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Taylor Polynomials of the Reciprocal Gamma Function

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ABSTRACT

Using a Taylor Series of Reciprocal Gamma instead of a Reciprocal Gamma has the advantage, as the series and the function agree broadly around 0. The more terms we include in our approximate function, the better the approximation to the true value. The coefficients of the Taylor series of the reciprocal Gamma have an interesting combinatorial interpretation. The geometrical representation of the Gamma reciprocal function demonstrates how accurately the Taylor polynomials represent the actual graph of the reciprocal Gamma function. The numerical computations in this paper are achieved in aid of a MATLAB software as well as graph the Gamma function, and the reciprocal Gamma function with its Taylor approximations.

Keywords: Taylor Series; Taylor Polynomial; Gamma Function; Reciprocal Gamma Function.

1. Introduction

In mathematics, the primary goal of series is to express a particular complex quantity as a single sum of simple terms; Furthermore, since the terms get smaller and smaller, we can take only the first few terms of the series to approximate the original quantity. A function can be represented by Taylor series as an infinite sum of terms calculated from the values of its derivatives at a single point. The series is also known as Maclaurin series if its center is at zero. In practice, we obtain the approximation of a function by using a finite number of terms from the series. Many physical applications benefit from expansion techniques, sometimes in unexpected ways [1].

The application of the Taylor series expansion to a special analytical function, namely $1/\Gamma(x)$ will be the main focus of our paper. We derive Taylor polynomials for the reciprocal Gamma function and also give numerical coefficients. All the results and the calculations presented in this paper are produced by means of a MATLAB software.

2. Basic Definitions

This section provides the basic definitions that has been used in our work; we have introduced the Taylor series formula, and presented a brief explanation of the fundamental analytical functions, namely, Gamma and zeta, as well as the Euler's constant, which are utilized in the majority of calculus definitions.

2.1.Taylor Series

Taylor expansion is one of the most interesting and useful ideas in mathematics. In general, solving most functions and polynomials is typically smooth process. Polynomials, on the other hand, are typically able to approximate functions, making them simpler to work with than nearly any other sort of function. Using the Taylor formula, we can find an equation for the polynomial expansion for nearly every smooth function.

Consider a function $f(x)$ that has a power series representation at a point $x = a$. Then the series has the following form [2], [3]:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \quad (1)$$

Once we have a power series for $f(x)$ with known coefficients $c_n = \frac{f^{(n)}(a)}{n!}$, $f(x)$ can be approximated by taking a finite partial sum of the series up to some cutoff term n. This partial sum is called a Taylor polynomial, denoted $T_n(x)$. In other words, if $f(x)$ has derivatives of all degrees at $x = a$, then the Taylor series for the function $f(x)$ at a is given by:

$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad (2)$$

$$= f(a) + f'(a)(x-a)$$

$$+\frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2.2. Gamma Function

The Gamma function is the one that is used the most frequently out of all the special functions. It typically comes up first because it appears in almost every

$$\Gamma(z) = \int_0^{\infty} y^{z-1} e^{-y} dy \quad (3)$$

integral or series representation of other complex mathematical functions. Here we use the integral formula to define it.

It is worth mentioning that the Gamma function is meromorphic on the whole complex plane, which means that it is a single-valued function and analytic everywhere in the complex plane except for non-positive integers [4].

2.3. Riemann Zeta Function

The Riemann zeta function, denoted as $\zeta(z)$which appears in definite integration, is a very significant special function of mathematics and physics. It can be defined by simple formulae in two ways: as a Dirichlet series, or as an Euler product. Here we will present only the first definition, and for the second, the reader can refer to [5], [6].

The Riemann Zeta function $\zeta(z)$ is defined by:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (4)$$

The Dirichlet series on the right hand converges for $Re(z) > 1$, and converges uniformly in any region $Re(z) \geq 1 + \delta,\ \delta > 0$.

