UNIVERSITY OF TRIPOLI

Faculty of Engineering

Department of Aeronautical Engineering

# Finite Strip Method for Buckling Analysis of Plates

By

Eng. Marwa K. Alzletni

02114401

Supervised by

Prof. Ali H. Essuri

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# Dedication

**"I thank my Allah Almighty my creator, my strong pillar, my source of inspiration, wisdom, knowledge, and understanding. He has been the source of my strength throughout this program and on His wings only have I soared".**

This is especially dedicated to **Prof. Ali Essuri** who helped and guided me to successfully complete this project work.

I dedicate this work and give special thanks to family and many friends. A special feeling of gratitude to my loving parents, for being there for me throughout the entire M. SC program.

I also dedicate this dissertation to my husband who has supported me throughout the process.

I will always appreciate all they have done, for helping me develop my skills and myself.

## Acknowledgement

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Finally, last but not least, thanks go to all my family and friends who supported me throughout my project. I would like to thank especially my parents whose help is simply invaluable.

**Thank you**

# Abstract

This project is dedicated to study buckling of plates and plate type structures using finite strip method.

The study considered derivation of elastic stiffness and geometric stiffness matrices for finite strips and then assembled to produce the assembly matrices required for the determinant equation which produces the Eigen-value as buckling modes and eigenvectors for buckling modes.

The computer program is produced to solve the determinant equation for several types of structures and the result when compared with the solution obtained by classical methods showed very good agreement.

# List of contents

## List of figures

## List of charts

## List of tables

# List of symbols

| Symbol | Definition |
|---|---|
| A | Length |
| b | Width |
| $E$ | Yong's Modulus |
| $U$ | Internal Energy |
| $V$ | External Energy |
| $u, v, w$ | Displacements in x, y and z Directions |
| $N$ | Shape Function |
| $N_x$ | Critical load per unit |
| $M_x, M_y$ | Bending Moment |
| $M_{xy}, M_{yx}$ | Twisting Moment |
| $\sigma_{cr}$ | Critical Stress |
| $D$ | Flexural Rigidity of Plate |
| $X, Y, Z$ | Body Forces |
| $L, m, n$ | Direction Cosines of the Angels |
| $G$ | Shear Modulus in Elasticity |
| $T$ | Temperature |
| $w$ | Deflection |
| $\upsilon$ | Poison's Ratio |
| $\varepsilon_x, \varepsilon_y, \varepsilon_z$ | Normal Strains in x, y and z Directions |
| $\gamma_{xy}, \gamma_{yz}, \gamma_{yz}$ | Shear Strains |
| $\sigma_x, \sigma_y, \sigma_z$ | Normal Stresses Parallel to x, y and z axis |
| $\tau_{xy}, \tau_{yz}, \tau_{zx}$ | Shear Stresses |
| $\alpha$ | Temperature expansion coefficient in longitudinal |
| $\phi_x, \phi_y, \phi_z$ | Surface Forces |

# Chapter I
# Buckling of Isotropic Plates

# 1 Buckling of Isotropic Plates

## 1.1 Literature Survey

1. The technique was first introduced in 1968 by W. Wittrick [1], The exact finite strip (, 1968), this method is based on large deformation theory and the de-stabilising effect of the in-plane stresses is considered. As a result, all modes of buckling regardless of the critical wave length can be singled out in the results of the analysis.
2. The approximation finite strip (Y. Cheung, 1968) [2], Cheung method was based on small deformation theory, and an approximate deflection pattern is assumed, in the longitudinal direction the deflection shape is approximate by a Fourier series and in the transverse direction a polynomial is assumed and only overall buckling is computed.
3. The finite strip for local instability (J.S. Przemieniecki, 1973), [3] the method which is also based on small deflection theory is particularly devised for use in study for the class of problems which fail by local instability.
4. Other references used finite strip are [4] ,[5]

In this project the finite strip method will be used for buckling analysis of plate type structure computer code is specially developed for handling a variety of buckling problems of thin walled structure.

## 1.2 Introduction [6]

There are two major factors leading to sudden failure of a mechanical component: material failure and structural instability.

For material failures you need to consider the yield stress for ductile materials and the ultimate stress for brittle materials. Tests on test specimen are used to determine material characteristics.

The geometry of such test specimens has been standardized. Thus, geometry is not specifically addressed in defining material properties, such as yield stress. Geometry enters the problem of determining material failure only indirectly as the stresses are calculated by analytic or numerical methods.

Long columns and thin plates and panels under axial load fail under buckling stress and their deformed shape known as buckling modes are shown in figure (1.1)

