

POINTWISE REGULARITY FOR SOLUTIONS OF DOUBLE OBSTACLE PROBLEMS ON METRIC SPACES

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Abstract

We study continuity at a given point for solutions of double obstacle problems. We obtain pointwise continuity of the solutions for discontinuous obstacles. We also show Hölder continuity for solutions of the double obstacle problems if the obstacles are Hölder continuous.

1. Introduction

The objective of this paper is to study continuity at a given point x_0 of the solution of the obstacle problem on certain metric spaces. The setting considered here is that of a complete metric space X endowed with a metric d and a Borel measure μ which is *doubling*, i.e., there exists a constant $C > 0$ such that for all balls $B = B(x, r) := \{y \in X : d(x, y) < r\}$ in X we have

$$0 < \mu(2B) \leq C\mu(B) < \infty,$$

where $\tau B = B(x, \tau r)$. The doubling property implies that X is complete if and only if X is proper, i.e., closed bounded sets are compact. We also require the space X to support a p -Poincaré inequality, see Section for the definition.

Nonlinear potential theory on metric measure spaces has been studied in many papers, see for example Björn-Björn [1], [2], Björn-Björn-Shanmugalingam [3], [4], J. Björn [7], Cheeger [8], Heinonen-Koskela [13], Kinnunen-Martio [16], Koskela-MacManus [19] and Shanmugalingam [23], [24].

There are several notions of Sobolev spaces in metric spaces; see for example Cheeger [8], Hajłasz [11] and Shanmugalingam [23], [24]. We shall follow the definition of Shanmugalingam [23], where Sobolev type spaces $\mathbf{N}^{1,p}(X)$ (called the Newtonian spaces) were defined as the collection of p -integrable functions with p -integrable upper gradients. The notion of upper gradients was introduced by Heinonen-Koskela [13], as a substitute for the modulus of the usual gradient. Koskela-MacManus [19] extended the concept

to p -weak upper gradients. Variational inequalities can then be used to define p -harmonic functions as minimizers of the variational integral

$$(1) \quad \int g_u^p d\mu,$$

where g_u denotes the minimal p -weak upper gradient of u , whose existence and uniqueness were proved in Shanmugalingam [23]. The existence and uniqueness of minimizers of (1) were shown in Shanmugalingam [24].

Let $1 < p < \infty$ and let Ω be a bounded open subset of X whose complement has positive capacity. We minimize the variational integral (1) on Ω among all functions which have prescribed boundary values f and lie between two given obstacles ψ_1 and ψ_2 . The minimizer is called a solution of the $\mathcal{H}_{\psi_1, \psi_2, f}$ -problem. The Dirichlet problem (the obstacle problem with $\psi_1 = -\infty$ and $\psi_2 = \infty$) for p -harmonic functions on metric spaces was studied e.g. in Björn-Björn [1], [2], Björn-Björn-Shanmugalingam [3], [4], Kinnunen-Shanmugalingam [17] and Shanmugalingam [24], [25].

In this paper we investigate the continuity at a given point x_0 of the solutions of the double obstacle problem. The obstacles in this context are to be regarded as quite general and irregular. In particular, they may be discontinuous. We show that if the obstacles are not continuous, but satisfy a Wiener type regularity condition, the solution is still continuous. This extends Theorem 5.4 from Malý-Ziemer [21] to metric spaces.

Although in \mathbb{R}^n p -harmonic functions are Lipschitz they need not be so in the general setting of metric spaces, see p. 149 in Koskela-Rajala-Shanmugalingam [20]. Therefore, as p -harmonic functions are solutions of special obstacle problems we can expect at most Hölder regularity of our solutions. Indeed, we show that the continuous solution of the single obstacle problem, with locally Hölder continuous obstacle, is locally Hölder continuous. For the double obstacle problem we prove that if the obstacles are locally Hölder continuous, then the continuous solution u is Hölder continuous at every point $x_0 \in \Omega$. This does not directly imply that u is locally Hölder continuous as Example 4.2 shows.

Hölder continuity of p -harmonic functions on metric spaces was obtained in Kinnunen-Shanmugalingam [17]. In the Euclidean case Hölder continuity for solutions of the obstacle problems, with Hölder continuous obstacles, was obtained in Kilpeläinen-Ziemer [14] and Michael-Ziemer [22].

2. Notation and preliminaries

A nonnegative Borel function g is said to be an *upper gradient* of an extended real-valued function f on X if for all rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$

parameterized by arc length ds , we have

$$(2) \quad |f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise.

If g is a nonnegative measurable function on X and if (2) holds for p -almost every curve, then g is a p -weak upper gradient of f .

By saying that (2) holds for p -almost every curve we mean that it fails only for a curve family with zero p -modulus, see Definition 2.1 in Shanmugalingam [23]. If f has an upper gradient in $L^p(X)$, then it has a *minimal p -weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p -weak upper gradient $g \in L^p(X)$ of f , $g_f \leq g$ a.e., see Corollary 3.7 in Shanmugalingam [24].

We shall use the following definition of Sobolev type spaces which is equivalent to the one introduced in Shanmugalingam [23].

DEFINITION 2.1. Let $u \in L^p(X)$, then we define

$$\|u\|_{\mathbf{N}^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p},$$

where g_u is the minimal p -weak upper gradient of u . The *Newtonian space* on X is the quotient space

$$\mathbf{N}^{1,p}(X) = \{u : \|u\|_{\mathbf{N}^{1,p}(X)} < \infty\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{\mathbf{N}^{1,p}(X)} = 0$.

