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Generalized homotopy in generalized topological spaces

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ABSTRACT

In 2002, Csàszàr introduced the concept of generalized topological spaces and studied the notion of generalized continuity. A generalized topology μ on a non-empty set X is a collection of subsets of X which satisfies two conditions, namely; $\emptyset \in \mu$ and μ is closed under arbitrary unions. In this article, we extend the notion of homotopy between topological spaces to the setting of generalized topological spaces in the sense of Csàszàr. We introduce the new notion of generalized homotopy ((μ, λ)-homotopy) between generalized topological spaces and study some of its basic properties. It is proved that the generalized homotopy is not an equivalence relation on the set of (μ, λ)-continuous functions in general.

Keywords: Generalized topology; generalized continuity; homotopy; homotopy equivalence.

1. Introduction

In [3], Csàszàr first introduced the concept of generalized topological spaces and studied the notion of generalized continuity. A generalized topology (briefly, GT) μ on a non-empty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions [3]. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). The elements of μ are called μ-open sets, and $X \in \mu$ must not hold [5]. Let M_μ denote the union of all μ-open sets, then (X, μ) is called strong iff $M_\mu = X$ [4].

A function $f: (X, \mu) \rightarrow (Y, \lambda)$ on GTS's is called (μ, λ)-continuous [3] if $m \in \lambda$ implies $f^{-1}(m) \in \mu$. A base for GT μ [7], denoted by $\mathcal{B}(\mu)$, is a collection of subsets of X with $\emptyset \in \mathcal{B}(\mu)$ such that the collection of all possible unions of elements of $\mathcal{B}(\mu)$ forms μ.

Let $K \neq \emptyset$ be an index set, for $k \in K$, let $X_k \neq \emptyset$ and let (X_k, μ_k) be a family of GTS's. Let $X = \prod_{k \in K} X_k$ represents the Cartesian product of the sets X_k. The collection of all sets of the form $\prod_{k \in K} A_k$, where $A_k \in \mu_k$ and with exception of finite number of indices k, $A_k = M_{\mu_k}$, forms a base for a GT μ called the product of GT's μ_k, and the GTS (X, μ) is called the product of the GTS's (X_k, μ_k) [6].

Significant research has been conducted on GTS's and generalized analogues of almost all basic topological notions have been introduced, for instance, generalized separation axioms [9], generalized axioms of countability [2], μ-compactness [8], and generalized topological groups [10]. However, no study has been done regarding homotopy theory in GTS's (as far as the author know).

For topological spaces X and Y, two continuous functions $f, g: X \rightarrow Y$ are called homotopic [1], if for all $x \in X$ there exists a continuous function $H: X \times I \rightarrow Y$ such that, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

The main goals of this paper is to show how the definition of homotopic topological spaces can be modified in order to define homotopic GTS's. In section 2, we introduce homotopy in GTS's, and we obtain several results regarding this concept.

2. Results and discussion

2.1. Homotopy in generalized topological spaces

Throughout this paper, by I we mean the closed interval [0,1]. Suppose that (I, ν) and (X, μ) are given GTS's, consider the generalized product topology $(X, \mu) \times (I, \nu) = (X \times I, \sigma)$, where $X \times I$ is the Cartesian product and σ is the product of GT's $\mu \times \nu$. We denote by $\mathcal{B}(\sigma)$ the base of GT σ which consists of all sets of the form $U \times V$ where $U \in \mu$ and $V \in \nu$.

Definition 2.1

Let (I, ν), (X, μ), and (Y, λ) be GTS's. Two (μ, λ)-continuous functions $f, g: (X, \mu) \to (Y, \lambda)$ are said to be (μ, λ)-homotopic, (notation: $f \simeq_{GH} g$), if there exists a (σ, λ)-continuous function $H: (X \times I, \sigma) \to (Y, \lambda)$, called (μ, λ)-homotopy, joining f and g such that for every $x \in X$; $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

By generalized homotopy we mean (μ, λ)-homotopy. Of course, every homotopy is a generalized homotopy.

Example 2.1

Let $X = Y = [-1,1]$, and let $\mu, \lambda,$ and ν be the usual subspaces topology of $\mathbb{R}$. Let $f, g: (X, \mu) \to (Y, \lambda)$ where $f(x) = -x$ and $g(x) = x$. For every $x \in X$ and $t \in I$, define $H: (X \times I, \sigma) \to (Y, \lambda)$ by $H(x, t) = (2t - 1)x$. Then:

1. For every λ-open set m, $H^{-1}(m) = m \times \{1\} \in \sigma$. Thus, H is (σ, λ)-continuous.
2. For every $x \in X$ we have:
   - $H(x, 0) = -x = f(x)$,
   - $H(x, 1) = x = g(x)$.

