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p-Harmonic Functions On Metric Spaces

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ABSTRACT

The main object of this paper is to present the p-harmonic functions in metric spaces and compare them with the harmonic functions in the Euclidean spaces. We obtain many useful properties for the p-harmonic functions in metric spaces. In particular the p-harmonic functions in metric spaces satisfy the strong maximum principle, the Harnack's inequality and are locally Hölder continuous.

Keywords: Harmonic functions; newtonian functions; hölder continuity; harnack's inequality; doubling measure; metric space; nonlinear; sobelev spaces; poincaré inequality.

1. Introduction

The nonlinear Dirichlet problem (in $\boldsymbol{R}^n$) is the p-energy minimization problem min

$$\int |\nabla u|^p \, dx \qquad (1)$$

with given boundary values, and the continuous minimizer is called p-harmonic function. The minimizers are solutions of the corresponding Euler-Lagrange equation, which is the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \qquad (2)$$

For $p = 2$ this is the well known Laplace equation $\Delta u = 0$. It has been shown that the harmonic functions in $\boldsymbol{R}^n$ satisfy the Harnack's inequality, the maximum principle and the local Hölder continuity, see e.g. [1] and [2].

In a metric space we have no partial derivatives but we have a substitute of the modulus of the gradient called upper gradient. Therefore we cannot study the p-Laplace equation (2), but by looking at the minimizing equation (1) we only use the scalar of the gradient and one can, even, use the weak gradient to find weak solutions of the problem. Therefore, we instead

deal with the p-minimization integral equation (1) with $|\nabla u|$ replaced by the minimal p-weak upper gradient of u. For more information about upper gradients and Sobolev spaces in metric spaces see, e.g. [3], [4] and [5,6].

The p-harmonic function on metric spaces is defined to be the continuous minimizer of the p-Dirichlet integral

$$\int g_u^p \, d\mu \qquad (3)$$

where, g_u is the minimal p-weak upper gradient of u. In the Euclidean case $g_u = |\nabla u|$, see Section 2, Lemma 2.2. The potential theory of minimizers is not linear for $p > 1$; this happens because the operation of taking an upper gradient is not linear.

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space with a metric d and a positive complete Borel measure μ which is doubling, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ in X, we have

$$0 < \mu(2B) \leq C\,\mu(B) < \infty.$$

We also assume that the space X supports a p- Poincaré inequality, which means that the local mean oscillation of every function is controlled by the $\boldsymbol{L}^p$-norm of its upper gradient.

The Dirichlet problem for p-harmonic functions on metric spaces was studied e.g. in [7,8], [9,10], [11] and [6,12]. Although in $\boldsymbol{R}^n$ p-harmonic functions are Lipschitz functions they need not to be so in the general setting of metric spaces, see p.149 in [13]. The Hölder continuity for p-harmonic functions on metric spaces, the strong maximum principle and Harnack's inequality were obtained in [11].

This paper is organized as follows. In section 2 we focus on preliminary notations and definitions needed in the rest of the paper. In section 3 we present the harmonic function on the Euclidean spaces and some of their properties.

In section 4, we consider the p-harmonic functions, as minimizers of the p-Dirichlet integral (3), on the setting of a complete metric measure space. In particular we prove existence and uniqueness of p-harmonic function for a given Newtonian boundary value. The strong maximum principle and the Harnack's inequality are also shown. Moreover, it has been shown that the Hölder continuity is satisfied for the p-harmonic functions.

2. Notation and preliminaries

Definition 2.1

A nonnegative Borel function g on X is said to be an upper gradient of an extended real–valued function f on X if for all rectifiable curve $\gamma : [0, l\gamma] \to X$ parametrized by the arc length ds, we have

$$f\big(\gamma(0)\big) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds \qquad (4)$$

When ever both $f\,(\gamma(0))$ and $f\,(\gamma(l\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (4) holds for p-almost every curve then g is a *p-weak upper gradient* of f. If f has an upper gradient in $\boldsymbol{L}^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in \boldsymbol{L}^p(X)$ in the sense that for every p-weak upper gradient $g \in \boldsymbol{L}^p(X)$ of f we have $g_f \leq g$ a.e., see Corollary 3.7 in [6].

By saying that (4) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in [5].

The upper gradient in not unique. In particular, from (4) every Borel function greater than g will be another upper gradient of f. Moreover, the operation of taking an upper gradient is not linear. However, we have the following useful property. If $a, b \in \boldsymbol{R}$ and g_1, g_2 are upper gradients of u_1, u_2, respectively. Then $|a|g_1 + |b|g_2$ is an upper gradient of $au_1 + bu_2$.

