Journal Of Education Sciences
Ten Issue - September 2022

A numerical solution of Time Fractional Partial Differential Equations

Samia Atallah[1], Ebtesam Alousta [2], and Rem Arebi [3]

1 Department of Mathematics ,Janzour Faculty of education,Tripoli University, Libya.

2 Department of Mathematics Faculty of science,University of Tripoli, Libya.

3 Department of Mathematics ,anzour Faculty of education ,University of Tripoli, Libya.

Abstract

Fractional partial differential equations (FPDEs) have many applications in areas such as diffusion processes, *electromagnetics*, electrochemistry, material science and turbulent flow. In recent years, people start to consider the numerical methods for solving fractional partial differential equations. The numerical methods include finite difference method, finite element method and the spectral method. In this paper, mainly consider the finite element method, for the time fractional partial differential equation. And consider time discretization. This paper obtained the optimal error estimates in time. The numerical examples demonstrate that the numerical results are consistent with the theoretical results.

Keywords:

- Fractional partial differential equations.
- Finite element method.
- Caputo fractional derivative.
- Riemann-Liouville fractional derivative.

1 Introduction

Time fractional partial differential equations have been used in various areas such as, diffusion processes material science, turbulent flow, electromagnetics, electrochemistry, etc.[1], [2], [3], [4], [**5**],. Analytical solutions of time fractional partial differential equations have been focused on using Green's functions or Fourier-Laplace transforms [6], [7], [8], [19],[20]. Numerical methods for fractional partial differential equations were considered by some authors. Lin and Xu [9] proposed the numerical solution for a time-fractional diffusion equation.

et al. [10] used the finite difference method in both space and time and analysed the stability condition. Sun and Wu [11] advised a finite difference method for the fractional diffusion-wave equation. Ervin and Roop [12] employed finite element method to get the variational solution of the fractional advection dispersion equation, where the fractional derivative based on the space, related to the nonlocal operator. Li et al. [13] studied a time fractional partial differential equation by using the finite element method and obtained error estimates in both semi-discrete Liu and fully discrete cases.

Jiang et al. [14] considered a high-order finite element method for the time fractional partial differential equations and proved the optimal order error estimates.In [15], an unconditionally stable finite element (FEM) approach for solving a one-dimensional Caputo-type fractional differential equation with singularity at the boundary was presented.

The paper is organized as follows. In Section 2, we introduce the basic definition of fractional caculus. In Section 3, we consider the finite element method, In Section 4, we obtained the optimal order error estimates in time discretization. Finally in Section 5, we give two numerical examples and show that the numerical results are consistent with the theoretical results.

2 Basic definitions

In this section, we set up notations, basic definitions and main properties of RiemannLiouville Integral, and the relation between Riemann-Liouville integral and Caputo fractional derivative is also given.

Definition 2.1 ([16] pp.33)

The Riemann-Liouville fractional integral of order $0 < \alpha < 1$, is denoted by the expression:

$$ {}_a^R D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau . \qquad (2.1)$$

Definition 2.2 ([16, pp.35])

Let $\alpha > 0$, the Riemann-Liouville fractional derivative is defined with $n-1 < \alpha \le n$ by,

$$ {}_a^R D_t^{\alpha} f(t) = D^n \left[D_t^{\alpha-n} f(t)\right] = D^n \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau, \qquad (2.2)$$

where $D^n = \frac{d^n}{dt^n}$ denotes the standard nth derivative.

Definition 2.3

The Caputo fractional derivative of order $\alpha > 0$ is takes the form:

$$ {}_a^C D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} [D^n f(\tau)]\,d\tau, & \text{where } n-1 < \alpha < n, \\ \frac{d^n}{dt^n} f(t), & \text{where } \alpha = n. \end{cases} \qquad (2.3)$$

The relationship between the Caputo derivative and the Riemann-Liouville derivative is the following, K. Diethelm [17],

Definition 2.4 [6]

The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$$

Definition 2.5 [6]

$$\beta(p,q) = \int_0^1 (1-u)^{p-1} u^{q-1}\,du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p,q \in \mathbb{R}_+. \qquad (2.4)$$

Definition 2.6 [6]

The Mittag-Leffler function is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha > 0, \beta > 0. \qquad (2.5)$$

There are some relationships to other functions given by:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z},$$

3 Finite element method for solving FPDEs

This section considers how to solve the one dimension time fractional partial differential equation by using finite element method.

Consider the time fractional partial differential equation with the Caputo type (see definition2.3)

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \qquad 0 \le x \le 1, \quad 0 < t \le T, \qquad (3.1)$$

Initial condition: $u(0,x) = u_0, \qquad 0 \le x \le 1, \qquad (3.2)$

Boundary condition: $u(t,0) = u(t,1) = 0, \qquad 0 < t < T, \; 0 < \alpha < 1. \qquad (3.3)$

We know that

$$ {}_0^C D_t^\alpha u(x,t) = {}_0^R D_t^\alpha [u(x,t) - u_0] $$

Hence the equations (3.1)-(3.3) reduces to

$$ {}_0^R D_t^\alpha [u(t,x) - u_0] - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad 0 \le x \le 1, \quad 0 < t < T, \qquad (3.4) $$

$$ u(t,0) = u(t,1) = 0, \qquad 0 \le t \le T. \qquad (3.5) $$

Here ${}_0^R D_t^\alpha u(t,x)$, denotes the Riemann-Liouville fractional derivative (see definition 2.2) with respect to the time variable t defined by

$$ {}_0^R D_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(\tau,x)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1, \qquad (3.6) $$

Where Γ denotes the Gamma function (see definition 2.4).

The variational form is to find $u(t) \in H_0^1(0,1)$ such that

$$ ({}_0^R D_t^\alpha [u(t,x) - u_0], v(x)) + (\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}) = (f,v), \quad \forall\, v \in H_0^1, \qquad (3.7) $$

$H_0^1(0,1) = H_0^1 = \{v(x) \mid v(x)\ and\ v'(x)$ are square integrable on $(0,1)$,

i.e, $\{v \in L^2(\Omega), v' \in L^2(\Omega)\ and\ v(0) = v(1) = 0\}$.