The space $\mathbf{N}^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p. 249 in Shanmugalingam [23].

The *capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|_{\mathbf{N}^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in \mathbf{N}^{1,p}(X)$ such that $u \geq 1$ on E .

We say that a property holds *quasi*everywhere (q.e.) in X , if it holds everywhere except on a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e., if $u, v \in \mathbf{N}^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [23] shows that if $u, v \in \mathbf{N}^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

From now on we assume that X supports a p -Poincaré inequality i.e., there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(z, r)$ in X , all

integrable functions u on X and all upper gradients g of u we have

$$\int_{B(z,r)} |u - u_{B(z,r)}| d\mu \leq Cr \left(\int_{B(z,\lambda r)} g^p d\mu \right)^{1/p},$$

where $u_{B(z,r)} := \int_{B(z,r)} u d\mu := \mu(B(z,r))^{-1} \int_{B(z,r)} u d\mu$.

Under the above assumptions, every function $u \in \mathbf{N}^{1,p}(X)$ is quasicontinuous, i.e., for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{X \setminus G}$ is continuous, see Theorem 1.1 in Björn-Björn-Shanmugalingam [5]. Moreover, when restricted to \mathbb{R}^n , the Newtonian space $\mathbf{N}^{1,p}(\mathbb{R}^n)$ is the refined Sobolev space $W^{1,p}(\mathbb{R}^n)$, as defined in Chapter 4 in Heinonen-Kilpeläinen-Martio [12].

We say that $f \in \mathbf{N}_{\text{loc}}^{1,p}(\Omega)$ if $f \in \mathbf{N}^{1,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$, where by $\Omega' \subset\subset \Omega$ we mean that the closure of Ω' is a compact subset of Ω . Let also

$$\mathbf{N}_0^{1,p}(\Omega) = \{u|_{\Omega} : u \in \mathbf{N}^{1,p}(X) \text{ and } u = 0 \text{ in } X \setminus \Omega\}.$$

The following definition is slightly different from the notation used in Kinnunen-Martio [16]. We use q.e. inequalities rather than a.e. inequalities as in [16], for more discussion see p. 265 in Farnana [10].

DEFINITION 2.2. Let $V \subset X$ be a nonempty bounded open set such that $C_p(X \setminus V) > 0$, let $f \in \mathbf{N}^{1,p}(V)$ and $\psi_i : V \rightarrow \bar{\mathbb{R}}, i = 1, 2$. Then we define the obstacle problem with obstacles ψ_1, ψ_2 and boundary values f by

$$\mathcal{K}_{\psi_1, \psi_2, f}(V) = \{v \in \mathbf{N}^{1,p}(V) : v - f \in \mathbf{N}_0^{1,p}(V) \text{ and } \psi_1 \leq v \leq \psi_2 \text{ q.e. in } V\}.$$

Furthermore, a function $u \in \mathcal{K}_{\psi_1, \psi_2, f}(V)$ is a *solution of the $\mathcal{K}_{\psi_1, \psi_2, f}(V)$ -problem* if

$$\int_V g_u^p d\mu \leq \int_V g_v^p d\mu \quad \text{for all } v \in \mathcal{K}_{\psi_1, \psi_2, f}(V).$$

We also let $\mathcal{K}_{\psi_1, \psi_2, f} = \mathcal{K}_{\psi_1, \psi_2, f}(\Omega)$, $\mathcal{K}_{\psi, f}(V) = \mathcal{K}_{\psi, \infty, f}(V)$ and $\mathcal{K}_{\psi, f} = \mathcal{K}_{\psi, f}(\Omega)$.

A function $u \in \mathbf{N}_{\text{loc}}^{1,p}(\Omega)$ is a *minimizer* in Ω if it is a solution of the $\mathcal{K}_{-\infty, u}(\Omega')$ -problem for every open $\Omega' \subset\subset \Omega$. Also u is *p-harmonic* in Ω if it is a continuous minimizer. Similarly, a function $u \in \mathbf{N}_{\text{loc}}^{1,p}(\Omega)$ is a *superminimizer* in Ω if it is a solution of the $\mathcal{K}_{u, u}(\Omega')$ -problem for every open $\Omega' \subset\subset \Omega$. In [16], Kinnunen and Martio showed that if u is a solution of the $\mathcal{K}_{\psi, f}$ -problem, then it is a superminimizer in Ω and its *lower semicontinuous regularization* $u^*(x) := \text{ess lim inf}_{y \rightarrow x} u(y)$ equals u q.e. in Ω and is the

unique lower semicontinuously regularized solution. Moreover u^* is superharmonic, see e.g. Kinnunen-Martio [16] for the definition of superharmonic functions.

We shall need the following results from Farnana [10]. We let $u_+ = \max\{u, 0\}$ be the positive part of u .