The following lemma gives a nontrivial example of upper gradient, see [8], Corollary 1.15.

Lemma 2.2

If $X = \boldsymbol{R}^n$ and $f \in C^1(\boldsymbol{R}^n)$, then $|\nabla f|$ is an upper gradient of f.

In [5], Newtonian space has been defined to be the collection of all $\boldsymbol{L}^p$- functions with $\boldsymbol{L}^p$-upper gradients. We will use the following equivalent definition.

Definition 2.3

Let $u \in \boldsymbol{L}^p(X)$, then we define

$$\| u \|_{N^{1,p}(X)} = \left(\int_X |u|^p\, d\mu + \int_X g_u^p\, d\mu \right)^{1/p} \qquad (5)$$

where the g_u is the minimal p-weak upper gradient of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u \colon \| u \|_{N^{1,p}\,(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\| u - v \|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p.249 in [5].

We shall need the following lemma about minimal p-weak upper gradient.

Lemma 2.4

If $u, v \in N^{1,p}(X)$, then $g_u = g_v$ a.e. on $\{x \in X \colon u(x) = v(x)\}$. Moreover, if $c \in \boldsymbol{R}$ is a constant, then $g_u = 0$ a.e. on $\{x \in X \colon u(x) = c\}$.

Definition 2.5

The Capacity of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \| u \|_{N^{1,p}(X)}$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

We say that a property holds *quasieverywhere* (q.e.) in X, if it holds everywhere except on a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u,\ v\ \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in [5] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e. in X.

From now on we assume that X supports a p–Poincaré inequality, i.e. there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(z,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)}\int_{B(z,r)} |u - u_{\mathrm{B(z,r)}}|\ d\mu \leq C\,r\left(\frac{1}{\mu(B)}\int_{B(z,\lambda r)} g^p\ d\mu\right)^{1/p},$$

To be able to compare the boundary values of Newtonian functions we need to define a Newtonian space with zero boundary values outside of Ω as follows

$$N_0^{1,p}(\Omega) = \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ q.e. in } X\backslash\Omega\}.$$

The next lemma is useful for proving that a function belongs to the $N_0^{1,p}(\Omega)$, see Lemma 5.3 in [7].

Lemma 2.6

Let $u \in N^{1,p}(\Omega)$ be such that $v \leq u \leq w$ q.e. in Ω for some $v, w \in N_0^{1,p}(\Omega)$. Then $u \in N_0^{1,p}(\Omega)$.

The following Poincaré type inequality is from [11], and well be needed.

Lemma 2.7

Assume that $\Omega \subset \mathrm{X}$ is a nonempty bounded open set with $C_p(X\backslash\Omega) > 0$. Then there exists a constant $C > 0$ such that for all $u \in N_0^{1,p}(\Omega)$ we have

$$\int_\Omega |u|^p\ d\mu \leq C \int_\Omega g_u^p\ d\mu.$$

The coming lemma will be needed. For a proof, see [14].

Lemma 2.8

Assume that g_j is a p-weak upper gradient of u_j, $j = 1,2,\cdots$, and that both sequences $\{u_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are bounded in $\boldsymbol{L}^p(X)$. Then there are $u, g \in \boldsymbol{L}^p(X)$, convex combinations $v_j = \sum_{i=j}^{N_j} a_{j,i}u_i$ with p-weak upper gradients $\bar{g}_j = \sum_{i=j}^{N_j} a_{j,i}g_i$ and strictly increasing sequence of indices $\{j_k\}_{k=1}^\infty$, such that:

(a) both $u_{j_k} \to u$ and $g_{j_k} \to g$ weakly in $\boldsymbol{L}^p(X)$;
(b) both $v_j \to u$ and $\bar{g}_j \to g$ in $\boldsymbol{L}^p(X)$;
(c) $v_j \to v$ q.e.;
(d) g is a p-weak upper gradient of u.

3. Harmonic functions on $\boldsymbol{R^n}$

Let $\Omega \subset \boldsymbol{R^n}$ be open and $u: \bar{\Omega} \longrightarrow \boldsymbol{R}$ is an unknown function. The Laplacian is defined by

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

and $\Delta u = 0$ is called the Laplace equation. The p-Laplace equation is defined by

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \tag{6}$$

and the Dirichlet problem is the p-energy minimization problem defined by

$$\min \int |\nabla u|^p\,dx \tag{7}$$

Definition 3.1

A C^2 function satisfying (6) is called (classical) p-harmonic function.