2.4. Euler's constant

Euler's constant (sometimes also called the Euler–Mascheroni constant) is a constant usually denoted by the lowercase Greek letter Gamma (γ) [4].

It is defined as the limiting difference between the harmonic series and the natural logarithm, denoted here by log:

$$\gamma = \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) \quad (5)$$

$$\gamma = \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx \quad (6)$$

Here, $\lfloor\ \rfloor$ represents the Step function. The numerical value of Euler's constant, to 50 decimal places, is:

0.57721566490153286060651209008240243104215933593992...

Euler's constant plays an important role in Calculus and occurs frequently in Number Theory (see for instance [7]). The constant γ is deeply related to the Gamma function $\Gamma(z)$.

In upcoming sections of the paper, we will see how we can benefit from both Riemann Zeta function and Euler's constant to obtain the coefficients of Taylor polynomials.

2.5. Reciprocal Gamma Function

In mathematics, the reciprocal Gamma function is defined as function $f(z) = \frac{1}{\Gamma(z)}$. The reciprocal Gamma function is an entire function and can be defined as:

$$\frac{1}{\Gamma(z)} = z\, e^{\left(\gamma z - \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k) z^k}{k} \right)} \quad (7)$$

where γ is the Euler-Mascheroni constant and ζ is the Riemann zeta function [8]. An asymptotic series for $\frac{1}{\Gamma(z)}$ is given by:

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \frac{1}{12}(6\gamma^2 - \pi^2)z^3 + \frac{1}{12}(2\gamma^3 - \gamma\pi^2 4\zeta(3))z^4 + \cdots \tag{8}$$

It can be written as:

$$\frac{\mathbf{1}}{\boldsymbol{\Gamma(z)}} = \sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{a_n z^n}, \tag{9}$$

where the the coefficient a_n for the z^n term can be computed recursively as;

$$\boldsymbol{a_n = n\, a_1 a_n - a_2 a_{n-1} + \sum_{k=2}^{\infty} (-1)^k \zeta(k)\, a_{n-k}} \tag{10}$$

According to the helpful Weierstrass formula, $\frac{1}{\Gamma(z)}$ is expressed as:

$$\frac{\mathbf{1}}{\boldsymbol{\Gamma(z)}} = \boldsymbol{z\, e^{(\gamma z)} \prod_{n>0} \left[\left(1 + \frac{z}{n}\right) e^{\left(-\frac{z}{n}\right)}\right]} \tag{11}$$

This identity implies the following relation;

$$\Gamma'(1) = -\gamma.$$

We will concentrate on using the real-valued function $\frac{1}{\Gamma(x)}$ instead of $\frac{1}{\Gamma(z)}$.

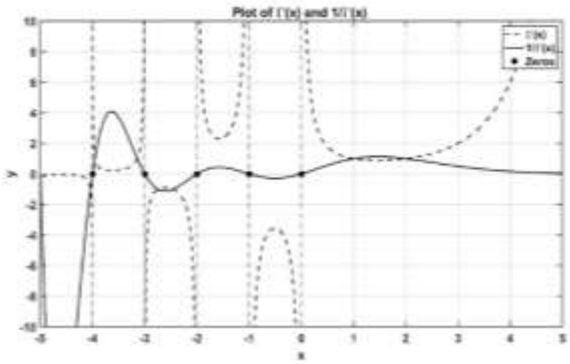
Zeros of a function

Let $f(x)$ be a function, then a number c, for which $f(c) = 0$, is called a zero of a function. Since the Gamma function $\Gamma(x)$ is nonzero, and has simple poles with residue $\frac{(-1)^n}{n!}$, its reciprocal $\frac{1}{\Gamma(x)}$ is an entire function with simple zeros at $x = -n(n = 0,1,2,3,\cdots)$, as shown in Fig.1.