The inner product in $L^2(0,1)$ is defined by

$$ (f,g) = \int_0^1 f(x) g(x) dx, \quad \forall\, f,g \in L^2(0,1). $$

The finite element method is to find a solution $u_h(t) \in S_h$. Such that

$$ {}_0^R D_t^\alpha [u(t,x) - u_0], \chi) + \left(\frac{\partial u_h}{\partial x}, \frac{\partial v}{\partial x}\right) = (f,\chi), \quad \forall\, \chi \in S_h \qquad (3.8) $$

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a time discretization

$$ {}_0^R D_t^\alpha y(t)|_{t=t_j} = \Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj} [\, y(t_j - t_k) - y(0)] + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g), \qquad (3.9) $$

Where

$$ \Gamma(2-\alpha)\omega_{kj} = \begin{cases} 1, & \text{for } k = 0, \\ -2k^{1-\alpha} + (k-1)^{1-\alpha} + (k+1)^{1-\alpha}, & \text{for } k = 1,2,\dots,j-1, \\ -(\alpha-1)k^{-\alpha} + (k-1)^{1-\alpha} - k^{1-\alpha}, & \text{for k = j,} \end{cases} $$

And the remainder term $R_j(g)$ satisfies

$$ \|R_j(g)\| \le C j^{\alpha-2} \sup_{0 \le t \le T} \|y''(t_j - t_j\omega)\|, \quad 0 < \omega < 1. $$

Denote $U^j \approx u_h(t_j)$ as the approximation of $u_h(t)$ at $t = t_j$.

Then we can define the following time discretization, with $f^j = f(t_j)$,

$$ (\Delta t^{-\alpha} \sum_{k=0}^j \omega_{kj} (U^{j-k} - u_0), \chi) + \left(\frac{\partial U^j}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^j, \chi), \quad j = 0,1,2,\dots \quad \forall \chi \in S_h \qquad (3.10) $$

Or

$$ \Delta t^{-\alpha} \omega_{0j} (U^j, \chi) + \left(\frac{\partial U^j}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^j, \chi) - (\Delta t^{-\alpha} \sum_{k=1}^j \omega_{kj} (U^{j-k} - u_0), \chi) \qquad (3.11) $$

$$ + \Delta t^{-\alpha} \omega_{0j} (u_0, \chi), \quad \forall \chi \in S_h, \text{ for } j = 0,1,2,\dots,n $$

Now find U^j, for $j = 0,1,2,\dots,n$.

Step 1: if we set j=0, then we will get $U^0 = u_0$

Step 2: we put j=1, then we have

$$ \Delta t^{-\alpha} \omega_{01} (U^1, \chi) + \left(\frac{\partial U^1}{\partial x}, \frac{\partial \chi}{\partial x}\right) = (f^1, \chi) - (\Delta t^{-\alpha} \sum_{k=1}^j \omega_{k1} (U^0 - u_0), \chi) $$

$$ + \Delta t^{-\alpha} \omega_{01} (u_0, \chi), \quad \forall \chi \in S_h. \qquad (3.12) $$

And we know that $U^1 = \sum_{\ell=1}^{M-1} \alpha_\ell \, \phi_\ell(x)$, where $\phi_1(x), \phi_2(x), \dots, \phi_{M-1}(x)$, are the basis functions of the finite element space S_h, and then we have

$$\Delta t^{-\alpha}\omega_{01}(\sum_{\ell=1}^{M-1} \alpha_\ell(\phi_\ell(x),\chi) + \sum_{\ell=1}^{M-1} \alpha_\ell \left(\frac{\partial\phi_\ell}{\partial x},\frac{\partial\chi}{\partial x}\right) = (f^1,\chi) - (\Delta t^{-\alpha}\omega_{11}(U^0 - u_0),\chi)$$
$$+\Delta t^{-\alpha}\omega_{01}(u_0,\chi),\quad \forall\chi \in S_h,$$

Choose $\chi = \phi_m(x)$, for $m = 1,2,\dots, M-1$, and we substitute into equation (3.12)

$$\Delta t^{-\alpha}\omega_{01}\left(\sum_{\ell=1}^{M-1} \alpha_\ell(\phi_\ell(x),\phi_m(x)\right) + \sum_{\ell=1}^{M-1} \alpha_\ell \left(\frac{\partial\phi_\ell(x)}{\partial x},\frac{\partial\phi_m(x)}{\partial x}\right) = \left(f^1,\phi_m(x)\right) -$$
$$(\Delta t^{-\alpha}\omega_{11}(U^0 - u_0),\phi_m(x)) + \Delta t^{-\alpha}\omega_{01}\left(u_0,\phi_m(x)\right), \qquad (3.13)$$

Then we get

$$\Delta t^{-\alpha}\omega_{01}(\mathrm{Mass} * \mathrm{V}^1) + \mathrm{stiff} * \mathrm{V}^1 = \mathrm{F}^1 - \Delta t^{-\alpha}\omega_{11}\mathrm{V}^0 + \Delta t^{-\alpha}\sum_{k=0}^{1}\omega_{k1}u^0 \qquad (3.14)$$

Denote

$$\mathrm{Mass}=\begin{bmatrix} (\phi_1,\phi_1) & (\phi_2,\phi_1) & \dots & (\phi_{M-1},\phi_1) \\ (\phi_1,\phi_2) & (\phi_2,\phi_2) & \dots & (\phi_{M-1},\phi_2) \\ \vdots & \vdots & & \vdots \\ (\phi_1,\phi_{M-1}) & (\phi_2,\phi_{M-1}) & \dots & (\phi_{M-1},\phi_{M-1}) \end{bmatrix}$$

$$\mathrm{Stiff}=\begin{bmatrix} \left(\frac{\partial\phi_1}{\partial x},\frac{\partial\phi_1}{\partial x}\right) & \left(\frac{\partial\phi_2}{\partial x},\frac{\partial\phi_1}{\partial x}\right) & \cdots & \left(\frac{\partial\phi_{M-1}}{\partial x},\frac{\partial\phi_1}{\partial x}\right) \\ \left(\frac{\partial\phi_1}{\partial x},\frac{\partial\phi_2}{\partial x}\right) & \left(\frac{\partial\phi_2}{\partial x},\frac{\partial\phi_2}{\partial x}\right) & \cdots & \left(\frac{\partial\phi_{M-1}}{\partial x},\frac{\partial\phi_2}{\partial x}\right) \\ \vdots & \vdots & & \vdots \\ \left(\frac{\partial\phi_1}{\partial x},\frac{\partial\phi_{M-1}}{\partial x}\right) & \left(\frac{\partial\phi_2}{\partial x},\frac{\partial\phi_{M-1}}{\partial x}\right) & \cdots & \left(\frac{\partial\phi_{M-1}}{\partial x},\frac{\partial\phi_{M-1}}{\partial x}\right) \end{bmatrix}$$