The more general p-harmonic function is the continuous (weak) solution of (6).

Theorem 3.2

If $u \in C^2(\bar{\Omega})$, then u solves (6) with a given boundary value function g, if and only if u is the minimizer of the Dirichlet problem (7). In other words, the p-harmonic functions are exactly the minimizers of (7), with a given boundary value function.

The existence and uniqueness of p-harmonic function on $\boldsymbol{R^n}$, with prescribed boundary values, can be found, e.g. as it shown in [1] and [2].

3.2.1. Properties of p- harmonic functions in $\boldsymbol{R^n}$.

The p-harmonic functions satisfy the following useful properties, see e.g. [2].

1. The strong maximum principle: A nonconstant p-harmonic function in a domain Ω cannot attain its supremum or infimum.
2. Liouville's Theorem

Theorem 3.3 Suppose that $u: \boldsymbol{R^n} \longrightarrow \boldsymbol{R}$ is p-harmonic and bounded. Then u is constant.

3. Harnack's inequality

Theorem 3.4 Let u be a nonnegative p-harmonic function in Ω. Then there exists a positive constant C such that

$$\sup_B u \leq C \inf_B u,$$

whenever B is a ball in Ω such that $2B \subset \Omega$.

4. p −Harmonic functions in metric spaces

Definition 4.1

Suppose that $\Omega \subset X$. A function $u \in N^{1,p}(\Omega)$ is a minimizer in Ω if for every function $v \in N^{1,p}(\Omega)$ with $u - v \in N_0^{1,p}(\Omega)$, we have

$$\int_\Omega g_u^p \, d\mu \leq \int_\Omega g_v^p \, d\mu,$$

where g_u and g_v are the minimal p-weak upper gradients of u and v respectively. We also say that a function u is p-harmonic if it is a continuous minimizer.

If u is a minimizer (or p-harmonic) and $\alpha, \beta \in \boldsymbol{R}$, then $\alpha u + \beta$ is a minimizer (or p-harmonic). Note, however, that the sum of two minimizers (or p-harmonic) functions need not to be a p-harmonic function and thus the theory is not linear. We instead have the minimum of two p-harmonic functions is a p-harmonic function.

The first question that arises is whether there exists such a minimizer and if it will be unique. In [6] it was shown that there exists a unique minimizer for every $u \in N^{1,p}(\Omega)$.

Theorem 4.2

Assume that Ω is bounded and that $C_p(X \setminus \Omega) > 0$. Let $f \in N^{1,p}(\Omega)$, then there exists a unique minimizer u with $u - f \in N_0^{1,p}(\Omega)$ (up to a set of capacity zero).

Proof. Let

$$I = \inf_v \int_\Omega g_v^p \, d\mu,$$

where the infimum is taken over all v such that $v - f \in N_0^{1,p}(\Omega)$. Since $f \in N^{1,p}(\Omega)$, we have $0 \leq I < \infty$. Then there will be a sequence $\{u_j\}_{j=1}^\infty$ with $u_j - f \in N_0^{1,p}(\Omega)$, and that

$$\int_\Omega g_{u_j}^p \, d\mu \searrow I, \quad \text{as } j \to \infty.$$

It follows that $\{g_{u_j}\}_{j=1}^\infty$ is bounded in $L^p(\Omega)$, since $\int_\Omega g_{u_j}^p \, d\mu \leq \int_\Omega g_{u_1}^p \, d\mu$. Hence from the p- Poincaré inequality for $N_0^{1,p}(\Omega)$, we get that

$$\begin{aligned}\int_\Omega |u_j - f|^p \, d\mu &\leq C \int_\Omega g_{u_j - f}^p \, d\mu \\ &\leq C \int_\Omega g_{u_j}^p \, d\mu + C \int_\Omega g_f^p \, d\mu \\ &\leq C \int_\Omega g_{u_1}^p \, d\mu + C \int_\Omega g_f^p \, d\mu.\end{aligned}$$

This shows that $\{u_j\}_{j=1}^\infty$ is bounded in $L^p(\Omega)$ and hence bounded in $N^{1,p}(\Omega)$. By Lemma 2.8, we find convex combinations $v_j = \sum_{k=j}^{N_j} a_{j,k} u_k$ with p-weak upper gradients $g_j = \sum_{k=j}^{N_j} a_{j,k} g_{u_k}$ and limit functions v and g such that $v_j \to v$ and $g_j \to g$ in $L^p(\Omega)$, $v_j \to v$ q.e., as $j \to \infty$, and g is a p-weak upper gradient of v. Thus $v \in N^{1,p}(\Omega)$.