Fig. 1. The Gamma function has poles at zero and the negative integers; whereas, the entire function $\boldsymbol{1/\Gamma(x)}$ with zeros at these points

3. The expansion of The Reciprocal Gamma Function

The Reciprocal Gamma $\frac{1}{\Gamma(x)}$ is an entire function and so it has a convergent Taylor series expansion. In this section, we present the tool that typically enables us to express $\frac{1}{\Gamma(x)}$ explicitly as a power series. For this special function, we will explain how to find a number of



Taylor polynomials with particular degrees.

The Taylor polynomial for a smooth function is the truncation at the order n of the Taylor series of the function. The first-degree Taylor polynomial $T_1(x)$ is the linear approximation of the function or tangent line at a point, while the quadratic approximation is typically represented by the second-degree Taylor polynomial $T_2(x)$.

Because the software takes longer time for $n > 30$, we limited ourselves in this paper to approximate $\frac{1}{\Gamma(x)}$ by calculating $T_n(x)$, where $n \in \{5,10,15,20,25,30\}$ as shown in Fig. 2.

$$\boldsymbol{T_5(x) = x + \gamma x^2 + \frac{1}{12}(6\gamma^2 - \pi^2)s x^3 + \frac{1}{12}\left(2\gamma^3 - \gamma\pi^2 4\zeta(3)\right)x^4} + \frac{1}{3}\zeta(3)x^5 \tag{12}$$

The estimated coefficients a_n are as follows:

$a_0 = 0$, $a_1 = 1$, $a_2 \approx 0.577216$, $a_3 \approx -0.655878$, $a_4 \approx -0.042003$, and $a_5 \approx 0.400686$.

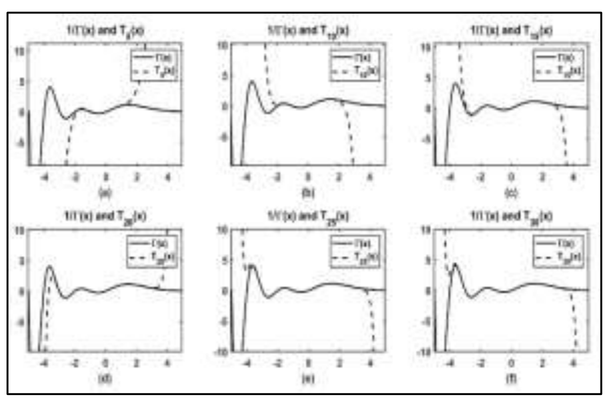


Fig. 2. The graphs of $\frac{1}{\Gamma(x)}$ along with the $\boldsymbol{T_n(x)}$; $\boldsymbol{n \in \{5, 10, 15, 20, 25, 30\}}$

In Fig. 2(a), the Taylor polynomial $T_5(x)$, matches $\frac{1}{\Gamma(x)}$ in its five derivatives and stay closer to its original curve within the interval $|x| < 0.5$, namely, the series converges for $|x| < 0.5$, i.e., for $-0.5 < x < 0.5$, and diverges for $x < -0.5$ and for $x > 0.5$.

The next Taylor polynomial $T_{10}(x)$ (in Fig. 2(b)) is even closer to $\frac{1}{\Gamma(x)}$ for even larger x. we observe that as the degree n get larger, the polynomial $T_n(x)$ approaches $\frac{1}{\Gamma(x)}$ more closely, over a wider domain (see Fig. 2(c), (d), (e) and (f)).

The convergence intervals where Taylor polynomials are identical to $\frac{1}{\Gamma(x)}$ can be tabulated in table 1. The $T_n(x)$ values are generated using a MATLAB program at different values of x. table 3. provides a summary of the results.

