 and δ = 40. And the sample sizes at 10, 30 and 60 with 5000 samples of each given size and then the estimates of β and δ were calculated. The values of the parameters were changed to β = 4.0 and $\delta = 60$ and the same process was repeated. Finally, the averages of $\hat{\beta}$ and $\hat{\delta}$ and their mean square errors were calculated. The results were organized (Tables 1-6). In these tables, graph1 and graph2 used expected ranks while graph3 and graph4 used median ranks.

Summary and Conclusion

This work was concerned with estimating the parameters of the two-parameter Weibull distribution and comparing the results of three methods of estimation. Specifically, we compare the results of the graphical method (which yields four sets of estimates), the maximum likelihood method, and Menon's method. The study was carried out using computer simulation. The estimates were calculated for 5000 samples using three sample sizes, and two sets of values for the parameters. The results can be summarized as follows:

(a) Estimation of β: comparing the four graphical methods among themselves, we find that Graph2 gives the best overall results. Indeed Graph2 shows the smallest MSE in four of the six tables. A comparison of Graph2 with MLE.

(b) shows that Graph2 gives better results in four out of six cases. Moreover, Graph2 is better than Menon's method in all cases. Finally, the MLE performs better than Menon's method in all cases.

(c) Estimation of δ: Among the four graphical estimates, Graph4 is the best in all six cases. This method is also better than the MLE in every case. While Graph4 does better than Menon's method in all cases, the latter is a very close second. Finally, Menon's method performs better than the MLE in all cases.

Overall, the recommendation would be to use the graphical method with $\ln t$ as the dependent variable. Whether one should use expected ranks or median ranks depends on the particular parameter being estimated.

Table 1. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 2.5, δ = 40 and $n = 10$.

Method	$\hat{\beta}$	MSE	$\hat{\delta}$	MSE
Graph1	2.16826	0.61274	41.14634	32.22774
Graph2	2.34325	0.60495	40.31985	28.98757
Graph3	2.41465	0.62365	40.59934	30.26144
Graph4	2.61078	0.74159	39.85169	28.49849
MLE	2.25575	0.59606	39.00353	30.37933
Menon	2.88805	1.03031	39.83452	28.60007

Table 2. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 2.5, δ = 40 and $n = 30$.

Method	$\hat{\beta}$	MSE	$\hat{\delta}$	MSE
Graph1	2.26665	0.23743	40.70539	10.90907
Graph2	2.36484	0.20493	40.25156	9.77734
Graph3	2.39858	0.21137	40.44557	10.40554
Graph4	2.50056	0.21252	40.02770	9.68688
MLE	2.31574	0.21735	41.75192	10.28313
Menon	2.62521	0.25183	39.98303	9.79446

Table 3. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 2.5, δ = 40 and $n = 60$.

Method	Average $\hat{\beta}$	MSE	Average $\hat{\delta}$	MSE
Graph1	2.34017	0.12470	40.47874	5.34917
Graph2	2.40449	0.10485	40.19720	4.88033
Graph3	2.42779	0.10995	40.31235	5.14381
Graph4	2.49201	0.10469	40.05281	4.83610
MLE	2.37233	0.11303	38.64855	5.08889
Menon	2.56914	0.11844	40.00487	4.87854

Table 4. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 4, δ = 60 and $n = 10$

Method	$\hat{\beta}$	MSE	$\hat{\delta}$	MSE
Graph1	3.50287	1.57330	60.96162	27.30112
Graph2	3.78933	1.62345	60.20712	25.24890
Graph3	3.90030	1.63737	60.46074	26.13047
Graph4	4.22272	2.03278	59.77011	25.18502
MLE	3.64610	1.56209	55.77761	28.42441
Menon	4.66788	2.80011	59.75307	25.30340

Table 5. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 4 , δ = 60 and $n = 30$.

Method	$\hat{\beta}$	MSE	$\hat{\delta}$	MSE
Graph1	3.63472	0.60539	60.60054	9.67728
Graph2	3.79244	0.52066	60.17821	8.86132
Graph3	3.84652	0.54208	60.35862	9.31684
Graph4	4.00984	0.54396	59.96960	8.82781
MLE	3.71358	0.55324	59.87639	9.65165
Menon	4.20985	0.65082	59.92770	8.92202

Table 6. Estimates of $\hat{\beta}, \hat{\delta}$ when β = 4, δ = 60 and $n = 60$.

Method	$\hat{\beta}$	MSE	$\hat{\delta}$	MSE
Graph1	3.74345	0.32511	60.43938	4.67957
Graph2	3.84702	0.27242	60.17437	4.29790
Graph3	3.88372	0.28723	60.28357	4.51810
Graph4	3.98692	0.27248	60.03949	4.27700
MLE	3.79524	0.29425	59.37104	4.65716
Menon	4.11008	0.30856	59.99522	4.32374

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Refernceses

Akdag, S. A. and Dinler, A. (2009). A new method to estimate Weibull parameters for wind energy applications, Energy Conversion and Management, **50**, 1761-1766.

Cohen, A. C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples, Technometrics, **7**, 579-588.

Doris L. G. (1989). A Primer of Reliability Theory, John Wiley and Sons, Australia.

Dubey, S. D. (1960). Contribution to Statistical Theory of Life Testing and Reliability, Ph.D. Dissertation, Michigan State University, University Microfilms, Inc., Ann Arbour, Michigan.

Dubey, S. D. (1967). Some Percentile Estimators for Weibull Parameters, Technometrics, **9**, 119-129.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C. and Wallis, J. R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form, Water Resources Research, **15**, 1049-1054.

Kao, J. H. K. (1959). A graphical estimation of mixed Weibull parameters in life-testing of electron tubes, Technometrics, **1**, 389-407.

Kao, J. H. K. (1960). A Summary of some new techniques on failure analysis, Proceedings of the Sixth National Symposium on Reliability and Quality Control in Electronics, Washington, D. C., 190-201.

Kenneth S. S. (2011). Reliability Data Analysis with Excel and Minitab, William A. Tony Publishers.

Lehmann, E. L. (1983). Theory of Point Estimation, John Wiley and Sons, New York.

Lemon, G. H. (1975). Maximum likelihood estimation for the three-parameter Weibull distribution based on censored samples, Technometrics, **17**, 247-254.

Mann, N. R. (1967). Tables for obtaining the best linear invariant estimates of parameters of the Weibull distribution, Technometrics, **9**, 629-645.

Mann, N. R. (1971). Best linear Invariant estimation for Weibull parameters under progressive censoring, Technometrics, **13**, 521-533.

Menon, M. V. (1963). Estimation of the shape and scale parameters of the Weibull distribution, Technometrics, **5**, 175-182.

Seguro, J. V. and Lamber, T. W. (2000). Modern estimation of the parameters of the Weibull wind speed distribution for wind energy analysis, Journal of Wind Engineering and Industrial Aerodynamics, **85**, 75-84.

Singh, V. P. (1987). On application of the Weibull distribution in hydrology. Water Resources Management, **1**, 33-43.

Singh, V. P., Cruise, J. F. and Ma, M. (1990). A comparative evaluation of the estimators of the Weibull distribution by Monte Carlo simulation, Journal of Statistical Computation and Simulation, **36**, 229-241.