We have to show that $v - f \in N_0^{1,p}(\Omega)$ and that v is the desired minimizer. To do that let $w_j \coloneqq v_j - f \in N_0^{1,p}(\Omega)$ and we can consider w_j to be zero outside of Ω. Let also $w = v - f, g'_j = g_j + g_f$ and $g' = g + g_f$ and that all three are considered to be zero outside of Ω. Then $w_j \to w$ and $g'_j \to g'$ in $L^p(X)$ and $w_j \to w$ q.e. in X, as $j \to \infty$. By Lemma 2.8, $w \in N^{1,p}(X)$ and thus $v - f \in N_0^{1,p}(\Omega)$.

Now, we have

$$I \leq \int_\Omega g_v^p \, d\mu \leq \int_\Omega g^p d\mu = \lim_{j\to\infty} \int_\Omega g_j^p \, d\mu = I.$$

This shows that $I = \int_\Omega g_v^p \, d\mu$ and hence v is the minimizer.

For uniqueness, assume that u_1, u_2 are two minimizers with the same boundary values f, i.e. $u_1 - f \in N_0^{1,p}(\Omega)$ and $u_2 - f \in N_0^{1,p}(\Omega)$.

Then also $u' = \frac{1}{2}(u_1 + u_2)$ has the boundary values f (in the weak sense) and

$$\begin{aligned}\|g_{u_1}\|_{L^p(\Omega)} \leq \|g_{u'}\|_{L^p(\Omega)} &\leq \left\|\tfrac{1}{2}(g_{u_1} + g_{u_2})\right\|_{L^p(\Omega)} \\ &\leq \tfrac{1}{2}\|g_{u_1}\|_{L^p(\Omega)} + \tfrac{1}{2}\|g_{u_2}\|_{L^p(\Omega)} \\ &= \|g_{u_1}\|_{L^p(\Omega)}.\end{aligned}$$

Hence $g_{u_1} = g_{u_2}$ a.e. in Ω by the strict convexity of the $L^p(\Omega)$. We shall show that $g_{u_1 - u_2} = 0$ a.e. in Ω. The p- Poincaré inequality for $N_0^{1,p}(\Omega)$, Lemma 2.7, then implies that $\|u_1 - u_2\|_{L^p(\Omega)} = 0$ and hence $u_1 = u_2$ q.e. in Ω.

To show that $g_{u_1-u_2} = 0$ a.e. in Ω, let $c \in \boldsymbol{R}$ and

$$u = \max\left\{u_1, \min\{u_2, c\}\right\}.$$

Then $u \in N^{1,p}(\Omega)$ and $u - f \geq u_1 - f \in N_0^{1,p}(\Omega)$. Also,

$$u - f \leq \max\{u_1, u_2\} - f = \max\{u_1 - f, u_2 - f\} \in N_0^{1,p}(\Omega).$$

Lemma 2.6 shows that $u - f \in N_0^{1,p}(\Omega)$.

Now, let $V_c = \{x \in \Omega: u_1(x) < c < u_2\}$ and note that $V_c \subset \{x \in \Omega: u(x) = c\}$ and hence $g_u = 0$ a.e. in V_c, by Lemma 2.4. The minimizing property of g_{u_1} then implies that

$$\int_\Omega g_{u_1}^p \, d\mu \leq \int_\Omega g_u^p \, d\mu \leq \int_{\Omega\setminus V_c} g_u^p \, d\mu \leq \int_{\Omega\setminus V_c} g_{u_1}^p \, d\mu, \tag{8}$$

since by Lemma 2.4 $g_u = g_{u_1} = g_{u_2}$ for a.e. $x \in \Omega\setminus V_c$. From (8) we conclude that $g_{u_2} = g_{u_1}$ a.e. in V_c for all $c \in \boldsymbol{R}$. Now

$$\{x \in \Omega: u_1(x) < u_2(x)\} \subset \bigcup_{c\in\boldsymbol{Q}} V_c$$

and hence $g_{u_2} = g_{u_1} = 0$ a.e. in $\{x \in \Omega: u_1(x) < u_2(x)\}$, and similarly for $\{x \in \Omega: u_2(x) < u_1(x)\}$. It follows that

$$g_{u_1-u_2} \leq \left(g_{u_1} + g_{u_2}\right)\chi_{\{x\in\Omega: u_1 \neq u_2\}} = 0 \text{ a.e. in } \Omega,$$

which implies that $u_1 = u_2$ q.e. in Ω, and hence the minimizer is unique.

