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Metrizable Spaces with Exactly One Non-Isolated Point

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Abstract

It has been proved that if X is an infinite set, $x_0 \in X$ and $\tau = \mathrm{P}(\mathrm{X} \backslash \{x_0\}) \cup \{\{x_0\} \cup \mathrm{F} : \mathrm{F} \in \mathcal{F}\}$ then (X, τ) is a metrizable space, where $\mathcal{F}$ is a free filter in $\mathrm{X} \backslash \{x_0\}$ with countable filter base and $\mathrm{P}(\mathrm{X} \backslash \{x_0\})$ is the power set of $\mathrm{X} \backslash \{x_0\}$ [3]. In this paper I will define a metric on X which induces the topology τ and show that every metrizable space with exactly one non-isolated point has to be of this form.

Keywords: Metrizable space; Isolated point; Free filter.

المستخلص

مثبت أنه إذا كانت $\mathbf{X}$ مجموعة غير منتهية, $\mathbf{x}_\circ \in \mathbf{X}$ و $\boldsymbol{\tau} = \mathbf{P}(\mathbf{X} \backslash \{\mathbf{x_0}\}) \cup \{\{\mathbf{x_0}\} \cup \mathbf{F} : \mathbf{F} \in \mathcal{F}\}$ حيث $\mathcal{F}$ مرشح حر في $\mathbf{X} \backslash \{\mathbf{x_0}\}$ له أساس عدِّي و $\mathbf{P}(\mathbf{X} \backslash \{\mathbf{x_0}\})$ مجموعة القوة للمجموعة $\mathbf{X} \backslash \{\mathbf{x_0}\}$ فإن $(\mathbf{X}, \boldsymbol{\tau})$ فضاء قابل للمترية [3]. في هذه الورقة سوف أعرف دالة مترية أو قياس يعطي التوبولوجيا $\boldsymbol{\tau}$ أعلاه. و سوف اثبت أن أي فضاء قابل للمترية به فقط نقطة واحدة غير معزولة يكون على هذا النحو.

Preliminaries

Throughout the paper I am assuming X is an infinite set. A space (X, τ) is said to be metrizable if there is a metric d defined on X induces the topology τ [1,5]. If (X, τ) is a metrizable space with a metric d, then for any $\varepsilon > 0$, $x \in X$, $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ is called an open disc with center x and radius ε. A point x of a topological space X is called an isolated point if $\{x\}$ is open in X.

A filter in a set X is a collection $\mathcal{F}$ of non-empty subsets of X such that if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$ and if $\mathrm{F} \in \mathcal{F}$, $\mathrm{G} \subseteq \mathrm{X}$ with $\mathrm{F} \subseteq \mathrm{G}$, then $\mathrm{G} \in \mathcal{F}$. A subcollection ℓ of a filter $\mathcal{F}$ is a filter base for $\mathcal{F}$ if for any $\mathrm{F} \in \mathcal{F}$ there exists $C \in \ell$ with $\mathrm{C} \subset \mathrm{F}$.

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A filter $\mathcal{F}$ is said to be free filter if $\bigcap_{F\in\mathcal{F}} F = \emptyset$. A collection ß$_n$ of subsets of a topological space X is said to be locally finite collection if every point in X has an open neighborhood which intersects only finitely many members of ß$_n$. A collection ß$_n$ of subsets of a topological space X is said to be σ-locally finite if ß$=\bigcup_{n=1}^{\infty}$ ß$_n$, where ß$_n$ is locally finite collection for all n [5].
The following theorem is theorem 23.9 of [5].

Theorem-1

A topological space X is metrizable if and only if it is T_3 and has a σ-locally finite base.
The following theorem is theorem 6 of [3].
If $x_\circ \in X$, $\tau = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F : F \in \mathcal{F}\}$, where $\mathcal{F}$ is a free filter in $X\backslash\{x_\circ\}$, $P(X\backslash\{x_\circ\})$ is the power set of $X\backslash\{x_\circ\}$, then (X,τ) is metrizable if and only if $\mathcal{F}$ has a countable filter base.
Clearly if $x_\circ$, X, τ as in the above theorem, then $x_\circ$ is the only non-isolated point in X.