Therefore, $H(x, t)$ is a (μ, λ)-homotopy between f and g.

Example 2. 2

Let $X = Y = \mathbb{R}$, $\mu = \lambda = \{\emptyset, \mathbb{R}\}$, and $\nu = \{\emptyset, I, (0,1)\}$. Let $f, g: (X, \mu) \to (Y, \lambda)$ where $f(x) = x$ and $g(x) = x - 1$. Define $H: (X \times I, \sigma) \to (Y, \lambda)$ such that $H(x, t) = x - t$, for every $x \in X$ and $t \in I$. Then:

1. The base $\mathcal{B}(\sigma) = \{\emptyset, \mathbb{R} \times I, \mathbb{R} \times (0,1)\}$. Since the only λ-open sets are $\emptyset, \mathbb{R}$, then $H^{-1}(\emptyset) = \emptyset \in \sigma$ and $H^{-1}(\mathbb{R}) = \mathbb{R} \times I \in \sigma$. Thus, H is (σ, λ)-continuous.
2. For every $x \in X$ we have:
   - $H(x, 0) = x = f(x)$,
   - $H(x, 1) = x - 1 = g(x)$.

Therefore, $H(x, t)$ is a (μ, λ)-homotopy between f and g.

Remark 2.1

In the classic homotopy theory, every continuous function f is homotopic to itself by choosing $H = f$. Self-homotopy is necessary in order for homotopy to be an equivalence relation on the set of functions between any two topological spaces. However, this does not apply to generalized homotopy in general. Consider the following example:

Example 2.3

Let $X = \{0,1\}$, $\mu = \{\emptyset, X\}$, and $\nu = \{\emptyset, (0,1)\}$. Let $f: (X, \mu) \to (X, \mu)$ where $f(x) = x$. Define $H: (X \times I, \sigma) \to (X, \mu)$ by $H(x, t) = x = f(x)$, for every $x \in X$ and $t \in I$. Since $\mathcal{B}(\sigma) = \{\emptyset, X \times (0,1)\}$, then $X \in \mu$ and $H^{-1}(X) = X \times I \notin \sigma$. Therefore, $H(x, t)$ is not (σ, μ)-cnotinouos, and thus is not a (μ, μ)-homotopy.

Proposition 2.1

Let $H: (X \times I, \sigma) \to (Y, \lambda)$ be (μ, λ)-homotopy. If (Y, λ) and $(X \times I, \sigma)$ are strong, then $H^{-1}(M_\lambda) = M_\sigma$.

Proof.

If (Y, λ) is strong, then $M_\sigma = X \times I$ and $M_\lambda = Y$ [4]. Since H is (μ, λ)-homotopy, then H is (σ, λ)-continuous and hence, $H^{-1}(M_\lambda) \subset M_\sigma$. On the other hand, we have $H(M_\sigma) \subset M_\lambda$ [4]. Therefore, $H^{-1}(H(M_\sigma)) \subset H^{-1}(M_\lambda)$ which implies that $M_\sigma \subset H^{-1}(M_\lambda)$. It follows that $H^{-1}(M_\lambda) = M_\sigma$. Q.E.D.

Theorem 2.1

Suppose that (X,μ), (Y,λ) and (I,ν) are all strong GTS's. Then the generalized homotopy is an equivalence relation on the set of (μ,λ)-continuous functions from (X,μ) to (Y,λ).

Proof.

Let (X,μ), (Y,λ) and (I,ν) be strong GTS's, and let $f,g,h:(X,\mu)\ \to (Y,\lambda)$ be (μ,λ)-continuous.

Reflexivity:

Since f is (μ,λ)-continuous, then $f^{-1}(m)=U\in\mu$ for every $m\in\lambda$. Define (μ,λ)-homotopy $H:(X\times I,\sigma)\ \to(Y,\lambda)$ by $H(x,t)\ =f(x)$, for every $x\in X$ and $t\in I$. Then $H^{-1}(m)=f^{-1}(m)\times V=U\times V\in\sigma$ where $V\in\nu$. Since the GT's are all strong, then by Proposition 2.1 we have $H^{-1}(M_\lambda)=M_\sigma=X\times I\in\sigma$. Thus, $f\simeq_{GH} f$ and the realation is reflexive.

Symmetry:

If $f\simeq_{GH} g$ by (μ,λ)-homotopy $H(x,t)$, then $g\simeq_{GH} f$ by (μ,λ)-homotopy $\overline{H}(x,t)=H(x,1-t)$. Observe that reversing the paths does not change the (σ,λ)-continuity of a (μ,λ)-homotopy. Therefore, the relation is symmetric.