Table 1. The intervals in which Taylor polynomials are the same as $\frac{1}{\Gamma(x)}$ itself

The number of terms	$\frac{1}{\Gamma(x)} \approx T_n(x)$	Domain
$n = 5$	$\sum_{n=1}^{5} a_n x^n$	$-0.5 < x < 0.5$
$n = 10$	$\sum_{n=1}^{10} a_n x^n$	$-1 < x < 1$
$n = 15$	$\sum_{n=1}^{15} a_n x^n$	$-1.5 < x < 1.5$
$n = 20$	$\sum_{n=1}^{20} a_n x^n$	$-2 < x < 2$
$n = 25$	$\sum_{n=1}^{25} a_n x^n$	$-2.5 < x < 2.5$
$n = 30$	$\sum_{n=1}^{30} a_n x^n$	$-3 < x < 3$

It can be noticed in table 3. that the Taylor approximations precisely match the reciprocal $\frac{1}{\Gamma(x)}$ as $x \to 0$, which reduces the relative error around the origin and rapidly increases as the value of |x| grows (see Fig. 2).

4. The Approximate Area under The Curve $\frac{1}{\Gamma(x)}$, $-3 \le x \le 3$

The integral of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ along the positive real axis, as observed in the litterers, has the value:

$$\int_0^\infty \frac{1}{\Gamma(x)} dx \approx 2.80777024$$

This constant is known as the Fransén–Robinson constant [9]. When we calculated the approximated integration in the domain $0 \le x \le 3$, we get:

$$\int_0^3 \frac{1}{\Gamma(x)} dx \approx \int_0^3 \mathrm{T}_{30}(x)\, dx = 2.377988248$$

We can compare the approximated area under the curve $\frac{1}{\Gamma(x)}$ on the interval [0,3] with the definite integral $\int_0^3 \mathrm{T}_{30}(\mathrm{x})\,\mathrm{dx}$ using Trapezoid rule method [10], [11].

As show in table 3, when the value of n is get larger, the two approximations get closer to each other.

Table 2. A comparison between $\frac{1}{\Gamma(x)}$ and $\mathrm{T}_{30}(x)$ using Trapezoid rule method

n	$Trap\left(\frac{1}{\Gamma(x)}\right)$	$Trap(T_{30}(x))$
3	2.250000000000000	2.248960569140906
6	2.347410764353472	2.346880153165195
9	2.364624388811608	2.364240152025899
12	2.370593931947559	2.370269209831074
15	2.373347501340530	2.373052157275474
18	2.374840776652307	2.374561926312747
21	2.375740331032812	2.375471617356713
24	2.376323837721993	2.376061782504875
27	2.376723734190222	2.376466280772617
30	2.377009701117583	2.376755557928740

Therefore, considering that the Taylor polynomial $\mathrm{T}_{30}(\mathrm{x})$ accurately estimates $\frac{1}{\Gamma(\mathrm{x})}$, we attempted within the interval $-3 \le \mathrm{x} \le 3$, to calculate the area under the estimated curve $\frac{1}{\Gamma(\mathrm{x})}$, and obtained the following results:

$$\int_{-3}^3 \frac{1}{\Gamma(x)} dx \approx \int_{-3}^3 \mathrm{T}_{30}(x)\, dx$$

$$= \int_{-3}^{-2} -\mathrm{T}_{30}(x)\, dx + \int_{-2}^{-1} \mathrm{T}_{30}(x)\, dx + \int_{-1}^{0} -\mathrm{T}_{30}(x)\, dx$$

$$+ \int_0^1 \mathrm{T}_{30}(x)\, dx + \int_1^2 \mathrm{T}_{30}(x)\, dx + \int_2^3 \mathrm{T}_{30}(x)\, dx$$