LEMMA 2.3. *Let $\psi, \psi', \phi, \phi' : \Omega \rightarrow \bar{\mathbf{R}}$ and $f, f' \in \mathbf{N}^{1,p}(\Omega)$. Let u be a solution of the $\mathcal{H}_{\psi, \phi, f}$ -problem and u' be a solution of the $\mathcal{H}_{\psi', \phi', f'}$ -problem. Assume that $(f - f')_+ \in \mathbf{N}_0^{1,p}(\Omega)$ and that $\psi \leq \psi'$ and $\phi \leq \phi'$ q.e. in Ω . Then $u \leq u'$ q.e. in Ω .*

LEMMA 2.4. *Let $\psi_i : \Omega \rightarrow \bar{\mathbf{R}}, i = 1, 2$, and $f \in \mathbf{N}^{1,p}(\Omega)$. Let u be a solution of the $\mathcal{H}_{\psi_1, \psi_2, f}$ -problem, $V \subseteq \Omega$ be open and $k \in \mathbf{R}$. If $\psi_2 \geq k$ q.e. in V , then $u_k = \min\{u, k\}$ is a superminimizer in V .*

The following result, which we will need later, is a combination of Theorem 4.2 and Remark 4.4 in Kinnunen-Shanmugalingam [17] and Theorem 4.4 and Remark 4.5 in Kinnunen-Martio [16]. It shows that the solution of the single obstacle problem is locally bounded from above if the obstacle is locally bounded from above.

THEOREM 2.5. *Let $B(x, 2r) \subset \Omega$ and $k \geq \psi$ q.e. in $B(x, 2r)$. Let u be a solution of the $\mathcal{H}_{\psi, f}(\Omega)$ -problem. Then for all $q > 0$ and $r > 0$,*

$$\operatorname{ess\,sup}_{B(x,r)} u \leq k + C \left(\int_{B(x,2r)} (u - k)_+^q d\mu \right)^{1/q}.$$

REMARK 2.6. The comparison Lemma 2.3 between a solution u of the $\mathcal{H}_{\psi_1, \psi_2, f}$ -problem and a solution v of the $\mathcal{H}_{\psi_1, f}$ -problem implies that $u \leq v$ q.e. in Ω , hence a.e. This together with Theorem 2.5 implies that u is essentially locally bounded from above if ψ_1 is essentially locally bounded from above. Similarly it follows that u is essentially locally bounded from below if ψ_2 is essentially bounded from below, using that $-u$ is a solution of the $\mathcal{H}_{-\psi_2, -\psi_1, -f}$ -problem.

3. Continuity in the presence of irregular obstacles

In this section we will initiate a study of the regularity of solutions of the double obstacle problems. Since we are concerned with pointwise regularity, we will consider a fixed point $x_0 \in \Omega$ throughout the rest of this section.

The relative capacity of $E \subset\subset B$ with respect to a ball B is

$$\operatorname{Cap}_p(E, B) = \inf_u \int_B g_u^p d\mu,$$

where the infimum is taken over all $u \in \mathbf{N}_0^{1,p}(B)$ such that $u \geq 1$ on E .

A set $E \subset X$ is p -thin at $x \in X$ if

$$\int_0^1 \left(\frac{\text{Cap}_p(E \cap B(x, t), B(x, 2t))}{\text{Cap}_p(B(x, t), B(x, 2t))} \right)^{1/(p-1)} \frac{dt}{t} < \infty,$$

and $G \subset X$ is p -finely open if $X \setminus G$ is p -thin at every $x \in G$. The complement of a p -finely open set is called p -finely closed. The p -fine open sets generate the p -fine topology which is finer than the metric topology, see J. Björn [7], Kinnunen-Latvala [15] and Korte [18]. In the Euclidean case the p -fine topology was studied in Malý-Ziemer [21].

DEFINITION 3.1. A point x_0 is called a *Wiener point* relative to the obstacles ψ_1 and ψ_2 if

$$(3) \quad \begin{aligned} p\text{-fine lim sup}_{x \rightarrow x_0} \psi_1(x) &= C_p\text{-lim sup}_{x \rightarrow x_0} \psi_1(x), \\ p\text{-fine lim inf}_{x \rightarrow x_0} \psi_2(x) &= C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x) \end{aligned}$$

and

$$(4) \quad C_p\text{-lim sup}_{x \rightarrow x_0} \psi_1(x) \leq C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x).$$

Here C_p -lim sup and C_p -lim inf stand for essential limits up to sets of capacity zero, while p -fine lim sup and p -fine lim inf stand for limits in the p -fine topology.

REMARK 3.2. The inequality (4) is necessary even if we just want to insert a function continuous at x_0 which lies q.e. between ψ_1 and ψ_2 .

A function u is p -finely continuous at x_0 if $u(x_0) = p\text{-fine lim sup}_{x \rightarrow x_0} u(x) = p\text{-fine lim inf}_{x \rightarrow x_0} u(x)$. If $u \in \mathbf{N}^{1,p}(\Omega)$ then u is p -finely continuous at q.e. $x \in \Omega$, see Theorem 4.6 in J. Björn [7].

We say that a function u is p -finely represented if u is defined everywhere in Ω and

$$p\text{-fine lim inf}_{y \rightarrow x} u(y) \leq u(x) \leq p\text{-fine lim sup}_{y \rightarrow x} u(y)$$

for each $x \in \Omega$. By Theorem 4.6 in J. Björn [7] every function $u \in \mathbf{N}^{1,p}(\Omega)$ can be p -finely represented, e.g.

$$x \mapsto p\text{-fine lim inf}_{y \rightarrow x} u(y) \quad \text{and} \quad x \mapsto p\text{-fine lim sup}_{y \rightarrow x} u(y)$$

are p -fine representatives of u . The following result extends Theorem 5.4 from Malý-Ziemer [21] to metric spaces. The proof below has been inspired by the proof in [21].

