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On Extremal Topology

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Abstract

Extremal topology was defined on an arbitrary set X as a maximal non-discrete topology [2]. In this paper we will prove that every extremal topology τ on a set X has to be of the form $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F: F \in \mathcal{F}\}$, for some $x \in X$ and some ultrafilter $\mathcal{F}$ in $X\backslash\{x\}$. Where $P(X\backslash\{x\})$ is the power set of $X\backslash\{x\}$. We also show that if $\mathcal{F}$ is a free ultrafilter then (X, τ) is a T_4 space.

Keywords: Extremal topology; Door Space; $T_{\frac{1}{2}}$ Space.

المستخلص

عُرّفت التوبولوجيا المتطرفة عل أنها أقوى توبولوجيا غير متقطعة على مجموعة X [2]، وسوف تثبت هذه الورقة أن أي توبولوجيا متطرفة على مجموعة X على النحو $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}$ لبعض $x \in X$ ولبعض المرشحات الفوقية $\mathcal{F}$ على $X\backslash\{x\}$, حيث $P(X\backslash\{x\})$ هي مجموعة القوة للمجموعة $X\backslash\{x\}$. أيضا سوف نثبت إذا كان $\mathcal{F}$ مرشح فوقي حر فإن (X, τ) يكون T_4.

Preliminaries

If X is a non-empty set, a non-empty collection $\mathcal{F}$ of subsets of X is called a filter in X if (i) $\emptyset \notin \mathcal{F}$, (ii) if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$, (iii) if $F \in \mathcal{F}$ and $G \subset X$ with $F \subset G$ then $G \in \mathcal{F}$.

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A filter $\mathcal{F}$ on X is said to be free filter provided $\bigcap_{F\in\mathcal{F}} F = \emptyset$ otherwise it is called a fixed filter.A filter $\mathcal{F}$ is called an ultrafilter if it is a maximal filter; that is if $\mathcal{G}$ is a filter containing $\mathcal{F}$, then $\mathcal{F} = \mathcal{G}$. A filter $\mathcal{F}$ is an ultrafilter in X if and only if for any $E \subseteq X$ either $E \in \mathcal{F}$ or $X\setminus E \in \mathcal{F}$ and an ultrafilter $\mathcal{F}$ is fixed ultrafilter if and only if there exists, $y \in X$ such that $\bigcap_{F\in\mathcal{F}} F = \{y\}$. A collection $\mathfrak{D}$ of subsets of X is a filter base for a filter in X if and only if: (i) $\emptyset \notin \mathfrak{D}$, (ii) For any $D_1, D_2 \in \mathfrak{D}$ there exists $D_3 \in \mathfrak{D}$ with $D_3 \subseteq D_1 \cap D_2$. Every filter is contained in an ultrafilter see [1] and [4]. A topological space X is said to be a door space if any subset of X is either open or closed [1]. A topological space X is called a $T_{\frac{1}{2}}$ space if every one-point set is either open or closed. Clearly if X is a door space then X is a $T_{\frac{1}{2}}$ space.

In [2] extremal topology was defined, and it was proved that for any $x, y \in X,\ x \neq y, \tau_{\{x,y\}} = P(X\setminus\{x\}) \cup \{\{x\} \cup A : A \subset P(X\setminus\{x\}), y \in A\}$ is an extremal topology and if X is finite then every extremal topology on X has to be of the form $\tau_{\{x,y\}}$ for some $x, y \in X,\ x \neq y$. In this paper we will generalize Theorem 1-2 and Theorem 2-1 of [2] and derive some other properties of extremal spaces.

The Main Results

We first give the following Theorem

Theorem 1:
If X is a non-empty set, $x \in X$ and $\tau = P(X\setminus\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}$ for some filter $\mathcal{F}$ in $X\setminus\{x\}$ then:

a) τ is a topology on X.
b) $(X,\ \tau)$ is a normal space.
c) $(X,\ \tau)$ is a door space and hence a $T_{\frac{1}{2}}$ space.
d) If $\mathcal{F}$ is a free filter in $X\setminus\{x\}$, then $(X,\ \tau)$ is a Hausdorff space and hence a T_4 space.

Proof:

a) Is trivial.

b) Let A, B be any two disjoint closed subsets. Then we have two cases:
case(i): If $x \notin A \cup B$ then A, B are two disjoint open sets.
case(ii): If $x \in A \cup B$, say $x \in A$, $x \notin B$. Then B is a clopen(closed and open) subset and so B^c and B are two disjoint open sets containing A and B respectively. Hence X is a normal space.

c) Let A be any subset of X. If $x \notin A$, then A is an open set. If $x \in A$, then for any $y \notin A$, $\{y\}$ is an open set containing y and disjoint from A. Hence A is a closed set. So (X, τ) is a door space.

d) If $\mathcal{F}$ is a free filter in $X\backslash\{x\}$, $y, z \in X, y \neq z$. Then if $y, z \in X\backslash\{x\}$, the sets $\{y\}, \{z\}$ are two disjoint open sets containing y and z respectively. If $z = x, y \neq x$, then since $\mathcal{F}$ is a free filter in $X\backslash\{x\}$, so there exists $\mathrm{F} \in \mathcal{F}$ with $y \notin \mathrm{F}$ and hence $\{x\} \cup \mathrm{F}, \{y\}$ are two disjoint open sets containing x and y respectively. Therefore X is a Hausdorff space and hence by (b) above X is a T_4 space.