The main results

The main result of this section is finding a metric which induces the topology given in theorem 1.2 above and it is given in the following theorem؛

Theorem-I

There is a metric induces the topology $\tau = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F : F \in \mathcal{F}\}$, where $x_\circ \in X$ and $\mathcal{F}$ is a free filter in $X\backslash\{x_\circ\}$ with countable filter base.

Proof

Let $\{C_n\}_{n=1}^{\infty}$ be a countable filter base for $\mathcal{F}$ and suppose $C_n \supsetneqq C_{n+1}$ for all n.
Since $\mathcal{F}$ is free, then $\bigcap_{n=1}^{\infty} C_n = \emptyset$. Let $d: X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 1, & if\ x,y \in X\backslash\{x_\circ\}, x \neq y\ and\ (x \notin C_1\ or\ y \notin C_1) \\ 1, & if\ (x = x_0, y \notin C_1)\ or\ (y = x_0, y \notin C_1) \\ \dfrac{1}{n+1}, & if\ (x = x_0, y \in C_1\ and\ n\ in\ the\ least\ integer\ such\ that\ y \notin C_n) \\ & or\ (y = x_0, x \in C_1\ and\ n\ in\ the\ least\ integer\ such\ that\ x \notin C_n) \\ \max\left\{\dfrac{1}{n+1}, \dfrac{1}{m+1}\right\}, & if\ x \neq y, x,y \in C_1\ and\ n,m\ are \\ & respectively\ the\ least\ integers\ such\ that\ x \notin C_n, y \notin C_m \\ 0, & if\ x = y, x,y \in X \end{cases}$$

To check that d is a metric:
clearly,
(i) $d(x,y) \geq 0$ for all $x,y \in X$,
(ii) $d(x,y) = 0$ if and only if $x = y$, and

(iii) $d(x,y) = d(y,x)$ for all $x, y \in X$ hold

(iv) to check the triangle inequality, let x, y, $z \in X$, then we have the following cases:

a) If x, y, $z \in X \backslash (C_1 \cup \{x_\circ\})$, then clearly the inequalities
$d(x,z) \le d(x,y) + d(y,z)$
$d(x,y) \le d(x,z) + d(z,y)$
$d(y,z) \le d(y,x) + d(x,z)$ hold.

b) If $x = x_\circ$, $y, z \notin C_1$ then the inequalities
$d(x_\circ, z) \le d(x_\circ, y) + d(y,z)$
$d(x_\circ, y) \le d(x_\circ, z) + d(z,y)$
$d(y,z) \le d(y,x_\circ) + d(x_\circ, z)$ hold.

c) If $\in X \backslash (C_1 \cup \{x_\circ\}$,y ,$z \in C_1$, then also the three inequalities hold.

d) If x, y, $z \in C_1$. Suppose $x \notin C_n$, $y \notin C_m$, $z \notin C_k$ where n, m, k are the least integers with these proporties. Suppose $n \le m \le k$, then all of the inequalities
$d(x,z) \le d(x,y) + d(y,z)$
$d(x,y) \le d(x,z) + d(z,y)$
$d(y,z) \le d(y,x) + d(x,z)$ hold

e) If $x = x_\circ$, $y, z \in C_1$
suppose $(x_\circ, y) = \frac{1}{n+1}$, $d(x_\circ, z) = \frac{1}{m+1}$ and $n \le m$, then the three triangle inequalities
$d(x_\circ, z) \le d(x_\circ, y) + d(y,z)$
$d(x_\circ, y) \le d(x_\circ, z) + d(z,y)$
$d(y,z) \le d(y,x_\circ) + d(x_\circ, z)$ hold.

f) If $x = x_\circ$, $y \in C_1$, $z \notin C_1$ then also the three triangle inequalities hold.

Hence d is a metric on X.

Next we will show that d induces the topology $\tau = P(X \backslash \{x_\circ\}) \cup \{\{x_\circ\} \cup F : F \in \mathcal{F}\}$.

If $x \notin C_1, x \ne x_\circ$ then $d(x,y) = d(y,x) = 1$ for any $y \in X, y \ne x$. So $B_1(x) = \{x\}$. If $x \in C_1, x \notin C_2$, then $d(x,x_\circ) = \frac{1}{3} = d(x_\circ, x)$, $d(x,y) = d(y,x) = 1$ if $y \notin C_1$ and $d(x,y) = \frac{1}{3} = d(y,x)$ if $y \in C_1$. So $B_{\frac{1}{3}}(x) = \{x\}$.