THEOREM 3.3. *Let $f \in \mathbf{N}^{1,p}(\Omega)$, $\psi_1 : \Omega \rightarrow [-\infty, \infty)$ and $\psi_2 : \Omega \rightarrow (-\infty, \infty]$. Let x_0 be a Wiener point relative to ψ_1 and ψ_2 and let u be a p -finely represented solution of the $\mathcal{K}_{\psi_1, \psi_2, f}$ -problem. Assume also that ψ_1 is bounded from above in a neighbourhood of x_0 and ψ_2 is bounded from below in a neighbourhood of x_0 . Then*

$$(5) \quad C_p\text{-}\limsup_{x \rightarrow x_0} \psi_1(x) \leq u(x_0) \leq C_p\text{-}\liminf_{x \rightarrow x_0} \psi_2(x).$$

Moreover u is real-valued continuous at x_0 .

Note that in Theorem 3.3 we allow $C_p\text{-}\limsup_{x \rightarrow x_0} \psi_1(x)$ to take the value $-\infty$ and $C_p\text{-}\liminf_{x \rightarrow x_0} \psi_2(x)$ to take the value ∞ .

PROOF. Let

$$\underline{u}(x) = p\text{-fine}\liminf_{y \rightarrow x} u(y) \quad \text{and} \quad \bar{u}(x) = p\text{-fine}\limsup_{y \rightarrow x} u(y).$$

Note that for every $k \in \mathbf{R}$ we have

$$p\text{-fine}\liminf_{y \rightarrow x} (\min\{u, k\})(y) = \min\{\underline{u}, k\}(x) =: \underline{u}_k(x),$$

$$p\text{-fine}\limsup_{y \rightarrow x} (\min\{u, k\})(y) = \min\{\bar{u}, k\}(x) =: \bar{u}_k(x)$$

and

$$\text{ess}\liminf_{y \rightarrow x} (\min\{u, k\})(y) = \min\{u^*, k\}(x) =: u_k^*(x),$$

where $u^*(x) = \text{ess}\liminf_{y \rightarrow x} u(y)$.

Next, as u is p -finely represented we have

$$(6) \quad \underline{u} \leq u \leq \bar{u} \quad \text{everywhere in } \Omega.$$

Choose $k < C_p\text{-}\liminf_{x \rightarrow x_0} \psi_2(x)$. Then there exists $r > 0$ such that $B(x_0, r) \subset \Omega$, ψ_2 is bounded from below in $B(x_0, r)$ and $k < \psi_2(x)$ for q.e. $x \in B(x_0, r)$. It follows that $u_k := \min\{u, k\}$ is a superminimizer in $B(x_0, r)$, by Lemma 2.4. Then the lower semicontinuous regularization u_k^* of u_k is a superharmonic function and hence p -finely continuous by Theorem 4.4 in J. Björn [7] (or Theorem 4.3 in Korte [18]). This and the fact that $u_k^* = u_k$ q.e.

in Ω imply that, for $x \in B(x_0, r)$,

$$\begin{aligned} \underline{u}_k(x) &= p\text{-fine lim inf}_{y \rightarrow x} u_k(y) = p\text{-fine lim inf}_{y \rightarrow x} u_k^*(y) = u_k^*(x) \\ &= p\text{-fine lim sup}_{y \rightarrow x} u_k^*(y) = p\text{-fine lim sup}_{y \rightarrow x} u_k(y) = \bar{u}_k(x). \end{aligned}$$

This and (6) show that $u_k = u_k^*$ everywhere in $B(x_0, r)$. Hence u_k is both lower semicontinuous and p -finely continuous at x_0 , whenever $k < C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x)$. As for $k = k_0 := C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x) < \infty$ we have $u_k \nearrow u_{k_0}$ uniformly as $k \nearrow k_0$. Then for every $\varepsilon > 0$,

$$u_{k_0}(x_0) \leq u_{k_0-\varepsilon}(x_0) + \varepsilon \leq \liminf_{x \rightarrow x_0} u_{k_0-\varepsilon}(x) + \varepsilon \leq \liminf_{x \rightarrow x_0} u_{k_0}(x) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this shows that u_{k_0} is lower semicontinuous at x_0 . Similarly we conclude that u_{k_0} is p -finely continuous at x_0 , using that u_k is p -finely continuous for all $k < k_0$ and that $u_k \nearrow u_{k_0}$ uniformly in Ω , as $k \nearrow k_0$.

Let k be a real number with

$$C_p\text{-lim sup}_{x \rightarrow x_0} \psi_1(x) \leq k \leq C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x).$$

Then we obtain

$$\begin{aligned} u(x_0) &\geq u_k(x_0) = p\text{-fine lim}_{x \rightarrow x_0} u_k(x) \\ &\geq p\text{-fine lim sup}_{x \rightarrow x_0} \psi_1(x) = C_p\text{-lim sup}_{x \rightarrow x_0} \psi_1(x). \end{aligned}$$

Similarly, as $-u$ is a p -finely represented solution of the $\mathcal{H}_{-\psi_2, -\psi_1, -f}$ -problem, we have that $u(x_0) \leq C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x)$ and hence (5) is established.