The following Theorem generalizes both 1-2 and 2-1 of [2]

Theorem 2 [3]:
A topology τ on X is an extremal if and only if there exists $x \in X$ such that $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}$ for some ultrafilter $\mathcal{F}$ in $X\backslash\{x\}$.

Proof:
$\Rightarrow$ Suppose τ is an extremal topology, then there exists $x \in X$ with $\{x\} \notin \tau$.

Let $\mathfrak{D} = \{D \subset X\backslash\{x\} : \{x\} \cup D \in \tau\}$, then $\mathfrak{D}$ is a filter base for a filter $\mathcal{g}$ in $X\backslash\{x\}$, and if $\bigcap_{D\in\mathfrak{D}} D = \emptyset$ then $\mathcal{g}$ is a free filter while if $\bigcap_{D\in\mathfrak{D}} D \neq \emptyset$ then $\mathcal{g}$ is a fixed filter. Let $\mathcal{F}$ be an ultrafilter in $X\backslash\{x\}$ containing $\mathcal{g}$ and τ^* be the topology generated by the collection $\tau \cup \{\{x\} \cup F : F \in \mathcal{F}\}$. Since $\{x\} \notin \tau^*$, then τ^* is not discrete and since $\tau \subset \tau^*$ and τ is an extremal so $\tau = \tau^*$ and hence $\{\{x\} \cup F : F \in \mathcal{F}\} \subset \tau$. Also since τ is extremal $A \in \tau$ for any $A \subset X\backslash\{x\}$, otherwise

$$\tau < A > = \{U \cup (V \cap A) : U, V \in \tau\}$$

is not a discrete topology containing τ and with $A \in \tau < A >$. Which is a contradiction. Hence

$$P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\} \subset \tau.$$

If $U \in \tau$, then if $x \notin U$ we then have $U \subset X\backslash\{x\}$ and so

$$U \in P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}.$$

If $x \in U$, then since $\mathcal{F}$ is an ultrafilter so either $U\backslash\{x\} \in \mathcal{F}$ or $X\backslash U \in \mathcal{F}$. If $U\backslash\{x\} \in \mathcal{F}$ then $U \in P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}$.

If $X\backslash U \in \mathcal{F}$ then $\{x\} \cup (X\backslash U) \in \tau$ and so $\{x\} = U \cap [\{x\} \cup (X\backslash U)] \in \tau$, which is a contradiction. So we have

$$\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F : F \in \mathcal{F}\}.$$

$\Longleftarrow$ Suppose $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F: F \in \mathcal{F}\}$ for some $x \in X$ and some ultrafilter $\mathcal{F}$ in $X\backslash\{x\}$. To show that τ is an extremal. Let $\acute{\tau}$ be a non-discrete topology with $\tau \subset \acute{\tau}$ and $\tau \neq \acute{\tau}$. Then there exists $w \in \acute{\tau}$ with $w \notin \tau$.
If $x \notin w$ then $w \in P(X\backslash\{x\}) \subset \tau$ and we have a contradiction.
If $x \in w$, then since $\mathcal{F}$ is an ultrafilter in $X\backslash\{x\}$, so either $w\backslash\{x\} \in \mathcal{F}$ or $X\backslash w \in \mathcal{F}$. If $w\backslash\{x\} \in \mathcal{F}$, then $w \in \tau$ and we have a contradiction. If $X\backslash w \in \mathcal{F}$ then $\{x\} \cup (X\backslash w) \in \tau \subset \acute{\tau}$, and so $w \cap [\{x\} \cup (X\backslash w)] = \{x\} \in \acute{\tau}$, a contradiction since $\acute{\tau}$ is not discrete so $\tau = \acute{\tau}$ and hence τ is an extremal topology on X.

Corollory 3:
If $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F: F \in \mathcal{F}\}$ is an extremal topology and $\mathcal{F}$ is a fixed ultrafilter in $X\backslash\{x\}$, then $\tau = \tau_{\{x,y\}}$ for some $y \neq x$.

Proof:
If $\mathcal{F}$ is a fixed ultrafilter, then there is $y \in X\backslash\{x\}$ with $\bigcap_{F\in\mathcal{F}} F = \{y\}$. So $\tau = P(X\backslash\{x\}) \cup \big\{\{x\} \cup F: F \in \mathcal{F}\big\} = P(X\backslash\{x\}) \cup \big\{\{x\} \cup F: y \in F, F \in \mathcal{F}\big\} = \tau_{\{x,y\}}$.

The following two corollaries are consequences of Theorem 2.

Corollary 4:
If τ is an extremal topology on a set X then:

a) (X, τ) is a normal space.
b) (X, τ) is a door space and hence a $T_{\frac{1}{2}}$ space.

Corollary 5:
If $\tau = P(X\backslash\{x\}) \cup \{\{x\} \cup F: F \in \mathcal{F}\}$ is an extremal topology on X and $\mathcal{F}$ is a free ultrafilter in $X\backslash\{x\}$, then (X, τ) is a T_2 space and hence a T_4 space.

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