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On Solving Some Fifth Order Nonlinear PDEs Using The Modified Kudryashov Method

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Abstract

In this paper , the modified Kudryashov method was applied to construct the exact travelling wave solutions for some fifth order nonlinear partial differential equations (PDEs), namely, the Kaup-Kupershmidt, the Ito, the Caudrey-Dodd-Gibbon, the Lax and the Sawada-Kotera equations.

Keywords: Modified Kudryashov method; Nonlinear PDEs; Exact solutions.

المستخلص

في هذه الورقة طبقت طريقة قدري شوف المطورة لبناء حلول موجية متحركة تامة لبعض المعادلات التفاضلية الجزئية غير الخطية من الرتبة الخامسة وهي معادلات كوب–كوبرشمدث ، و أيتو، وكودري–دود– جيبون، و الرخو، و سوادا كوتيرا.

Introduction

In the last four decades or so , seeking exact solutions of nonlinear PDEs has been of great importance, since the nonlinear complex physical phenomena related to the nonlinear PDEs arise in many fields of physics, mechanics, biology, chemistry and engineering The investigation of exact solutions of nonlinear PDEs, as mathematical models of the phenomena, will help us to understand the mechanism that governs.

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these physical models or provide better understanding of the problems and the possible applications. To these ends, a vast variety of powerful and direct methods for finding the exact significant solutions of nonlinear PDEs have been derived, such as the inverse scattering transform [1], the Hirota method [2], the truncated Painleve expansion method [4] , the Backlund transform method [1], the simplest equation method [6] , the Jacobi elliptic function method [7] , the tanh-function method [12], the modified simple equation method [3], the Kudryashov method [5,13,14], the multiple exp-function algorithm method [8] , the transformed rational function method [9], the Frobenius decomposition technique [10] , the local fractional variation iteration method [17] and the local fractional series expansion method [18] and so on.

The objective of this paper is to demonstrate the efficiency of the modified Kudryashov method for finding exact solutions of some nonlinear evolution equations in the mathematical physics, namely, the Kaup-Kupershmidt, the Ito, the Caudrey-Dodd-Gibbon, the Lax and the Sawada-Kotera equations.

Description of the Modified Kudryashov Method

Suppose we have a nonlinear evolution equation in the form

$$F(u, u_t, u_x, u_{xx}, ...) = 0 \tag{1}$$

Where F is polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [11]:

Step 1. Using the wave transformation

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t \tag{2}$$

To reduce Eq. (1) to the following ODE :

$$P(u, u', u'', ...) = 0, \tag{3}$$

Where P is a polynomial in $u(\xi)$ and its total derivatives, while k, ω are constants and the prime notation in (3) denotes differentiation with respect to ξ.

Step 2. We suppose that Eq. (3) has the formal solution

$$u(\xi)=\sum_{n=0}^{N}a_n Q^n(\xi), \quad (4)$$

where $a_n \ (n=0,1,...,N)$ are constants to be determined, such that $a_N \neq 0,$ *and* $Q(\xi)$ is the solution of the equation

$$Q'(\xi)=\left[Q^2(\xi)-Q(\xi)\right]\ln(a) \quad (5)$$

Eq. (5) has the solutions

$$Q(\xi)=\frac{1}{1\pm a^{\xi}} \quad (6)$$

Where $a>0,\ a\neq 1$ is a number. If $a=e,$ then we have the modified Kudryashov method which has been applied by many authors, see for example [5].

Step 3. We determine the positive integer N in Eq. (4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (3).

Step 4. Substitute Eq. (4) along with Eq. (5) into Eq. (3), we calculate all the necessary derivatives $u', u'', ...$ of the function $u(\xi).$ As a result of this substitution, we get a polynomial of $Q^i(\xi),\ (i=0,1,2,...).$ In this polynomial we gather all terms of same powers of $Q^i(\xi)$ and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters $a_n \ (n=0,1,...,N),$ k and $\omega.$ Consequently, we obtain the exact solutions of Eq. (1).

Remark 1. The obtained solutions can depended on the symmetrical hyperbolic Lucas functions and Fibonacci functions proposed by Stakhov and Rozin [15]. The symmetrical Lucas sine, cosine, tangent and cotangent functions are respectively, defined as

$$sLs(\xi) = a^{\xi} - a^{-\xi}, \quad cLs(\xi) = a^{\xi} + a^{-\xi}, \quad tLs(\xi) = \frac{a^{\xi} - a^{-\xi}}{a^{\xi} + a^{-\xi}} = \frac{sLs(\xi)}{cLs(\xi)},$$

$$ctLs(\xi) = \frac{a^{\xi} + a^{-\xi}}{a^{\xi} - a^{-\xi}} = \frac{cLs(\xi)}{sLs(\xi)} \tag{7}$$

Also, these functions satisfy the following formulas:

$$\left[cLs(\xi)\right]^2 - \left[sLs(\xi)\right]^2 = 4 \tag{8}$$

$$\left[cFs(\xi)\right]^2 - \left[sFs(\xi)\right]^2 = \frac{4}{5} \tag{9}$$

The obtained solutions in this paper can be obtained in terms of the symmetrical hyperbolic Lucas functions.