Next, we show that u is real-valued at x_0 . As ψ_1 is bounded from above in a neighbourhood of x_0 , there exist $r > 0$ and $M \in \mathbb{R}$ such that $\psi_1 \leq M$ in $B = B(x_0, r)$. Applying Remark 2.6 to the ball B implies that $u \leq M' < \infty$ q.e. in B for some $M' \in \mathbb{R}$. Since u is p -finely represented we have $u \leq \bar{u} \leq M' < \infty$ everywhere in B . Similarly we get that $u > -\infty$ in a neighbourhood of x_0 .

Finally we show that u is continuous at x_0 . As u is real-valued at x_0 we can take $k = u(x_0) \leq C_p\text{-lim inf}_{x \rightarrow x_0} \psi_2(x)$. Then u_k is lower semicontinuous at x_0 , by the the first part of the proof. It then follows that

$$u(x_0) = u_k(x_0) \leq \liminf_{x \rightarrow x_0} u_k(x) \leq \liminf_{x \rightarrow x_0} u(x).$$

Hence u is lower semicontinuous at x_0 . Using this and the fact that $-u$ is a p -finely represented solution of the $\mathcal{K}_{-\psi_2, -\psi_1, -f}$ -problem, we get that

$$\begin{aligned} u(x_0) &= -(-u)(x_0) \geq -\liminf_{x \rightarrow x_0} (-u)(x) \\ &= \limsup_{x \rightarrow x_0} u(x) \geq \liminf_{x \rightarrow x_0} u(x) \geq u(x_0), \end{aligned}$$

which shows the continuity of u at x_0 .

From Theorem 3.3 we get the following immediate corollary. It extends Theorem 3.9 from Farnana [10] where the continuity of u in Ω was proved for real-valued continuous obstacles.

COROLLARY 3.4. *Let $\psi_1 : \Omega \rightarrow [-\infty, \infty)$ and $\psi_2 : \Omega \rightarrow (-\infty, \infty]$ be continuous (as \mathbb{R} -valued functions) at $x_0 \in \Omega$. Let u be a p -finely represented solution of the $\mathcal{K}_{\psi_1, \psi_2, f}$ -problem, where $f \in \mathbf{N}^{1,p}(\Omega)$. Then u is continuous at x_0 (as a real-valued function). In particular if ψ_1 and ψ_2 are extended real-valued continuous on Ω , then u is real-valued continuous on Ω .*

4. Hölder continuity of the solutions

It was shown by Kinnunen-Shanmugalingam [17] that p -harmonic functions are locally Hölder continuous. In this section we extend this result to obstacle problems with Hölder continuous obstacles. Similar results in \mathbb{R}^n have been obtained by Kilpeläinen-Ziemer [14] and Michael-Ziemer [22]. Although, in \mathbb{R}^n , p -harmonic functions are Lipschitz they need not be so in the general setting of metric spaces, see p. 149 in Koskela-Rajala-Shanmugalingam [20].

We shall need the following result, for a proof see Theorem 9.2 in A. Björn-Marola [6].

THEOREM 4.1. *If u is a nonnegative superminimizer in Ω , then there are $q > 0$ and $C > 0$ independent of u such that*

$$\left(\int_{B(x, 2r)} u^q d\mu \right)^{1/q} \leq C \operatorname{ess\,inf}_{B(x, r)} u$$

for every ball $B(x, r)$ such that $B(x, 20\lambda r) \subset \Omega$.

Here λ is the dilation constant in the Poincaré inequality.

We say that f is Hölder continuous at x_0 if there exist $r > 0$, $\alpha > 0$ and $C > 0$ such that, for all $x \in B(x_0, r)$, we have

$$|f(x) - f(x_0)| \leq Cd(x, x_0)^\alpha.$$

We also say that f is *locally Hölder continuous* in Ω if for each $x_0 \in \Omega$ there exist $r > 0$, $\alpha > 0$ and $C > 0$ (depending on x_0) such that for all $x, y \in B(x_0, r)$, we have

$$|f(x) - f(y)| \leq Cd(x, y)^\alpha.$$

Note that Hölder continuity of f at every $x_0 \in \Omega$ does not imply, in general, that f is locally Hölder continuous as the following example shows.

EXAMPLE 4.2. The function

$$f(x) = \begin{cases} x \sin e^{1/x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

is continuously differentiable in $\mathbb{R} \setminus \{0\}$. Hence it is Hölder continuous (even Lipschitz continuous) at every point in $\mathbb{R} \setminus \{0\}$. As $|f(x)| \leq |x|$ at every $x \in \mathbb{R}$ we also have that f is Hölder continuous at $x = 0$. Thus f is Hölder continuous at every $x \in \mathbb{R}$. But f is not Hölder continuous in any neighbourhood of zero. If f were Hölder continuous in a neighbourhood of zero then there would exist $\varepsilon > 0$, $C > 0$ and $\alpha > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in B(0, \varepsilon).$$