$$\mathrm{F}^1 = \begin{pmatrix} (f^1,\phi_1) \\ (f^1,\phi_2) \\ \vdots \\ (f^1,\phi_{M-1}) \end{pmatrix},\quad \mathrm{V}^1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{M-1} \end{pmatrix},\quad \mathrm{V}^0 = \begin{pmatrix} (U^0,\phi_1) \\ (U^0,\phi_2) \\ \vdots \\ (U^0,\phi_{M-1}) \end{pmatrix},\quad u^0 = \begin{pmatrix} (u_0,\phi_1) \\ (u_0,\phi_2) \\ \vdots \\ (u_0,\phi_{M-1}) \end{pmatrix}$$

Step 3: Let us compute U^2.

To compute U^2 we set $j = 2$ into equation (2.7), then we have

$$\Delta t^{-\alpha}\omega_{02}(U^2,\chi) + \left(\frac{\partial U^2}{\partial x},\frac{\partial\chi}{\partial x}\right) = (f^2,\chi) - (\Delta t^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\,(U^{2-k} - u_0),\chi) \qquad (3.15)$$
$$+\Delta t^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\,(u_0,\chi),\quad \forall\chi \in S_h.$$

Let $U^2 = \sum_{\ell=1}^{M-1} \alpha_\ell\,\phi_\ell(x)$ as we have done before. Then the equation (3.15) is equivalent to

$$\Delta t^{-\alpha}\omega_{02}(\sum_{\ell=1}^{M-1} \alpha_\ell(\phi_\ell(x),\phi_m(x))) + \sum_{\ell=1}^{M-1} \alpha_\ell \left(\frac{\partial\phi_\ell}{\partial x},\frac{\partial\phi_m(x)}{\partial x}\right) = \left(f^2,\phi_m(x)\right) -$$
$$(\Delta t^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\,(U^{2-k} - u_0),\phi_m(x)) + (\Delta t^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\left(u_0,\phi_m(x)\right), \quad (3.16)$$

And finally we get

$$\Delta t^{-\alpha}\omega_{02}(\mathrm{Mass} * \mathrm{V}^2) + \mathrm{stiff} * \mathrm{V}^2 = \mathrm{F}^2 - (\Delta t)^{-\alpha}\sum_{k=1}^{2}\omega_{k2}\,\mathrm{V}^{2-\mathrm{k}} + (\Delta t)^{-\alpha}\sum_{k=0}^{2}\omega_{k2}u^0$$

Denote

$$\mathrm{F}^2 = \begin{pmatrix} (f^2,\phi_1) \\ (f^2,\phi_2) \\ \vdots \\ (f^2,\phi_{M-1}) \end{pmatrix},\quad \mathrm{V}^2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{M-1} \end{pmatrix},\quad \mathrm{V}^0 = \begin{pmatrix} (U^0,\phi_1) \\ (U^0,\phi_2) \\ \vdots \\ (U^0,\phi_{M-1}) \end{pmatrix},\quad u^0 = \begin{pmatrix} (u_0,\phi_1) \\ (u_0,\phi_2) \\ \vdots \\ (u_0,\phi_{M-1}) \end{pmatrix},$$

Step 4: We continue this process to obtain $\boldsymbol{U^n \approx u_h(t_n)}$, the approximation solution of $\boldsymbol{u_h(t_n)}$ at time $t = t_n$ for $n = 0,1,2,\dots$.

Journal Of Education Sciences
Ten Issue - September 2022

$$\Delta t^{-\alpha}\omega_{0n}(U^n,\chi)+\left(\frac{\partial U^n}{\partial x},\frac{\partial \chi}{\partial x}\right)=(f^n,\chi)-(\Delta t^{-\alpha}\textstyle\sum_{k=1}^{M-1}\omega_{kn}\,(U^{n-k}-u_0),\chi)$$
$$+\Delta t^{-\alpha}\textstyle\sum_{k=1}^{M-1}\omega_{kn}\,(u_0,\chi),\quad \forall\chi\in S_h. \qquad (3.17)$$

To calculate U^n we have to follow the same steps as in step 2 and 3. Based on the idea above, we can design the algorithm of the finite element method and solve the system by using MATLAB software.

4 Time discretization

In this section will consider the error estimate of the finite element approximation and the stability result of the following fractional partial differential equation with the Riemann-Liouville type

$$ {}_0^RD_t^\alpha[u(t,x)-u_0]-\frac{\partial^2 u(t,x)}{\partial x^2}=f(t,x),\quad 0\le x\le 1,\quad t>0, \qquad (4.1)$$
$$u(t,0)=u(t,1)=0,\qquad 0\le t\le 1,\ 0<\alpha<1. \qquad (4.2)$$

Define $A=\frac{\partial^2}{\partial x^2}, D(A)=H_0^1\cap H^2=\{u\ |u',u''\in L_2(0,1),u(0)=u(1)=0\}$,

Where $L_2(0,1)=\left\{f:\int_0^1 f^2dx<\infty\right\}$, then the system (4.1)-(4.2) can be written in the abstract form

FODE: $\quad {}_0^RD_t^\alpha[u(t)-u_0]+Au(t)=f(t),\ 0\le x\le 1,\ \ t>0. \qquad (4.3)$

First let us consider the error estimates for the time discretization of the abstract problem (4.3).