Applications

In this section, we apply the modified Kudryashov method to find the exact solutions of the following nonlinear PDEs:

Example 1. The Kaup-Kupershmidt (KK) Equation
This equation is well known [16] and has the form

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{10}$$

Let us now solve equation (10) using the modified Kudryashov method. To this end, we use the wave transformation (2) to reduce equation (10) to the following ODE:

$$\omega u' + 20ku^2u' + 25k^3u'u'' + 10k^3uu^{(3)} + k^5u^{(5)} = 0. \tag{11}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2.$ Consequently, equation (11) has the formal solution

$$u = a_0 + a_1Q + a_2Q^2 \tag{12}$$

where a_0, a_1 and a_2 are constants to be determined such that $a_2 \neq 0.$ From equation (12), we get

$$u' = (\ln a)(a_1 + 2Qa_2)Q(Q-1), \tag{13}$$

$$u'' = (\ln a)^2 Q(Q-1)\left[(-1+2Q)a_1 + 2Q(3Q-2)a_2\right], \tag{14}$$

$$u^{(3)} = (\ln a)^3 Q(Q-1)\left[(1-6Q+6Q^2)a_1 + 2Q(4-15+12Q^2)a_2\right], \tag{15}$$

$$u^{(4)} = (\ln a)^4 Q(Q-1)[(-1+14Q-36Q^2+24Q^3)a_1 + 2Q(-8+57Q-108Q^2+60Q^3)a_2], \tag{16}$$

$$u^{(5)} = (\ln a)^5 Q(Q-1)[(1-30Q+150Q^2-240Q^3+120Q^4)a_1 +2Q(16-195Q+660Q^2-840Q^3+360Q^4)a_2]$$

(17)

Substituting (12)-(17) into (11) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain algebraic system of equations ,on solving the obtained algebraic equations using the Maple or Mathematica, we get the following results:

Case 1.

$$a_0 = -k^2(\ln a)^2,\ a_1 = 12k^2(\ln a)^2,\ a_2 = -12k^2(\ln a)^2,\ \omega = -11k^5(\ln a)^4. \tag{18}$$

From (6), (7), (12), (18) , we obtain the following exact solutions of Eq. (10)

$$u_1(x,t) = -k^2(\ln a)^2 + 12\left(\frac{k\ln(a)}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{19}$$

$$u_2(x,t) = -k^2(\ln a)^2 - 12\left(\frac{k\ln(a)}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{20}$$

Case 2.

$$a_0=\frac{1}{8}k^2(\ln a)^2,\ a_1=\frac{3}{2}k^2(\ln a)^2,\ a_2=\frac{-3}{2}k^2(\ln a)^2,\ \omega=\frac{-1}{16}k^5(\ln a)^4. \qquad (21)$$

$$u_3(x,t)=\frac{-1}{8}k^2(\ln a)^2+\frac{3}{2}\left(\frac{k\ln(a)}{cLs\left(\frac{\xi}{2}\right)}\right)^2,$$

(22)

$$u_4(x,t)=\frac{-1}{8}k^2\left(\ln a\right)^2-\frac{3}{2}\left(\frac{k\ln(a)}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \qquad (23)$$

Example 2. The Ito Equation

This equation is well known [16] and has the form

$$u_t+2u^2u_x+6u_xu_{xx}+3uu_{3x}+u_{5x}=0. \qquad (24)$$

Let us solve equation (24) by using the modified Kudryashov method. To this end, we use the wave transformation (2) to reduce equation (24) to the following ODE:

$$\omega u'+2ku^2u'+6k^3u'u''+3k^3uu^{(3)}+k^5u^{(5)}=0. \qquad (25)$$

Balancing $u^{(5)}$ with u^2u' yields $N=2.$ Consequently , equation (24) has the formal solution (12). Substituting (12)-(17) into (24) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain algebraic system of equations ,on solving the obtained algebraic equations using the Maple or Mathematica, we get the following result:

$$a_0=\frac{-5}{2}k^2(\ln a)^2,\ a_1=30k^2(\ln a)^2,\ a_2=-30k^2(\ln a)^2,\ \omega=-6k^5(\ln a)^4 \qquad (26)$$