Let

$$x = \frac{1}{\log(2\pi n + \frac{\pi}{2})} \quad \text{and} \quad y = \frac{1}{\log(2\pi n - \frac{\pi}{2})}$$

for $n \in \mathbb{N}$ sufficiently large so that $0 < x < y < \varepsilon$. Then we have

$$(7) \quad |f(x) - f(y)| = x + y > y > \frac{1}{\log(2\pi n)}.$$

On the other hand we have

$$(8) \quad \begin{aligned} |x - y| = y - x &= \frac{\log(2\pi n + \frac{\pi}{2}) - \log(2\pi n - \frac{\pi}{2})}{\log(2\pi n + \frac{\pi}{2}) \log(2\pi n - \frac{\pi}{2})} \\ &\leq 2 \frac{\log(1 + \frac{1}{4n}) - \log(1 - \frac{1}{4n})}{(\log(2\pi n))^2} \\ &\leq \frac{2}{n}. \end{aligned}$$

It follows from (7) and (8) that for sufficiently large n we have

$$|f(x) - f(y)| \geq \frac{1}{\log(2\pi n)} > C \left(\frac{2}{n}\right)^\alpha \geq C|x - y|^\alpha,$$

a contradiction. Hence f is not Hölder continuous in any neighbourhood of zero.

THEOREM 4.3. *Let $\psi : \Omega \rightarrow \mathbb{R}$ be locally Hölder continuous and $f \in \mathbf{N}^{1,p}(\Omega)$. Let u be the continuous solution of the $\mathcal{K}_{\psi,f}$ -problem. Then u is locally Hölder continuous in Ω .*

PROOF. Let $B = B(x_0, R)$ and $20\lambda B \subset \Omega$, where λ is the dilation constant from the p -Poincaré inequality, be such that ψ is Hölder continuous in $2B$, with exponent α and constant C_0 . Let also, for $r > 0$,

$$\underline{u}(r) = \inf_{B(x_0,r)} u \quad \text{and} \quad \bar{u}(r) = \sup_{B(x_0,r)} u$$

and similarly for ψ . As $u - \underline{u}(20\lambda R)$ is a nonnegative superminimizer in $20\lambda B$ it follows from the weak Harnack inequality, Theorem 4.1, that there are $q > 0$ and $C_1 > 0$ such that

$$\begin{aligned} \underline{u}(R) - \underline{u}(20\lambda R) &= \inf_B (u - \underline{u}(20\lambda R)) \\ (9) \qquad \qquad \qquad &\geq \frac{1}{C_1} \left(\int_{2B} (u - \underline{u}(20\lambda R))^q d\mu \right)^{1/q}. \end{aligned}$$

We consider two cases. First assume that $\bar{\psi}(2R) \leq \underline{u}(20\lambda R)$ which implies that $\underline{u}(20\lambda R) \geq \psi$ in $2B$. The weak Harnack inequality, Theorem 2.5, then implies that, for some $C_2 > 0$, we have

$$\bar{u}(R) - \underline{u}(20\lambda R) = \sup_B (u - \underline{u}(20\lambda R)) \leq C_2 \left(\int_{2B} (u - \underline{u}(20\lambda R))^q d\mu \right)^{1/q}.$$

This and (9) imply that

$$(10) \qquad \bar{u}(R) - \underline{u}(20\lambda R) \leq C_1 C_2 (\underline{u}(R) - \underline{u}(20\lambda R)).$$

Next suppose that $\bar{\psi}(2R) > \underline{u}(20\lambda R)$. It follows that

$$(u - \bar{\psi}(2R))_+ \leq (u - \underline{u}(20\lambda R))_+ = u - \underline{u}(20\lambda R), \quad \text{in } 20\lambda B.$$

The weak Harnack inequality, Theorem 2.5, then imply that

$$\begin{aligned} \bar{u}(R) - \bar{\psi}(2R) &\leq C_2 \left(\int_{2B} (u - \bar{\psi}(2R))_+^q d\mu \right)^{1/q} \\ &\leq C_2 \left(\int_{2B} (u - \underline{u}(20\lambda R))^q d\mu \right)^{1/q}, \end{aligned}$$

for some $C_2 > 0$. This and (9) give

$$\bar{u}(R) - \bar{\psi}(2R) \leq C_1 C_2 (\underline{u}(R) - \underline{u}(20\lambda R)).$$

As ψ is α -Hölder continuous in $2B$ we have $\underline{u}(2R) \geq \underline{\psi}(2R) \geq \bar{\psi}(2R) - C_0(2R)^\alpha$. Then it follows that

$$\begin{aligned} \bar{u}(R) - \underline{u}(20\lambda R) &= \bar{u}(R) - \underline{u}(2R) + \underline{u}(2R) - \underline{u}(20\lambda R) \\ &\leq \bar{u}(R) - \underline{\psi}(2R) + \underline{u}(R) - \underline{u}(20\lambda R) \\ &\leq \bar{u}(R) - \bar{\psi}(2R) + C_0(2R)^\alpha + \underline{u}(R) - \underline{u}(20\lambda R) \\ &\leq (C_1 C_2 + 1)(\underline{u}(R) - \underline{u}(20\lambda R)) + C_0(2R)^\alpha. \end{aligned}$$

So in both cases we have, with $C_3 = C_1 C_2 + 1$,

$$(11) \quad \bar{u}(R) - \underline{u}(20\lambda R) \leq C_3(\underline{u}(R) - \underline{u}(20\lambda R)) + C_0(2R)^\alpha.$$

Now, suppose that

$$\underline{u}(R) - \underline{u}(20\lambda R) \leq \frac{1}{1 + C_3} (\bar{u}(R) - \underline{u}(20\lambda R)).$$