Lemma 4.3 (comparison principle)

Assume that Ω is bounded and that $C_p(X\setminus\Omega) > 0$. Let $u_1, u_2 \in N^{1,p}(\mathrm{X})$ be two minmizers in Ω such that $u_1 \leq u_2$ q.e. in $\partial\Omega$. Then $u_1 \leq u_2$ in Ω.

Theorem 4.5 **(**Harnack inequality)

Suppose that u is a nonnegative minimizer in Ω. Then there exists a constant $C \geq 1$,

only depending on p, C_μ and the constants in the p-Poincaré inequality, such that

$$\operatorname*{ess\,sup}_{\mathrm{B}} u \leq \mathrm{C} \operatorname*{ess\,inf}_{\mathrm{B}} u$$

for every $B \subset 50\lambda B \subset \Omega$.

Theorem 4.6 (The strong maximum principle)

If Ω is connected, u is p-harmonic in Ω and u attains its maximum in Ω then u is a constant in Ω. *Proof.*

We may assume that the maximum is 0. Let $A = \{x \in \Omega: u(x) = 0\}$, a relatively closed subset of Ω, since u is continuous. Let further $x \in A$. Then we can find a ball $B \ni x$ such that $50\lambda B \Subset \Omega$. As $-u$ is a nonnegative p-harmonic function in Ω, we have by the Harnack's inequality (Theorem 4.5) that

$$-\inf_B u = \sup_B(-u) \leq -C\, u(x) = 0,$$

i.e. $0 \leq \inf_B u \leq \sup_B u = 0$. Hence $B \subset A$, i.e. A is open. Since Ω is connected, A must be the only nonempty relatively closed open subset of Ω, viz. Ω itself. Thus $u \equiv 0$.

The following theorem provide us with a continuous minimizer i.e. it ensures the existence of p-harmonic functions and that they are locally Hölder continuous in the domain of its harmonicity.

Theorem 4.7

Let u be a minimizer in Ω. Then u can be modified on a set of capacity zero so that it becomes locally α-Hölder continuous in Ω, where $0 < \alpha < 1$. More precisely, if $B = B(x_0, r_0) \subset 2B \Subset \Omega$ is a ball, then for all $x, y \in B$,

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C\left(\sup_{2\mathrm{B}} \tilde{u} - \inf_{2\mathrm{B}} \tilde{u}\right)\frac{d(x,y)^\alpha}{r_0^\alpha},$$

where $\tilde{u} = u$ q.e. in Ω, and C and α only depend on p, C_μ and the constants in the p-Poincaré inequality.

Theorem 4.8

Let u be a p-harmonic function on Ω and $G \Subset G' \Subset \Omega$. Then for $x, y \in \Omega$,

$$|u(x) - u(y)| \leq C\left(\sup_{\mathrm{G}'} u - \inf_{\mathrm{G}'} u\right)\frac{d(x,y)^\alpha}{r_0^\alpha},$$

where C and $0 < \alpha < 1$ only depend on G, G', p, C_μ and the constants in the p-Poincaré inequality.

Corollary 4.9 **(**Liouville's theorem**)**

If u is p-harmonic and bounded from below on all of X, then u is constant.

Proof. Let $v = u - \inf_X u \geq 0$. Then by the Harnack's inequality, for $x \in X$, we have

$$v(x) \leq \sup_{B(x,r)} v \leq \mathrm{C} \inf_{B(x,r)} v \to 0, \quad \text{as} \quad r \to \infty.$$

Thus $v \equiv 0$, and u is constant.

Corollary 4.10 (Harnack's inequality)

If Ω is connected and $E \Subset \Omega$, then there is a constant C such that for all nonnegative p-harmonic function u on Ω we have

$$\sup_{\mathrm{E}} u \leq C \inf_{\mathrm{E}} u.$$

5. Conclusion

In this study we assume that Ω is open and bounded, $X = (X, d, \mu)$ is a complete metric space with a metric d and a positive complete Borel measure μ which is doubling. We also assume that the space X supports a p-Poincaré inequality. We define the p-harmonic functions in metric spaces and investigate their properties. The finding showed that the p-harmonic functions in metric spaces keep the most useful properties as those in the Euclidean spaces. In particular, it has been shown that, the p-harmonic functions in metric spaces satisfy the strong maximum principle, the Harnack's inequality and they are locally Hölder continuous.

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