From (6) , (7) , (12) , (26) , , we obtain the following exact solutions of Eq. (25)

$$u_1(x,t)=\frac{-5}{2}k^2(\ln a)^2+30\left(\frac{k\ln a}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{27}$$

$$u_2(x,t)=\frac{-5}{2}k^2(\ln a)^2-30\left(\frac{k\ln a}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{28}$$

Example 3. The Caudrey-Dodd-Gibbon Equation (CDG)
This equation is well known [16] and has the form :

$$u_t+180u^2u_x+30u_xu_{xx}+30uu_{3x}+u_{5x}=0. \tag{29}$$

Let us solve equation (29) by the modified Kudryashov method. To this end , we use the wave transformation (2) to reduce equation (29) to the following ODE:

$$\omega u'+180ku^2u'+30k^3u'u''+30k^3uu^{(3)}+k^5u^{(5)}=0. \tag{30}$$

Balancing $u^{(5)}$ with u^2u' yields $N=2.$ Consequently, equation (29) has the formal solution (12) . Substituting (12)-(17) into (29) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain algebraic system of equations ,on solving the obtained algebraic equations using the Maple or Mathematica, we get the following results:

Case 1.

$$a_1=k^2(\ln a)^2,\ a_2=-a_1,\ \omega=-k^5(\ln a)^4-180ka_0^2-30k^3a_0(\ln a)^2. \tag{31}$$

In this case, we deduce the following exact solutions of Eq. (29)

$$u_1(x,t) = a_0 + \left(\frac{k \ln a}{cLs\left(\frac{\xi}{2}\right)} \right)^2 , \tag{32}$$

$$u_2(x,t) = a_0 - \left(\frac{k \ln a}{sLs\left(\frac{\xi}{2}\right)} \right)^2 , \tag{33}$$

Case 2.

$$a_0 = \frac{-1}{6} k^2 (\ln a)^2, \; a_1 = 2k^2 (\ln a)^2, \; a_2 = -2k^2 (\ln a)^2, \; \omega = -k^5 (\ln a)^4. \tag{34}$$

In this case, we deduce the following exact solutions of Eq. (29)

$$u_3(x,t) = \frac{-1}{6} k^2 (\ln a)^2 + 2 \left(\frac{k \ln a}{cLs\left(\frac{\xi}{2}\right)} \right)^2 , \tag{35}$$

$$u_4(x,t) = \frac{-1}{6} k^2 (\ln a)^2 - 2 \left(\frac{k \ln a}{sLs\left(\frac{\xi}{2}\right)} \right)^2 , \tag{36}$$

Example 4. The Lax Equation

This equation is well known [16] and has the form :

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{37}$$

Let us solve equation (37) by the modified Kudryashov method. To this end , we use the wave transformation (2) to reduce equation (37) to the following ODE:

$$\omega u' + 30ku^2u' + 20k^3u'u'' + 10k^3uu^{(3)} + k^5u^{(5)} = 0. \tag{38}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2.$ Consequently, equation (37) has the formal solution (12) . Substituting (12)-(17) into (37) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain algebraic system of equations ,on solving the obtained algebraic equations using the Maple or Mathematica, we get the following results:

Case 1.

$$a_1 = 2k^2(\ln a)^2,\ a_2 = -a_1,\ \omega = -k^5(\ln a)^4 - 30ka_0^2 - 10k^3a_0(\ln a)^2. \tag{39}$$

In this case, we deduce the following exact solutions of Eq. (37)

$$u_1(x,t) = a_0 + 2\left(\frac{k\ln a}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{40}$$

$$u_2(x,t) = a_0 - 2\left(\frac{k\ln a}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{41}$$

Case 2.

$$a_0 = \frac{-1}{2}k^2(\ln a)^2,\ a_1 = 6k^2(\ln a)^2,\ a_2 = -6k^2(\ln a)^2,\ \omega = \frac{-7}{2}k^5(\ln a)^4. \tag{42}$$

In this case, we deduce the following exact solutions of Eq. (37)

$$u_3(x,t) = \frac{-1}{2}k^2(\ln a)^2 + 6\left(\frac{k\ln a}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{43}$$

$$u_4(x,t) = \frac{-1}{2}k^2(\ln a)^2 - 6\left(\frac{k\ln a}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \qquad (44)$$