We use this and (11) together with the fact that $\underline{u}(R) \geq \underline{u}(20\lambda R)$ and $\bar{u}(R) \leq \bar{u}(20\lambda R)$ to conclude that

$$(12) \quad \begin{aligned} \bar{u}(R) - \underline{u}(R) &\leq C_3(\underline{u}(R) - \underline{u}(20\lambda R)) + C_0(2R)^\alpha \\ &\leq \frac{C_3}{1+C_3} (\bar{u}(20\lambda R) - \underline{u}(20\lambda R)) + C_0(2R)^\alpha. \end{aligned}$$

On the other hand if $\underline{u}(R) - \underline{u}(20\lambda R) > \frac{1}{1+C_3} (\bar{u}(R) - \underline{u}(20\lambda R))$, we have

$$\begin{aligned} \bar{u}(R) - \underline{u}(R) &= \bar{u}(R) - \underline{u}(20\lambda R) - (\underline{u}(R) - \underline{u}(20\lambda R)) \\ &\leq \bar{u}(R) - \underline{u}(20\lambda R) - \frac{1}{1+C_3} (\bar{u}(R) - \underline{u}(20\lambda R)) \\ &\leq \frac{C_3}{1+C_3} (\bar{u}(20\lambda R) - \underline{u}(20\lambda R)). \end{aligned}$$

Letting $\text{osc}_{\tau_B} u = \sup_{\tau_B} u - \inf_{\tau_B} u$, we thus have in both cases that

$$(13) \quad \text{osc}_B u \leq \gamma \text{osc}_{20\lambda B} u + C_0(2R)^\alpha,$$

where $\gamma = \frac{C_3}{1+C_3} < 1$ is independent of R and α . To complete the proof we iterate inequality (13). Choose $0 < R < 1/2$ and

$$0 < \alpha' < \min\{\alpha, -\log \gamma / \log 20\lambda\}.$$

Then, we get that $(20\lambda)^{\alpha'} \gamma < 1$, which will be needed below, and that

$$(14) \quad \text{osc}_B u \leq \gamma \text{osc}_{20\lambda B} u + C_0(2R)^{\alpha'}.$$

Let $r \in (0, R)$ be arbitrary and $j \in \mathbb{N}$ be such that $(20\lambda)^{j-1} \leq R/r < (20\lambda)^j$. Applying (14) to the balls $B_i = B(x_0, (20\lambda)^i r)$, $i = 0, 1, 2, \dots, j - 1$, we obtain

$$\begin{aligned} \text{osc}_{B_0} u &\leq \gamma \text{osc}_{B_1} u + C_0(2r)^{\alpha'} \\ &\leq \gamma^2 \text{osc}_{B_2} u + C_0 2^{\alpha'} ((20\lambda)^{\alpha'} \gamma + 1) r^{\alpha'} \\ &\dots\dots\dots \\ &\leq \gamma^{j-1} \text{osc}_{B_{j-1}} u + C_0 2^{\alpha'} (((20\lambda)^{\alpha'} \gamma)^{j-2} + \dots + 1) r^{\alpha'} \\ &\leq \gamma^{j-1} \text{osc}_{B_{j-1}} u + \frac{C_0 2^{\alpha'}}{1 - (20\lambda)^{\alpha'} \gamma} r^{\alpha'}, \end{aligned}$$

where we use the fact that $(20\lambda)^{\alpha'} \gamma < 1$. By the choice of j we conclude that

$$\begin{aligned} \gamma^{j-1} &= (20\lambda)^{(j-1) \log \gamma / \log 20\lambda} = (20\lambda)^{\alpha''} ((20\lambda)^j)^{-\alpha''} \\ &\leq (20\lambda)^{\alpha''} \left(\frac{R}{r}\right)^{-\alpha''} = (20\lambda)^{\alpha''} \left(\frac{r}{R}\right)^{\alpha''}, \end{aligned}$$

where $\alpha'' = -\log \gamma / \log 20\lambda > 0$. Hence we get, using that $r/R < 1$ and $\alpha' \leq \alpha''$,

$$(15) \quad \begin{aligned} \text{osc}_{B(x_0, r)} u &\leq (20\lambda)^{\alpha''} \left(\frac{r}{R}\right)^{\alpha''} \text{osc}_{B(x_0, R)} u + \frac{C_0 2^{\alpha'}}{1 - (20\lambda)^{\alpha'} \gamma} r^{\alpha'} \\ &\leq \left(\frac{(20\lambda)^{\alpha''}}{R^{\alpha''}} \text{osc}_{B(x_0, R)} u + \frac{C_0 2^{\alpha'}}{1 - (20\lambda)^{\alpha'} \gamma}\right) r^{\alpha'}. \end{aligned}$$

Now, fix $r \in \mathbb{R}$ such that $0 < 2r < R < 1$. Then, for $x, y \in B(x_0, r)$, we consider two cases. First let $r \leq d(x, y) < 2r < 1$. This and (15) imply that

$$(16) \quad \begin{aligned} |u(x) - u(y)| &\leq \text{osc}_{B(x_0, r)} u \\ &\leq \left(\frac{(20\lambda)^{\alpha''}}{R^{\alpha''}} \text{osc}_{B(x_0, R)} u + \frac{C_0 2^{\alpha'}}{1 - (20\lambda)^{\alpha'} \gamma}\right) d(x, y)^{\alpha'}. \end{aligned}$$