Let $0=t_0<t_1<\cdots<t_n=1$ be the time partition of [0, 1]. Then, for fixed $t_j, t_j=\frac{j}{n}$, $j=1,2,\dots,n,\ \Delta t=1/n$, is the time step, we have

$$ {}_0^RD_t^\alpha[u(t)-u_0]|_{t=t_j}=\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{j}\alpha_{kj}[\,u(t_j-t_k)-u_0]+\ R_j(g),$$

Where

$$\alpha(1-\alpha)j^{-\alpha}\alpha_{kj}=\begin{cases}-1, & \text{for } k=0,\\ 2k^{1-\alpha}-(k-1)^{1-\alpha}-(k+1)^{1-\alpha}, & \text{for } k=1,2,\dots,j-1,\\ (\alpha-1)k^{-\alpha}-(k-1)^{1-\alpha}+k^{1-\alpha}, & \text{for k = j,}\end{cases}$$

and

$$\|R_j(g)\|\le Cj^{\alpha-2}\sup_{0\le t\le 1}\|u_{tt}''(t_j-t_jt)\|,$$

where

$$\sup_{0\le t\le 1}\|u_{tt}''(t_j-t_jt)\|=\|u_{tt}''(t_j-t_jt)\|_{L_\infty}.$$

$$\frac{t_j^{-\alpha}}{\Gamma(-\alpha)}\left[\textstyle\sum_{k=0}^{j}\alpha_{kj}[\,u(t_j-t_k)-u_0]+\ R_j(g)\right]+Au(t_j)=f(t_j), \qquad (4.4)$$

Rewriting equation (4.4) when $k=0$ we obtain

$$[\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A]u(t_j)=t_j^\alpha\Gamma(-\alpha)f_j-\textstyle\sum_{k=1}^{j}\alpha_{kj}u(t_j-t_k)+\sum_{k=0}^{j}\alpha_{kj}u_0-\ R_j(g) \qquad (4.5)$$

Denote $U^j\approx u(t_j)$ as the approximation of $u(t_j)$. We can define the following time stepping method

$$[\alpha_{0j}+t_j^\alpha\Gamma(-\alpha)A]U^j=t_j^\alpha\Gamma(-\alpha)f_j-\textstyle\sum_{k=0}^{j}\alpha_{kj}U^{j-k}+\sum_{k=0}^{j}\alpha_{kj}U^0 \qquad (4.6)$$

Let $\varepsilon^j=U^j-u(t_j)$ denotes the error. Then we have the following error estimate:

Theorem 4.1 Let U^j and $u(t_j)$ be the solution of (4.1)-(4.2), then we have

$$\varepsilon^j\le C\Delta t^{2-\alpha}+\|u(t_0)-U^0\|,\qquad \text{where } \varepsilon_0=\|u(t_0)-U^0\|$$

Proof:

Subtracting (4.6) from (4.4), we get the error equation

$$(\alpha_{0j}+t_j^{\alpha}\Gamma(-\alpha)A)\varepsilon_j = -\sum_{k=0}^{j}\alpha_{kj}\,\varepsilon^{j-k} - R_j, \quad (4.7)$$

Rewriting (4.7), then we have

$$\varepsilon^{j} = (-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\left(\sum_{k=0}^{j}\alpha_{kj}\,\varepsilon^{j-k} + R_j\right), \quad (4.8)$$

Where

$$\| R_j\| \le \sup_{0\le t\le 1}\|u''_{tt}(t_j - t_j t)\|,$$

Taking the L_2 norm for (4.8), we get

$$\|\varepsilon^{j}\| \le \|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\|\left[\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\| + \| R_j\|\right] \quad (4.9)$$

Note that A is a positive definite elliptic operator. The eigenvalues of A are $\lambda_j = j^2\pi^2$, $j = 1,2,3,\ldots$. For any function $g(x)$ we have, by spectral method,

$$\|g(A)\| = \sup_{\lambda>0}|g(\lambda)|$$

From (4.9), we have

$$\|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\| = \left\|\left(\frac{1}{\alpha(1-\alpha)j^{-\alpha}} - t_j^{\alpha}\Gamma(-\alpha)A\right)^{-1}\right\|$$

$$= \left\|\alpha(1-\alpha)j^{-\alpha}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)A\right)^{-1}\right\|$$

$$\|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\| = \alpha(1-\alpha)j^{-\alpha}\sup_{\lambda>0}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)\lambda\right)^{-1}$$

Here $\alpha(1-\alpha)j^{-\alpha}\alpha_{kj} =$

$$\begin{cases} -1, & \text{for } k = 0,\\ 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}, & \text{for } k = 1,2,\ldots,j-1,\\ (\alpha-1)k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha}, & \text{for k = j.}\end{cases} \quad (4.10)$$

Since $\Gamma(-\alpha) > 0$, we find that

$$\sup_{\lambda>0}\left(1-\alpha(1-\alpha)j^{-\alpha}t_j^{\alpha}\Gamma(-\alpha)\lambda\right)^{-1} \le 1.$$

Hence

$$\|(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)^{-1}\| \le \alpha(1-\alpha)j^{-\alpha}.$$

Thus (4.9) implies that

$$\|\varepsilon^{j}\| \le \alpha(1-\alpha)j^{-\alpha}\left[\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\| + Cj^{\alpha}n^{-2}\sup_{0\le t\le 1}\|u''_{tt}(t_j - t_j t)\|\right], \quad (4.11)$$

Where we use the fact, noting that $t_n = n\cdot\Delta t = 1$,

$$u''_{tt}(t_j - t_j t) = u''(t_j - t_j t)\cdot t_j^2 = j^2\,\Delta t^2 u''(t_j - t_j t)$$
$$= j^2\,n^{-2}u''(t_j - t_j t).$$

Further (4.9) can be written into the form

$$\|\varepsilon^{j}\| \le \alpha(1-\alpha)j^{-\alpha}Cn^{-2}\|u''\|_{L_\infty} + \alpha(1-\alpha)j^{-\alpha}\sum_{k=0}^{j}\alpha_{kj}\,\|\varepsilon_{j-k}\|.$$

Denote $a = \alpha(1-\alpha)j^{-\alpha}Cn^{-2}\|u''\|_{L_\infty}$.

Choose: $j = 1$.

Then we have

$$\|\varepsilon^{1}\| \le a + \alpha(1-\alpha)1^{-\alpha}\alpha_{11}\|\varepsilon_0\|$$

Journal Of Education Sciences
Ten Issue - September 2022

$$= a \cdot d_1 + r_1\|\varepsilon_0\|,$$

Here $d_1 = 1, r_1 = \alpha(1-\alpha)1^{-\alpha}\alpha_{11}$.

