The Libyan Journal of Science (An International Journal): Volume 20, 2017

On the Metrizabiliy of Extremal Spaces

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Abstract

A topological space (X, τ) is called an extremal space if there is no topology on X strictly finer than τ which is not discrete. It was proved that (X, τ) is an extremal space if and only if

$$\tau = P(X\setminus\{x\}) \cup \{\{x\} \cup F : F \in \mathfrak{F}\},$$

for some $x \in$ X and some ultrafilter $\mathfrak{F}$ in X$\setminus\{x\}$ [3, 4]. Where P(X$\setminus\{x\}$) is the power set of X$\setminus\{x\}$. In this paper I will prove that any extremal space is not metrizable.

Keywords: Extremal space; 1^{st} Countable space; Metrizable space.

المستخلص

يسمى الفضاء(X, τ) فضاء متطرف إذا لم توجد توبولوجيا على X أقوى من τ. لقد أثبت أن (X, τ) فضاء متطرف إذا و إذا كان فقط

$$\tau = P(X\setminus\{x\}) \cup \{\{x\} \cup F : F \in \mathfrak{F}\},$$

لبعض $x \in$ X ولبعض المرشحات الفوقية على X\{x} [4, 3]. حيث P(X\{x}) مجموعة القوة للمجموعة X\{x}. في هذه الورقة سوف أثبت أن أي فضاء متطرف يكون غير قابل للمترية.

Preliminaries

Throughout the paper I am assuming X is an infinite set. A space (X, τ) is metrizable if there exists a metric d defined on X such that the topology τ_d induced by d is equivalent to τ [1, 5]. A space X is 1^{st} countable if any point in X has a

Accepted for publication: 13/8/2017

countable local open base and if X is T_1, then X is 1^{st} countable if and only if for any $x \in X$ there exists a countable collection $\{U_n\}_{n=1}^{\infty}$ of open sets containing x such that $U_{n+1} \subsetneq U_n$ for all n and $\bigcap_{n=1}^{\infty} U_n = \{x\}$ [1, 5]. A filter $\mathfrak{F}$ in a set X is a collection $\mathfrak{F}$ of non-empty subsets of X such that if $F_1, F_2 \in \mathfrak{F}$ then $F_1 \cap F_2 \in \mathfrak{F}$ and if $F \in \mathfrak{F}$ and $G \subseteq X$ with $F \subseteq G$, then $G \in \mathfrak{F}$. A filter $\mathfrak{F}$ is called an ultrafilter if there is no filter strictly finer than $\mathfrak{F}$ if and only if for any $E \subseteq X$ either $E \in \mathfrak{F}$ or $X \backslash E \in \mathfrak{F}$. A collection $\mathcal{C}$ of non-empty subsets of X is a filter base for a filter in X if for any $C_1, C_2 \in \mathcal{C}$ there exist $C_3 \in C$ such that $C_3 \subseteq C_1 \cap C_2$. A filter $\mathfrak{F}$ is said to be free filter if $\bigcap_{F \in \mathfrak{F}} F = \phi$, otherwise $\mathfrak{F}$ is called a fixed filter [4, 5].

Lemma 1:

If X is a set and $\{C_n\}_{n=1}^{\infty}$ is a collection of non-empty subsets of X such that $C_{n+1} \subsetneq C_n$ for all n, and $\bigcap_{n=1}^{\infty} C_n = \phi$, then $\{C_n\}_{n=1}^{\infty}$ is a filter base for a free filter on X which is not an ultrafilter.

Proof:

Let $\mathfrak{F} = \{ F \subseteq X: C_n \subseteq F$ for some $n \in$ IN$\}$, then clearly $\mathfrak{F}$ is a free filter with filter base the collection $\{C_n\}_{n=1}^{\infty}$. To show $\mathfrak{F}$ is not an ultrafilter, for each n choose $x_n \in C_n \backslash C_{n+1}$ and let $E = \{x_1, x_2, \ldots\}$, then $E \subseteq X$ with $E \notin \mathfrak{F}$ and $X \backslash E \notin \mathfrak{F}$. So $\mathfrak{F}$ is not an ultrafilter.