$=0.69841999+0.27577304+0.18372071$

$+0.54123573 +1.085142658+0.75160986$

$= 3.53590199.$

Table 3. The Taylor approximations of $\frac{1}{\Gamma(x)}$, when $-3 \le x \le 3$

x \ F	$T_5(x)$	$T_{10}(x)$	$T_{15}(x)$	$T_{20}(x)$	$T_{25}(x)$	$T_{30}(x)$	$\frac{1}{\Gamma(x)}$
$x = -3.0$	-23.9674	23.9013	0.8807	-0.0538	0.0023	0.0021	0
$x = -2.8$	-15.1204	10.7514	-0.6201	-0.8907	-0.8779	-0.8779	-0.8783
$x = -2.6$	-8.8768	4.1787	-1.0565	-1.1278	-1.1252	-1.1252	-1.1253
$x = -2.4$	-4.6628	1.3459	-0.8862	-0.9030	-0.9025	-0.9025	-0.9025
$x = -2.2$	-1.9892	0.4229	-0.4501	-0.4536	-0.4535	-0.4535	-0.4535
$x = -2.0$	-0.4454	0.3098	0.0006	-0.0000	0.0000	0.0000	0.0000
$x = -1.8$	0.3075	0.4111	0.3138	0.3137	0.3137	0.3137	0.3137
$x = -1.6$	0.5426	0.4594	0.4328	0.4328	0.4328	0.4328	0.4328
$x = -1.4$	0.4740	0.3821	0.3760	0.3760	0.3760	0.3760	0.3760
$x = -1.2$	0.2630	0.2072	0.2061	0.2061	0.2061	0.2061	0.2061
$x = -1.0$	0.0246	0.0001	0.0000	-0.0000	0.0000	0.0000	0.0000
$x = -0.8$	-0.1665	-0.1742	-0.1743	-0.1743	-0.1743	-0.1743	-0.1743
$x = -0.6$	-0.2689	-0.2705	-0.2705	-0.2705	-0.2705	-0.2705	-0.2705
$x = -0.4$	-0.2684	-0.2686	-0.2686	-0.2686	-0.2686	-0.2686	-0.2686
$x = -0.2$	-0.1718	-0.1718	-0.1718	-0.1718	-0.1718	-0.1718	-0.1718
$x = 0.0$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x = 0.2$	0.2178	0.2178	0.2178	0.2178	0.2178	0.2178	0.2178
$x = 0.4$	0.4510	0.4508	0.4508	0.4508	0.4508	0.4508	0.4508
$x = 0.6$	0.6736	0.6715	0.6715	0.6715	0.6715	0.6715	0.6715
$x = 0.8$	0.8710	0.8589	0.8589	0.8589	0.8589	0.8589	0.8589
$x = 1.0$	1.0459	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
$x = 1.2$	1.2251	1.0884	1.0891	1.0891	1.0891	1.0891	1.0891
$x = 1.4$	1.4659	1.1230	1.1271	1.1271	1.1271	1.1271	1.1271
$x = 1.6$	1.8622	1.1023	1.1192	1.1192	1.1192	1.1192	1.1192
$x = 1.8$	2.5510	1.0143	1.0735	1.0737	1.0737	1.0737	1.0737
$x = 2.0$	3.7190	0.8178	0.9992	1.0000	1.0000	1.0000	1.0000
$x = 2.2$	5.6088	0.4075	0.9038	0.9076	0.9076	0.9076	0.9076
$x = 2.4$	8.5252	-0.4486	0.7895	0.8053	0.8050	0.8050	0.8050
$x = 2.6$	12.8419	-2.2115	0.6427	0.7008	0.6994	0.6994	0.6995
$x = 2.8$	19.0077	-5.7376	0.4081	0.6026	0.5962	0.5961	0.5965
$x = 3.0$	27.5529	-12.533	-0.0757	0.5253	0.4982	0.4979	0.5000

5. Conclusion

Expanding the reciprocal Gamma function into a Taylor series is the simplest method for evaluating it. As a result, in this paper, rather than focusing on the function itself, we chose to use its Taylor expansions to simplify the numerical computation of this function; which has been used to determine the area under the curve (see Example 3.1). This concept can be applied to a variety of other analytical functions; for instance, the Bessel function, which is widely used in a variety of mathematical fields like probability, statistics, physics, engineering and in this regard, some new findings have been presented by other recent studies (see [12]).

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