Choose $j = 2$, we get

$$\|\varepsilon^2\| \le a + \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\,\|\varepsilon_{2-k}\| + \alpha_{22}\|\varepsilon_0\|\right]$$

$$\|\varepsilon^2\| \le a + \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\,(ad_{2-k} + r_{2-k}\|\varepsilon_0\|) + \alpha_{22}\|\varepsilon_0\|\right]$$

$$= a\left[1 + \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, d_{2-k}\right]$$

$$+ \alpha(1-\alpha)2^{-\alpha}\left[\sum_{k=1}^{2-1} \alpha_{k2}\, r_{2-k}\right]\|\varepsilon_0\|$$

$$= ad_2 + r_2\|\varepsilon_0\|.$$

Here

$$d_2 = 1 + \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, d_{2-k}$$

$$r_2 = \alpha(1-\alpha)2^{-\alpha}\sum_{k=1}^{2-1} \alpha_{k2}\, r_{2-k}, r_0 = 1$$

In general, we obtain

$$\|\varepsilon^j\| \le ad_j + r_j\|\varepsilon_0\|, \qquad j = 1,2,3, \ldots, \qquad (4.12)$$

Next we will find d_j and r_j, where

$$d_j = 1 + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1} \alpha_{kj}\, d_{j-k}, \quad j = 2,3, \ldots,$$

$$r_j = \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j} \alpha_{kj}\, r_{j-k}, \quad j = 1,2,3, \ldots, \quad r_0 = 1.$$

Lemma 4.1 [18] for $0 < \alpha < 1$, let the sequence $\{d_j\}$, $j = 1,2, \ldots$ be given by $d_1 = 1$ and

$$d_j = 1 + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1} \alpha_{kj}\, d_{j-k}, \quad j = 2,3, \ldots,$$

Then,

$$1 \le d_j \le \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} j^{\alpha}, \quad j = 1,2, \ldots.$$

Lemma 4.2 Assume that if $r_0 = 1,\ r_j = \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j} \alpha_{kj}\, r_{j-k}, \quad j = 1,2,3, \ldots,$

Then,

$$r_j \le 1.$$

Proof:

Step 1: If we have $r_0 = 1$. Then

$$r_1 = \alpha(1-\alpha)1^{-\alpha}\alpha_{11}r_0 = \alpha(1-\alpha)\alpha_{11} = (\alpha-1)1^{-\alpha} + 1^{1-\alpha} = \alpha < 1.$$

Step 2: Assume that $r_j < \alpha < 1$, thet

$$r_{j+1} = \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\, r_{j-k} \le \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\cdot 1$$

$$r_{j+1} \ = \alpha(1-\alpha)j^{-\alpha}\left[\sum_{k=0}^{j}\alpha_{kj} - \alpha_{0j}\right] = \alpha(1-\alpha)j^{-\alpha}\left[-\frac{1}{\alpha}+\frac{1}{\alpha(1-\alpha)j^{-\alpha}}\right]$$

$$= \alpha(1-\alpha)j^{-\alpha}\left[-\frac{1}{\alpha}+\frac{1}{\alpha(1-\alpha)j^{-\alpha}}\right] \ < \ \alpha(1-\alpha)j^{-\alpha}\frac{1-(1-\alpha)j^{-\alpha}}{\alpha(1-\alpha)j^{-\alpha}}$$

Hence, we get

$$r_{j+1} < \ \alpha(1-\alpha)j^{-\alpha}\frac{1}{\alpha(1-\alpha)j^{-\alpha}} = 1.$$

The proof of the Lemma 4.2 is complete.

By using Lemma 4.1 and Lemma 4.2, we obtain from (4.12) the follows

$$\|\varepsilon^j\| \le \ ad_j + r_j\|\varepsilon_0\| \le \alpha(1-\alpha)Cn^{-2}\|u''\|_{L_\infty}\cdot d_j + r_j\|\varepsilon_0\|$$

$$\le \alpha(1-\alpha)Cn^{-2}\|u''\|_{L_\infty}\frac{\sin\pi\alpha}{\pi\alpha(1-\alpha)}j^{\alpha} + \|\varepsilon_0\| \le 1 \le C\Delta t^{2-\alpha} + \|\varepsilon_0\|.$$

The proof of the Theorem 4.1 is complete.

Second: we will consider a stability result of the time discretization of the FPDEs (4.1) and (4.2).

Theorem 4.2 Let U^j be the approximate solution of (4.6), then we have

$$\|U^j\| \le 2\|U^0\| + \frac{\sin\pi\alpha}{\pi}|\Gamma(-\alpha)|t_j^{\alpha}\|f\|_{L_\infty}$$

Before proving this Theorem we have the following steps:

Step 1: Substituting by the expression $\sum_{k=0}^{j}\alpha_{kj} \ = -\frac{1}{\alpha}$, into (4.6), we get

$(\alpha_{0j}+t_j^{\alpha}\Gamma(-\alpha)A)U^j \ = t_j^{\alpha}\Gamma(-\alpha)f_j - \sum_{k=1}^{j}\alpha_{kj}U^{j-k} - \frac{1}{\alpha}U^0, \ \ \text{for } j = 1,2,3,\ldots.$

Or

$$(-\alpha_{0j} - t_j^{\alpha}\Gamma(-\alpha)A)U^j \ = \sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^{\alpha}\Gamma(-\alpha)f_j, \qquad (4.13)$$

Multiplying on both sides of (4.13) by $\alpha(1-\alpha)j^{-\alpha}$, and use the fact,

$$\alpha(1-\alpha)j^{-\alpha}(-\alpha_{0j}) = 1,$$

Then we obtain the follows

$$[1+\alpha(1-\alpha)j^{-\alpha}(-t_j^{\alpha})\Gamma(-\alpha)A]U^j = \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^{\alpha}\Gamma(-\alpha)f_j\right).$$

Step 2: Assume that $u_j \ = \alpha(1-\alpha)j^{-\alpha}(-t_j^{\alpha})\Gamma(-\alpha)$, then we get

$$U^j \ = (1+u_jA)^{-1}\alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}U^{j-k} + \frac{1}{\alpha}U^0 - t_j^{\alpha}\Gamma(-\alpha)f_j\right).$$

We denote that the norm $\|(1+u_jA)^{-1}\| = \sup_{\lambda>0}|(1+u_j\lambda)^{-1}| < 1$, then

$$\|U^j\| \le \left\|(1+u_jA)^{-1}\right\|\alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}\|U^{j-k}\| + \frac{1}{\alpha}\|U^0\| + |t_j^{\alpha}\Gamma(-\alpha)|\|f\|_{L_\infty}\right)$$

$$\leq \alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=1}^{j}\alpha_{kj}\left\|U^{j-k}\right\|\right)+(1-\alpha)j^{-\alpha}\|U^0\|$$
$$+\alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}$$
$$\leq \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\left\|U^{j-k}\right\|+(1-\alpha)j^{-\alpha}\|U^0\|+a, \qquad (4.14)$$

Here

$a=\alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}$.