Example 5. The Sawada-Kotera (SK) Equation
This equation is well known [16] and has the form :

$$u_t + 5u^2u_x + 5u_xu_{xx} + 5uu_{3x} + u_{5x} = 0. \qquad (45)$$

Let us solve equation (45) by the modified Kudryashov method. To this end , we use the wave transformation (2) to reduce equation (45) to the following ODE:

$$\omega u' + 5ku^2u' + 5k^3u'u'' + 5k^3uu^{(3)} + k^5u^{(5)} = 0. \qquad (46)$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2.$ Consequently, equation (46) has the formal solution (12) . Substituting (12)-(17) into (46) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain algebraic system of equations ,on solving the obtained algebraic equations using the Maple or Mathematica, we get the following results:

Case 1.

$$a_1 = 6k^2(\ln a)^2,\ a_2 = -a_1,\ \omega = -k^5(\ln a)^4 - 5ka_0^2 - 5k^3a_0(\ln a)^2. \qquad (47)$$

In this case, we deduce the following exact solutions of Eq. (45)

$$u_1(x,t) = a_0 + 6\left(\frac{k\ln a}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \qquad (48)$$

$$u_2(x,t)=a_0-6\left(\frac{k\ln a}{sLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{49}$$

Case 2.

$$a_0=-k^2(\ln a)^2,\ a_1=12k^2(\ln a)^2,\ a_2=-a_1,\ \omega=-k^5(\ln a)^4. \tag{50}$$

In this case, we deduce the following exact solutions of Eq. (45)

$$u_3(x,t)=-k^2(\ln a)^2+12\left(\frac{k\ln a}{cLs\left(\frac{\xi}{2}\right)}\right)^2, \tag{51}$$

$$u_4(x,t)=-k^2(\ln a)^2-12\left(\frac{k\ln a}{sLs\left(\frac{\xi}{2}\right)}\right)^2. \tag{52}$$

Physical Explanations of the Obtained Solutions

In this section we have presented some graphs of these solutions by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations. The solution obtained in this paper are bell-shaped soliton solutions and singular bell-shaped soliton solution. Using mathematical software Maple or Mathematica, the plots of some obtained solutions of equations (19) and (28) have been shown in Figs.1-2.

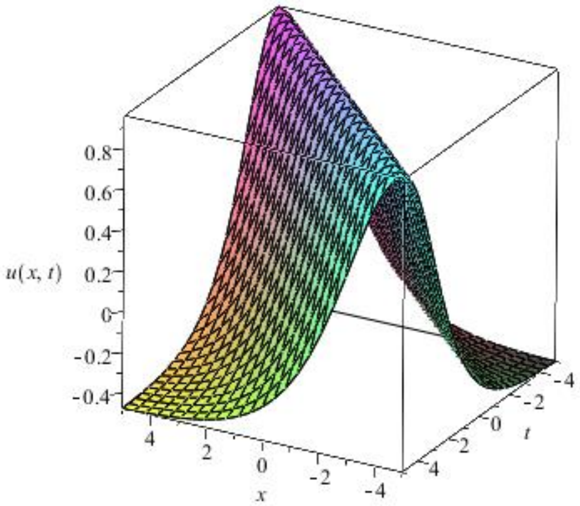

Fig.1. The plot of solution (19) when $k=1, \omega=1, a=2$.

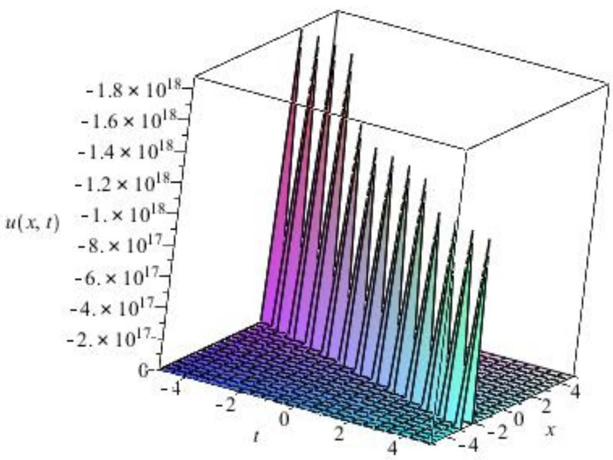

Fig.2. The plot of solution (28) when $k=2, \omega=1, a=3$.

Conclusion

In summary , we have presented the modified Kudryashov method and used it to construct more general exact solutions of nonlinear PDE's with the aid of Maple 14. This method provides a powerful mathematical tool for obtaining more general exact solutions of many nonlinear PDE's in mathematical physics. Applying this method to the indicated equations , we have successfully obtained many new exact travelling wave solutions. To our knowledge, these solutions have not been reported in the former literature. Furthermore ,this method is valid for a large number of nonlinear equations with variable coefficients. Finally, all solutions obtained in this article have been checked with the Maple 14 by putting them back into the original equation.

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