If $x \in C_2, x \notin C_3$, then $d(x,x_\circ) = \frac{1}{4} = d(x_\circ, x)$, $d(x,y) = d(y,x) = \frac{1}{4}$ if $y \notin C_1$. and $d(x,y) = d(y,x) = \frac{1}{4}$ if $y \in C_1$. So $B_{\frac{1}{4}}(x) = \{x\}$.

In general if $x \in C_n$, $x \notin C_{n+1}$ then $B_{\frac{1}{n+2}}(x) = \{x\}$.

If $x = x_\circ$, then $d(y,x_\circ) = d(x_\circ, y) = 1$ if $y \notin C_1$, $d(x_\circ, y) = d(y, x_\circ) = \frac{1}{n+1}$ if $y \in C_1$ and n is the least integer such that $x \notin C_n$.

So $B_{\frac{1}{n}}(x_\circ) = \left\{ y \in X : d(x_\circ, y) < \frac{1}{n} \right\} = C_n \cup \{x_\circ\}$; that is $B_1(x_\circ) = C_1 \cup \{x_\circ\}$, $B_{\frac{1}{2}}(x_\circ) = C_2 \cup \{x_\circ\}$ and so on.

Therefore as a base for the metric topology we have the collection

$$\square = \{B_1(x): x \in X\backslash C_1, x \neq x_\circ\} \cup \{\bigcup_{n=1}^{\infty} \{B_{\frac{1}{n+2}}: x \in C_n \backslash C_{n+1}\}\} \cup \{B_{\frac{1}{n}}(x_\circ)\}_{n=1}^{\infty}$$
$$= \{\{x\}: x \in X\backslash\{x_\circ\}\} \cup \{\{x_\circ\} \cup C_n\}_{n=1}^{\infty}$$

and this base induces the topology $\tau = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F: F \in \mathcal{F}\}$, where $\mathcal{F}$ is a free filter in $X\backslash\{x_\circ\}$ with $\{C_n\}_{n=1}^{\infty}$ as a filter base.

The next theorem shows that if X is a metrizable space with only one isolated point, then the metric topology will be as in the above theorem.

Theorem-II

A space (X, τ) is metrizable with only one non-isolated point $x_\circ$ if and only if $\tau = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F: F \in \mathcal{F}\}$, where $\mathcal{F}$ is a free filter in $X\backslash\{x_\circ\}$ with countable filter base.

proof $\Longrightarrow$:

Let X be a metrizable space with $x_\circ$ as the only non-isolated point.

Let $C_n = B_{\frac{1}{n}}(x_\circ)\backslash\{x_\circ\}$ for all n.

then $C_n \neq \emptyset$, $C_n \subseteq X\backslash\{x_\circ\}$, $C_{n+1} \subsetneq C_n$ for all n and $\bigcap_{n=1}^{\infty} C_n = \emptyset$.

Let $\mathcal{F}$ be the free filter with filter base $\{C_n\}_{n=1}^{\infty}$.

By theorem 1.2 if $\tau^* = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F: F \in \mathcal{F}\}$, then (X, τ^*) is a metrizable space.

To show $\tau^* = \tau$, let $U = \{x_\circ\} \cup F$ be an open neighborhood of $x_\circ$ in (X, τ^*), then there exists m such that $\{x_\circ\} \cup C_m \subset \{x_\circ\} \cup F$, so $\{x_\circ\} \cup C_m = B_{\frac{1}{m}}(x_\circ) \subseteq \{x_\circ\} \cup F$. Therefore $\{x_\circ\} \cup F \in \tau$ and so $\tau^* \subseteq \tau$. Also for any n, $B_{\frac{1}{n}}(x_\circ) = \{x_\circ\} \cup C_n \in \tau^*$ consequently $\tau \subseteq \tau^*$ and hence $\tau^* = \tau$.

$\Longleftarrow$: If $\tau = P(X\backslash\{x_\circ\}) \cup \{\{x_\circ\} \cup F: F \in \mathcal{F}\}$, where $\mathcal{F}$ is a free filter in $X\backslash\{x_\circ\}$ with countable filter base, then by the last theorem (X, τ) is metrizable with $x_\circ$ as the only non-isolated point. Type equation here.

2.3 Remark

The next research is to find a metric for any metrizable space with finitely many non-isolated points.

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