Denote that when $j=1$,

$$\|U^1\|\leq a+\alpha(1-\alpha)1^{-\alpha}\alpha_{11}\|U^0\|+(1-\alpha)1^{-\alpha}\|U^0\|.$$

Suppose that $d_1=1,\;\; b_1=(1-\alpha)1^{-\alpha},\;\; r_1=\alpha(1-\alpha)1^{-\alpha},$

Then we have

$$\|U^1\|\leq ad_1+b_1\|U^0\|+r_1\|U^0\|.$$

In general, we can write that

$$\left\|U^j\right\|\leq ad_j+b_j\|U^0\|+r_j\|U^0\|,\;\; j=1,2,3,\ldots. \qquad (4.15)$$

Here

$$\begin{cases} d_1=1,\\ d_j=1+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}d_{j-k},\;\; j=2,3,4,\ldots, \end{cases}$$

$$\begin{cases} b_1=(1-\alpha)1^{-\alpha},\\ b_j=(1-\alpha)j^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}b_{j-k},\;\; j=2,3,4,\ldots, \end{cases}$$

and

$$\begin{cases} r_0=1,\\ r_j=\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}r_{j-k},\;\; j=1,2,3,\ldots. \end{cases}$$

Step 3: Suppose that, for some fixed numbers $j=1,2,3,\ldots$,

$\left\|U^j\right\|\leq ad_j+b_j\|U^0\|+r_j\|U^0\|$.

Then by (4.15), we have

$$\left\|U^{j+1}\right\|\leq\;\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\left\|U^{j-k}\right\|+(1-\alpha)j^{-\alpha}\|U^0\|+a$$
$$\leq\;\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\left[ad_{j-k}+b_{j-k}\|U^0\|+r_{j-k}\|U^0\|\right]$$
$$+(1-\alpha)j^{-\alpha}\|U^0\|+a$$
$$=\left[\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}d_{j-k}\right]$$
$$+\left[(1-\alpha)j^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}b_{j-k}\right]\|U^0\|$$

$$+\left[\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}r_{j-k}\right]\|U^0\|$$
$$= ad_{j+1}+b_{j+1}\|U^0\|+r_{j+1}\|U^0\|.$$

Which shows that (4.15) holds.

Lemma 4.3: Assume that, for $0<\alpha<1$,

Choose: $j=1$, then $b_1=(1-\alpha)j^{-\alpha}$,

$$b_j=(1-\alpha)j^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}b_{j-k}, \quad \text{for } j=2,3,4,\dots,$$

Then we have

$$b_j\le 1.$$

Proof: we know that

$$b_1=(1-\alpha)j^{-\alpha}<1.$$

By mathematics induction principle, suppose that

$b_j<1$, then we have

$$b_{j+1}=(1-\alpha)(j+1)^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}\,b_{j-k}$$
$$\le(1-\alpha)(j+1)^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\left(\sum_{k=0}^{j-1}\alpha_{kj}-\alpha_{0j}\right)$$
$$\le(1-\alpha)(j+1)^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\left(-\frac{1}{\alpha}+\frac{1}{\alpha(1-\alpha)j^{-\alpha}}\right)$$
$$\le(1-\alpha)(j+1)^{-\alpha}+\alpha(1-\alpha)j^{-\alpha}\frac{1-(1-\alpha)j^{-\alpha}}{\alpha(1-\alpha)j^{-\alpha}}$$
$$\le(1-\alpha)(j+1)^{-\alpha}+1-(1-\alpha)j^{-\alpha}<1.$$

The proof of the Lemma 4.3 is complete.

Proof of the Theorem 4.2: by the expression (4.15), we have

$$\|U^j\|\le ad_j+b_j\|U^0\|+r_j\|U^0\|,\quad j=1,2,3,\dots,$$

here d_j, b_j and r_j are given before.

Using Lemma 4.1-4.3, we obtain

$$\|U^j\|\le ad_j+b_j\|U^0\|+r_j\|U^0\|\le a\frac{\sin\pi\alpha}{\pi\alpha(1-\alpha)}j^{-\alpha}+\|U^0\|+\|U^0\|$$
$$=\alpha(1-\alpha)\Delta t^{\alpha}|\Gamma(-\alpha)|\|f\|_{L_\infty}\frac{\sin\pi\alpha}{\pi\alpha(1-\alpha)}j^{-\alpha}+2\|U^0\|$$
$$\le 2\|U^0\|+\frac{\sin\pi\alpha}{\pi}|\Gamma(-\alpha)|t_j^{\alpha}\|f\|_{L_\infty}.$$

The proof of the Theorem 4.2 is complete.

5 Numerical simulation

In this section, we will consider two numerical examples.

Example 5.1 Consider the time fractional partial differential equation, with $0<\alpha<1$,

$$\frac{\partial^\alpha}{\partial t^\alpha}u(t,x)-\frac{\partial^2}{\partial x^2}u(t,x)=f(t,x) \tag{5.1}$$

$$\mathbf{I.c}: \; u(0,x) = u_0, \qquad 0 \le x \le 1, \qquad (5.2)$$

$$\mathbf{B.c}: \; u(t,0) = u(t,1) = 0, \qquad t > 0, \quad 0 < \alpha < 1 \qquad (5.3)$$

The exact solution is

$$u(t,x) = \sin(\pi t)\sin(\pi x).$$

The write hand side of the function

$$f(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \pi(t-s)^{-\alpha} \cos(\pi s)\sin(\pi x)\, ds - \pi^2 \sin(\pi t)\sin(\pi x)$$

We choose $\alpha = 0.2, \quad \Delta x = h = 0.01, \quad T = 1, \quad \Delta t = k = 1/32, \quad N = T/\Delta t$.

Let U^n denote the approximate solution and $u(t_n)$ denote the exact solution at $t = t_n$.

Let $\varepsilon^n = U^n - u(t_n)$ denote their error at $t = t_n$. We plot the exact solution, approximate solution at $t_N = 1$, in Figure 1. We plot the error at $t_N = 1$ in Figure 2.