A collection $\mathfrak{B}$ of subsets of a topological space X is said to be locally finite collection if every point in X has an open neighborhood which intersects only finitely many members of $\mathfrak{B}$. A collection $\mathfrak{B}$ of subsets of a topological space X is said to be σ-locally finite if $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$, where $\mathfrak{B}_n$ is locally finite collection for all n.

The following Theorem was independently proved by Nagata and Sinirnov gives a necessarily and sufficient condition for a space to be metrizable [5].

Theorem 2:

A topological space is metrizable if and only if it is T_3 and has σ-locally finite base.

The following Theorem is Corollary 5 of [4].

Theorem 3:

If $\tau = P(X \backslash \{x\}) \cup \{\{x\} \cup F: F \in \mathfrak{F}\}$, for some $x \in X$ and some free filter in $X \backslash \{x\}$, then (X, τ) is T_4, where $P(X \backslash \{x\})$ is the power set of $X \backslash \{x\}$.

The Main Results

The following Theorem says that some extremal spaces does not has the first countability property.

Theorem 4:

If (X, τ) is an extremal T_1 space, then (X, τ) is not 1^{st} countable.

Proof:

Suppose (X, τ) is an extremal space, then by [4] there exists x∈X such that τ = P(X\\$\{x\}$) ∪ $\{\{x\}\cup F: F\in \mathfrak{F}\}$ for some ultrafilter in X\\$\{x\}$.

If (X, τ) is T_1, then $\mathfrak{F}$ must be free ultrafilter and hence by Theorem3, (X, τ) is a T_4 space. If (X, τ) is 1^{st} countable space, then there exists a countable collection $\{F_n\}_{n=1}^{\infty}$ of subsets of X\\$\{x\}$, such that $F_n \in \mathfrak{F}$, $F_{n+1} \subsetneq F_n$ for all n and $\{\{x\}\cup F_n\}_{n=1}^{\infty}$ is countable local base at x with $\bigcap_{n=1}^{\infty} F_n = \phi$.

So $\mathfrak{F}$ has a countable filter base $\{F_n\}_{n=1}^{\infty}$ with empty intersection and so by Lemma1, $\mathfrak{F}$ is not an ultrafilter which is a contradiction.

Now, since every metrizable space is 1^{st} countable and T_1 so we have the following corollary.

Corollary 5:

If (X, τ) is an extremal space, then (X, τ) is not metrizable.

Now, if $x_0 \in$ X, $\mathfrak{F}$ is a free filter in X\\$\{x_0\}$, and τ = P(X\\$\{x_0\}$) ∪ $\{\{x_0\}\cup F$: $F\in \mathfrak{F}\}$, the following theorem gives a necessarily and sufficient condition for the space (X, τ) to be metrizable.

Theorem 6:

If $x_0 \in$ X, τ = P(X\\$\{x_0\}$) ∪ $\{\{x_0\}\cup F: F\in \mathfrak{F}\}$, where $\mathfrak{F}$ is a free filter in X\\$\{x_0\}$, then (X, τ) is metrizable iff $\mathfrak{F}$ has a countable filter base.

Proof:

$\Longrightarrow$ Suppose (X, τ) is metrizable, then (X, τ) is 1^{st} countable space and since (X, τ) is T_1 there exists a countable collection $\{\mathrm{U}_n\}_{n=1}^{\infty}$of open sets containing x_0 such that $\mathrm{U}_{n+1} \subsetneq \mathrm{U}_n$ for all n and $\bigcap_{n=1}^{\infty} \mathrm{U}_n = \{x_0\}$. For each n, let $C_n = \mathrm{U}_n \backslash \{x_0\}$, then $\{C_n\}_{n=1}^{\infty}$ is countable collection of subsets of X\\$\{x_0\}$, with $C_{n+1} \subsetneq C_n$ for all n, and $\bigcap_{n=1}^{\infty} C_n = \phi$. Since any open set containing x_0 has the form $\{x_0\}\cup F$ for some $F \in \mathfrak{F}$ and so there exist k such that $x_0 \in U_k = \{x_0\} \cup C_k \subset \{x_0\} \cup F$, it follows that the collection $\{C_n\}_{n=1}^{\infty}$ is a countable filter base for $\mathfrak{F}$.