Transitivity:

Let $f\simeq_{GH} g$ by (μ,λ)-homotopy $F(x,t)$, and let $g\simeq_{GH} h$ by (μ,λ)-homotopy $G(x,t)$. Define $H:(X\times I,\sigma)\to(Y,\lambda)$ by

$$H(x,t)=\begin{cases} F(x,2t) & \text{if}\quad 0\le t\le 1/2\\ G(x,2t-1) & \text{if}\quad 1/2\le t\le 1\end{cases}$$

Then H is a (μ,λ)-homotopy between f and h. To prove this, note the following:

- $t=0,\ H(x,0)=F(x,0)=f(x)$,
- $t=\frac{1}{2},\ F(x,1)=g(x)=G(x,0)$,
- $t=1,\ H(x,1)=G(x,1)=h(x)$.

Sine H is formed by two (σ,λ)-continuous functions F and G, then H is well-defined (σ,λ)-continuous and thus, a (μ,λ)-homotopy. Therefore, the relation is transitive. Q.E.D.

The (μ,λ)-homotopy class of a (μ,λ)-contiuous function $f:(X,\mu)\ \to(Y,\lambda)$ is denoted by $[f]_{GH}$.

Theorem 2.2

Let $f,g:(X,\mu)\ \to(Y,\lambda)$ be (μ,λ)-continuous, and let $f\simeq_{GH} g$. If $p:(X',\mu')\to(X,\mu)$ is a surjective (μ',μ)-continuous function, then $f\circ p\ \simeq_{GH} g\circ p$.

Proof.

Suppose that $f\simeq_{GH} g$ and that $H(x,t)$ is the (μ,λ)-homotopy between them. Define $G:(X'\times I,\acute{\sigma})\to(Y,\lambda)$ by $G(x',t)=H(p(x'),t)=H\circ p$ for every $x'\in X'$ and $t\in I$. Then G is (σ,λ)-continuous and,

- $G(x',0)=H(p(x'),0)=f(p(x'))$,
- $G(x',1)=H(p(x'),1)=g(p(x'))$.

Therefore, G is a (μ,λ)-homoytopy between $f\circ p$ and $g\circ p$. Q.E.D.

Remark 2.2

Using Theorem 2.2, we can define the composition of (μ,λ)-homotopy classes by:

$$[f]_{GH}\circ[g]_{GH}=[f\circ g]_{GH}.$$

Definition 2.2

Let (X,μ) and (Y,λ) be GTS's. A (μ,λ)-continuous function $f:(X,\mu)\to(Y,\lambda)$ is called a (μ,λ)-homotopy equivalence if there exists a (μ,λ)-continuous function $g:(Y,\lambda)\to(X,\mu)$ such that $g\circ f:(X,\mu)\to(X,\mu)$ and $f\circ g:(Y,\lambda)\to(Y,\lambda)$ are (μ,λ)-homotopic to the identity functions id_x and id_y respectively. In this case, the function g is called a (μ,λ)-homotopy inverse to f.

Definition 2.3

We say that a GTS (X,μ) is (μ,λ)-homotopic to a GTS (Y,λ) if there exists a (μ,λ)-homotopy equivalence between them.

Theorem 2.3

Let $f: (X, \mu) \to (Y, \lambda)$ be invertible (μ, λ)-continuous function. Then f is a (μ, λ)-homotopy equivalence.

Proof.

Since f is invertible, then we can take $g(y) = f^{-1}(y)$ to be a (μ, λ)-homotopy inverse to f and the result follows. Q.E.D.

Example 2.4

Let $X = \{0,1,2\}$, $Y = \{1\}$, $\mu = \{\emptyset, \{1\}, \{1,2\}\}$, and $\lambda = \{\emptyset, \{1\}\}$. Let $f: (X, \mu) \to (Y, \lambda)$ where $f(x) = 1$ for every $x \in X$. Let $g: (Y, \lambda) \to (X, \mu)$ where $g(y) = 1$ for every $y \in Y$. Then f is (μ, λ)-continuous since $f^{-1}(\{1\}) = \{1\} \in \mu$. Also, g is (λ, μ)-continuous since $g^{-1}(\{1\}) = \{1\} \in \lambda$, and $g^{-1}(\{1,2\}) = \{1\} \in \lambda$. For the composition of functions, we have:

- $g(f(0)) = g(1) = 1,$
- $g(f(1)) = g(1) = 1,$
- $g(f(2)) = g(1) = 1,$
- $f(g(1)) = f(1) = 1.$

Therefore, $g \circ f \simeq_{GH} id_x$ and $f \circ g \simeq_{GH} id_y$. It follows that f is a (μ, λ)-homotopy equivalence.

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