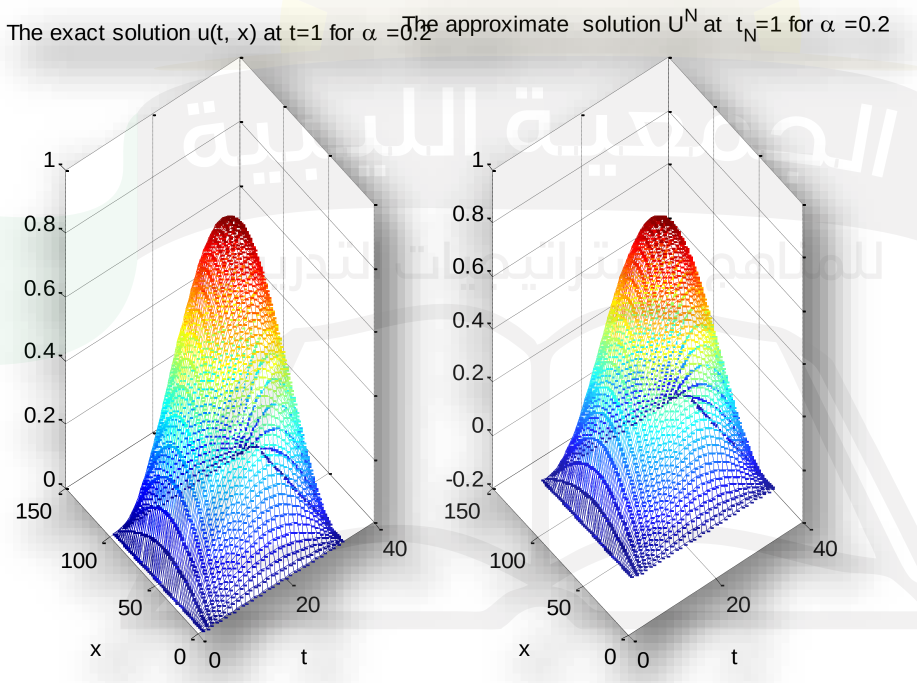


Figure 1. The approximate and exact solutions at $t_N = 1$

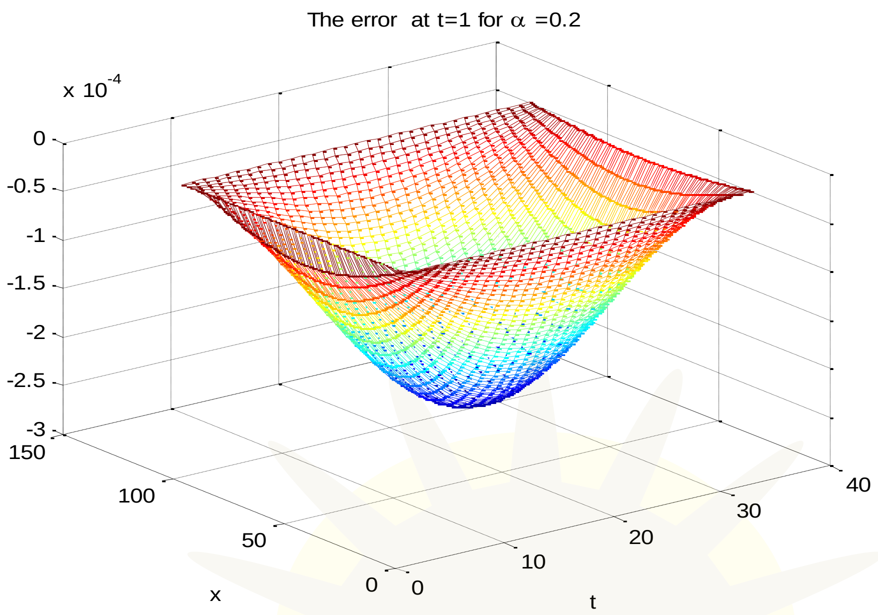


Figure 2: The error at $t_N = 1$

Example 5.2 Consider, with $0 < \alpha < 1$

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) - \frac{\partial^2}{\partial x^2} u(t,x) = f(t,x), \quad t > 0, \quad 0 < x < 1 \tag{5.4}$$

$$\textbf{I. c}: \; u(0,x) = u_0, \tag{5.5}$$

$$\textbf{B. c}: \; u(t,0) = u(t,1) = 0, \tag{5.6}$$

The exact solution is

$$u(t,x) = t^2 \sin(2\pi x).$$

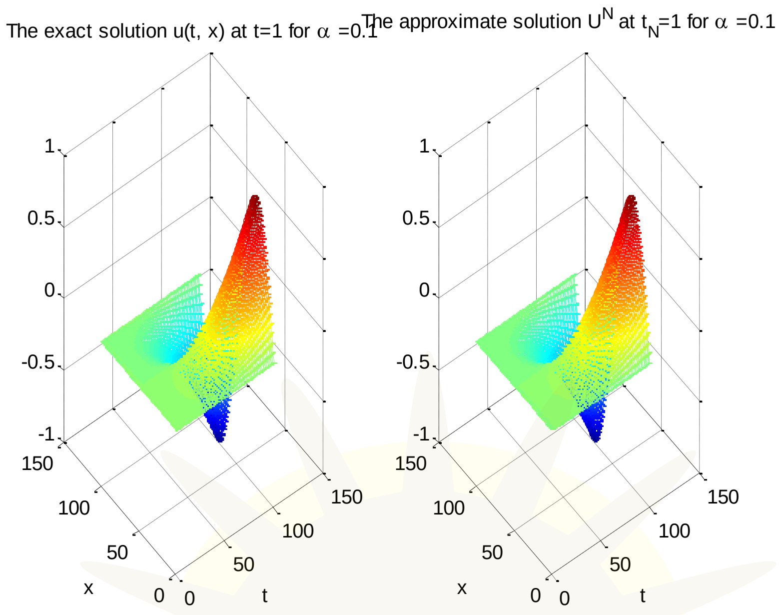
The right hand side functions

$$f(t,x) = 2t^{2-\alpha} \sin(2\pi x) / \Gamma(3-\alpha) - 4\pi^2 \sin(2\pi x) t^2.$$

We choose $\alpha = 0.2, \quad \Delta x = 0.01, \quad T = 1, \; \Delta t = 0.01, \; N = T/\Delta t.$

Let U^n denote the approximate solution and $u(t_n)$ denote the exact solution at $t = t_n$.