$\Leftarrow$ Let $\mathfrak{F}$ be a free filter in $X \setminus \{x_0\}$, with a countable filter base $\{C_n\}_{n=1}^{\infty}$, then $\bigcap_{n=1}^{\infty} C_n = \phi$ and we can assume $C_{n+1} \subsetneq C_n$ for all n. Let

$$\mathfrak{B}_1 = \{\{x\}, x \neq x_0, x \notin C_1\} \cup \{\{x_0\} \cup C_1\}$$

$$\mathfrak{B}_2 = \{\{x\}, x \neq x_0, x \notin C_2\} \cup \{\{x_0\} \cup C_2\}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\mathfrak{B}_n = \{\{x\}, x \neq x_0, x \notin C_n\} \cup \{\{x_0\} \cup C_n\}$$

$$\vdots \qquad\qquad\qquad \vdots$$

We claim that $\mathfrak{B}_n$ is a locally finite collection of open sets for all n. Clearly $\mathfrak{B}_n$ is a collection of open sets, and for any $y \in X$. If $y \notin C_n$, $y \neq x_0$, then $\{y\}$ is an open neighbourhood of y intersects only one member of $\mathfrak{B}_n$. If $y \in C_n$ or $y = x_0$, then $\{x_0\} \cup C_n$ is an open neighbourhood of y intersects only one member of $\mathfrak{B}_n$.

Therefore $\mathfrak{B}_n$ is a locally finite collection for all n. Let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ then $\mathfrak{B}$ is a σ-locally finite collection. To show $\mathfrak{B}$ is a base for the topology $\tau = P(X \setminus \{x_0\}) \cup \{\{x_0\} \cup F: F \in \mathfrak{F}\}$, clearly $\{x\} \in \mathfrak{B}$ for all $x \in X \setminus \{x_0\}$.

If $z \in \{x_0\} \cup F$ for some $F \in \mathfrak{F}$, then if $z \neq x_0$ we have $z \in \{z\} \subset \{x_0\} \cup F$ and if $z = x_0$, then there exist $C_n \in \mathfrak{F}$ with $\{x_0\} \cup C_n \subset \{x_0\} \cup F$. So $\mathfrak{B}$ is a base for the topology τ and since $\mathfrak{B}$ is σ-locally finite collection and (X, τ) is T_4 so by Theorem2, (X, τ) is a metrizable space.

Example:

Let $X = IR^n$, O is the origin of IR^n. Let $C_n = \{x \in IR^n: 0 < \| x \| < \frac{1}{n}\}$ for all $n \in IN$. Then $\{C_n\}_{n=1}^{\infty}$ is a collection of non-empty subsets of $IR^n \setminus \{O\}$, with $C_{n+1} \subsetneq C_n$ for all n, and $\bigcap_{n=1}^{\infty} C_n = \phi$. Let

$$\mathfrak{B}_1 = \{\{x\}, x \neq O, x \notin C_1\} \cup \{\{O\} \cup C_1\}$$

$$\mathfrak{B}_2 = \{\{x\}, x \neq O, x \notin C_2\} \cup \{\{O\} \cup C_2\}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\mathfrak{B}_n = \{\{x\}, x \neq O, x \notin C_n\} \cup \{\{O\} \cup C_n\}$$

$$\vdots \qquad\qquad\qquad \vdots$$

And let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$, then $\mathfrak{B}$ is a σ-locally finite base for the topology $\tau = P(IR^n \setminus \{O\}) \cup \{\{O\} \cup F: F \in \mathfrak{F}\}$, where $\mathfrak{F}$ is the free filter in $IR^n \setminus \{O\}$ with filter base the collection $\{C_n\}_{n=1}^{\infty}$. Hence by the last Theorem (IR, τ) is a metrizable space.

References

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