Next, if $d(x, y) < r < 1$, then choose $s > 0$ such that $s/2 < d(x, y) < s < r < R/2$. Then (15) with $B(x_0, r)$ and $B(x_0, R)$ replaced by $B(y, s)$ and

$B(y, R/2)$ implies that

$$\begin{aligned}
 |u(x) - u(y)| &\leq \operatorname{osc}_{B(y,s)} u \\
 &\leq \left(\frac{(20\lambda)^{\alpha''}}{(R/2)^{\alpha'}} \operatorname{osc}_{B(y,R/2)} u + \frac{C_0 2^{\alpha'}}{1 - (20\lambda)^{\alpha'} \gamma} \right) s^{\alpha'} \\
 (17) \quad &\leq 2^{\alpha'} \left(\frac{(20\lambda)^{\alpha''}}{R^{\alpha'}} \operatorname{osc}_{B(x_0,R)} u + \frac{C_0}{1 - (20\lambda)^{\alpha'} \gamma} \right) s^{\alpha'} \\
 &\leq 4^{\alpha'} \left(\frac{(20\lambda)^{\alpha''}}{R^{\alpha'}} \operatorname{osc}_{B(x_0,R)} u + \frac{C_0}{1 - (20\lambda)^{\alpha'} \gamma} \right) d(x, y)^{\alpha'}.
 \end{aligned}$$

Thus we conclude from (16) and (17) that we can find $C > 0$ such that for $x, y \in B(x_0, r)$ we have

$$|u(x) - u(y)| \leq C d(x, y)^{\alpha'},$$

where C depends only on R , $\alpha'' = -\log \gamma / \log 20\lambda$, the oscillation of u on $B(x_0, R)$ and the Hölder exponent of ψ . Since $x_0 \in \Omega$ was arbitrary we see that u is locally Hölder continuous in Ω .

For the double obstacle problem we have the following weaker result.

THEOREM 4.4. *Let $\psi_i : \Omega \rightarrow \mathbf{R}$, $i = 1, 2$, be locally Hölder continuous and $f \in \mathbf{N}^{1,p}(\Omega)$. Let u be the continuous solution of the $\mathcal{H}_{\psi_1, \psi_2, f}$ -problem. Then u is Hölder continuous in Ω at every $x_0 \in \Omega$.*

Note that Hölder continuity of u at every $x_0 \in \Omega$ does not imply, in general, that u is locally Hölder continuous as Example 4.2 shows.

PROOF. Assume first that $\psi_1(x_0) < \psi_2(x_0)$. Note that either $u(x_0) < \psi_2(x_0)$ or $u(x_0) > \psi_1(x_0)$. If $u(x_0) < \psi_2(x_0)$ find $\varepsilon > 0$ and $B = B(x_0, r)$ such that $u(x_0) + \varepsilon \leq \psi_2(x_0) - \varepsilon$ and that

$$\sup_B u \leq \inf_B u + \varepsilon =: k \quad \text{and} \quad \inf_B \psi_2 \geq \psi_2(x_0) - \varepsilon.$$

It follows that $k \geq u$ in B and that

$$k = \inf_B u + \varepsilon \leq u(x_0) + \varepsilon \leq \psi_2(x_0) - \varepsilon \leq \inf_B \psi_2 \leq \psi_2 \quad \text{in } B.$$

Lemma 2.4 then implies that $u = \min\{u, k\}$ is a superminimizer in B . Thus u is a solution of the $\mathcal{H}_{u,u}(B)$ -problem. Two applications of the comparison Lemma 2.3 between the continuous solution of the $\mathcal{H}_{\psi_1, u}(B)$ -problem and u , once as u is the continuous solution of the $\mathcal{H}_{\psi_1, \psi_2, u}(B)$ -problem and the other as u is the solution of the $\mathcal{H}_{u,u}(B)$ -problem, then implies that u is the

continuous solution of the $\mathcal{K}_{\psi_1, u}(B)$ -problem. It follows from Theorem 4.3 that there exists a ball $B(x_0, r') \subset B$ such that

$$|u(x) - u(y)| \leq Cd(x, y)^\alpha \quad \text{for all } x, y \in B(x_0, r').$$

Similarly, if $u(x_0) > \psi_1(x_0)$ we conclude that $-u$ is the continuous solution of the $\mathcal{K}_{-\psi_2, -u}(B(x_0, s))$ -problem, for some $s > 0$. Hence u is Hölder continuous in $B(x_0, s')$, for some $0 < s' < s$.

Next, if $\psi_1(x_0) = \psi_2(x_0)$, then $u(x_0) = \psi_1(x_0) = \psi_2(x_0)$, by continuity. Fix $B = B(x_0, r)$ such that ψ_1 and ψ_2 are Hölder continuous in B . Then using that ψ_1 is Hölder continuous in B we get that, for $x \in B$,

$$u(x_0) - u(x) \leq \psi_1(x_0) - \psi_1(x) \leq C_1 d(x_0, x)^{\alpha_1},$$

for some $C_1 > 0$ and $\alpha_1 > 0$. Similarly, using that ψ_2 is Hölder continuous in B , we obtain

$$u(x) - u(x_0) \leq \psi_2(x) - \psi_2(x_0) \leq C_2 d(x_0, x)^{\alpha_2},$$

for some $C_2 > 0$ and $\alpha_2 > 0$. Hence u is Hölder continuous at x_0 .

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