Let $\varepsilon^n = U^n - u(t_n)$ denote the error at $t = t_n$. We plot the exact solution and the approximate solution at $t_N = 1$ in Figure 3, and we plot the error at $t_N = 1$ in Figure 4.



Figur 3. The approximate and exact solutions at $t_N = 1$

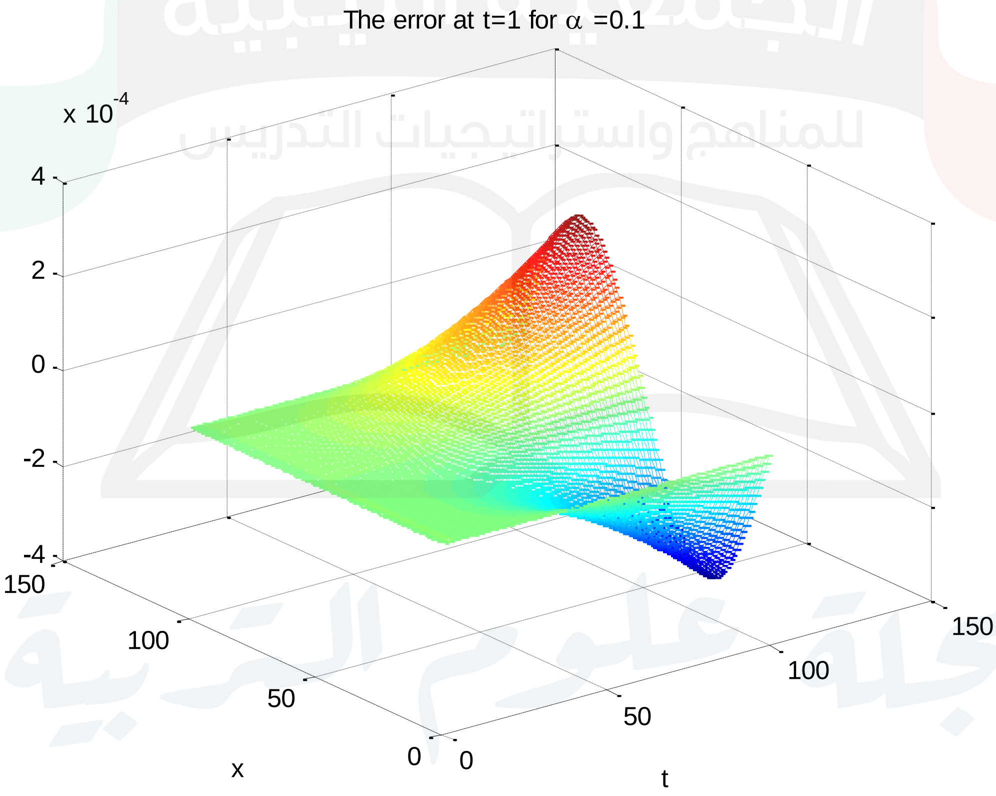


Figure 4. The error at $t_N = 1$

6 Conclusion

In this paper we discuss the finite element method for the time fractional partial differential equations. We introduce the finite element method for solving time fractional partial differential equation. We obtain the error estimates in the L_2-norm between the exact solution and the approximate solution in fully discrete case. The numerical examples show that the numerical results are consistent with the theoretical results.

7 Bibliography

[1] F.Amblard, A. C. Maggs, B. Yurke, N. Pargellis, and S. Leibler, Subdiffusion and anomalous local viscoelasticity in actin networks, Phys. Rev. Lett.77(1996) 4470.

[2] E.Barkai, R. Metzler, and J. Klafter, Form continuous time random walks to the fractional Fokker-Planck equation, Phys. Rev. E 61 (2000), 132-138.

[3] B. Henry and S. Weatne, Fractional reaction-diffusion, Physica A 276 (2000), 448-455.

[4] R. R. Nigmatullin, Realization of generalized transfer equation in a medium with fractal geometry, Physica B 133(1986), 425-430.

[5] N. J. Ford and J. Xiao and Y. Yan, A finite element method for the time fractional partial differenti al equations, 14(2011), 454-474.

[6] I. Podlubny, Fractional Differential Equations, volume 198 of Mathematics in Science and Engineering. Academic Press Inc., San Diego, CA, (1999).

[7] F.Liu, V.Anh, I. Turner, and P. Zhuang, Time fractional advection dispersion equation, J. Comput. Appl. Math. 13 (2003), 233-245.

[8]W. R.Schneider and W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989), 134-144.

[9]Lin, Y, Xu, C: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. 225, (2007)1533-1552.

[10] F. Liu, S. Shen, V. Anh and I. Turner, Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation, ANZIAMJ 46(2005), 488-504.

[11] Z. Z. Sun and X. N. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math 56(2006), 193-209.

[12] V. J. Ervin and J. P. Roop, Variational solution of fractional advection dispersion equations on bounded domain in R^d, Numerical Meth. P. D. E. 23(2007), 256-576.

[13] C. Li, Z. Zhao, and Y. Chen, Numerical approximation and error estimates of a time fractional order diffusion equation, Proceedings of the ASME 2009 International Design Engineering Technical Conference and Computers and information in Engineering Conference IDETC/CIE, San Diego, Californian, USA, (2009).

[14] Y.Jiang and J. Ma, High-order finite element methods for time-fractional partial differential equations, J. Comput. Appl. 235(2011), 3285-3290.

[15] H, Q, Huang, G, Zhan, H: A finite element solution for the fractional advection-dispersion equation. Adv. WaterResour. 31, (2008)1578-1589.

[16] S.G.Samko, A.A.Kilbas, and O.I.Maricher, Fractional Integrals and Derivatives: Theory and Applications. Gorden and Breach Science Publishers, (1993).

[17] K. Diethelm, Fractional Differential Equations, Theory and Numerical Treatment.
TU Braunschweig, Braunschweig, (2003).

[18] K. Diethelm, Generalized compound quadrature formulae for finite-part integrals, IMA Journal of Numerical Analysis, 17(1997), 479.

[19] F.Mainardi, Fractional calculus and waves in linear viscoelasticity, Imperial College Press, London , (2010).

[20] Analytic Solution to the System of Time Fractional Partial Differential Equation Via Laplace Transform, universal Journal of Applied Mathematics and Computations 5 (2017) 1-9.