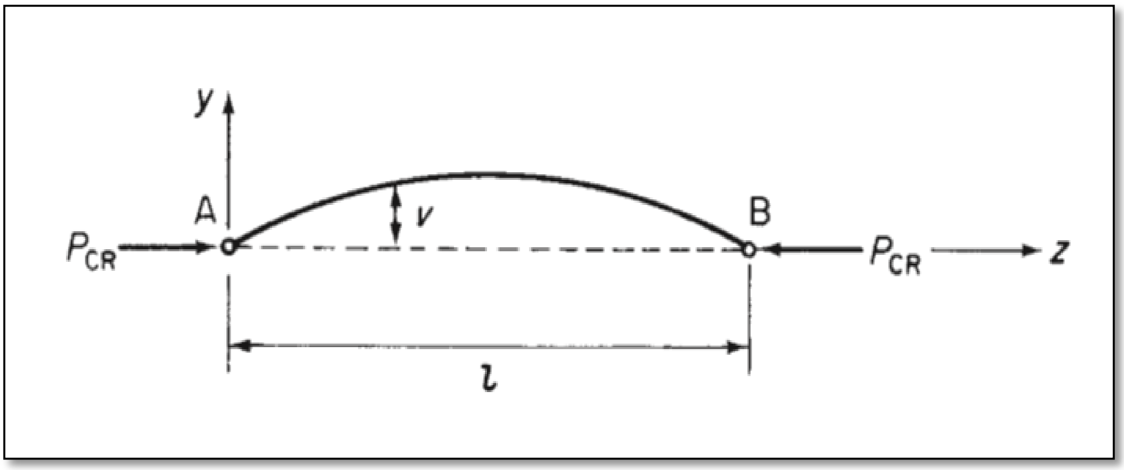



**Figure 1-1: Determination of Buckling Load for a Pin-Ended Colum.[6]**

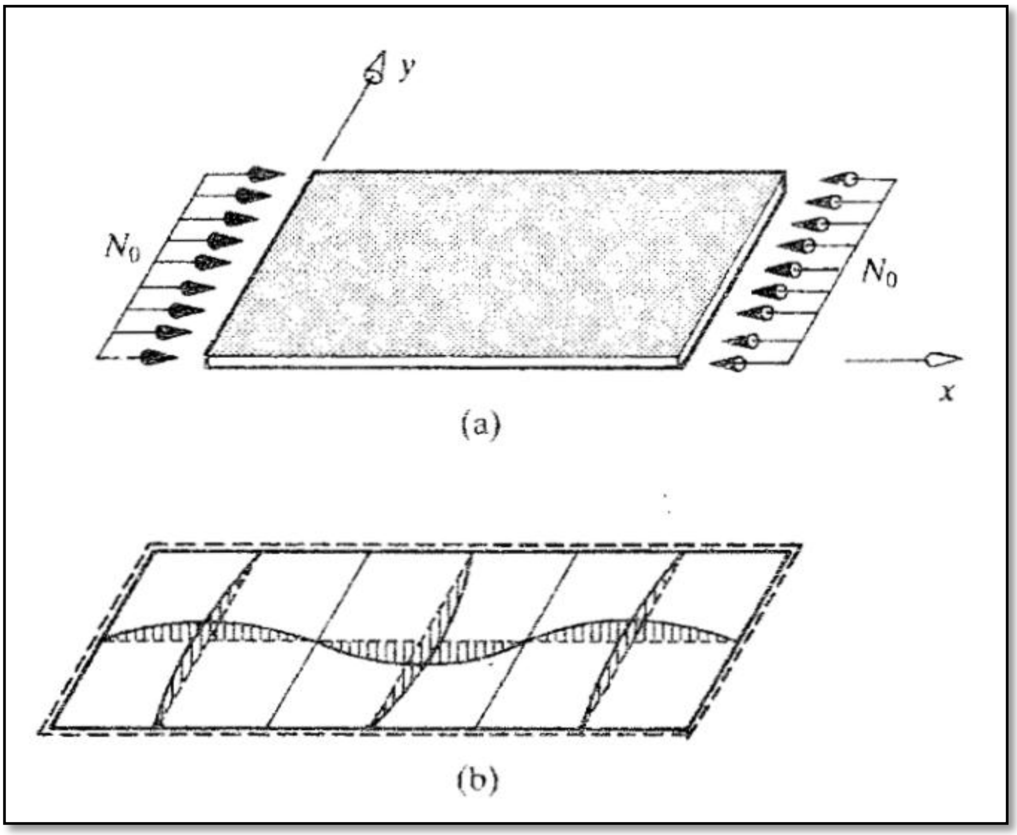



**Figure 1-2(a) A Simply Supported Plate with a Uniform in a Plane Edge Loading in the x Direction, (b) The Buckling of a Long Thin Flat Plate. [7]**

## 1.3 Revision of Theory of Elasticity Fundamental

### 1.3.1 Equation of Strain and Displacements in 3D [7]

- **Normal Strain**

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \quad 1.1$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \quad 1.2$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \quad 1.3$$

- **Shear Strains**

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \quad 1.4$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \qquad 1.5$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \qquad 1.6$$

**Where**:

$u, v, w$ are displacements in $x, y$, and z direction of orthogonal coordinate system axis and function of $x, y$, and z coordinates .

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ are normal strains.

$\gamma_{xy}, \gamma_{yz}, \gamma_{yz}$ are shear strains.

### 1.3.2 Stresses – Strain (Hooke's Low)

For 3D – Isotropic material (absent temperature):

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right] \qquad 1.7$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon(\sigma_x + \sigma_z)\right] \qquad 1.8$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] \qquad 1.9$$

**Where**

$\sigma_x, \sigma_y, \sigma_z$ Normal components of stress parallel to $x, y$ and $z$ axis.

$E$ is young's modulus and poison's ratio $\upsilon$ and introducing $G$ as shear modulus of elasticity.

$$\because \quad G = \frac{E}{2(1+\upsilon)} \qquad , and\ from \qquad \gamma = \frac{\tau}{G}$$

**Then**

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad 1.10$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \qquad 1.11$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad 1.12$$

**Where**

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ are shear stresses.

Hooke's low in presence of temperature [8] in explicit form become:

$$e_x = \varepsilon_x + \alpha\, T = \frac{1}{E}\left[\sigma_x - v\left(\sigma_y + \sigma_z\right)\right] + \alpha\, T \qquad 1.13$$

$$e_y = \varepsilon_y + \alpha\, T = \frac{1}{E}\left[\sigma_y - v(\sigma_x + \sigma_z)\right] + \alpha\, T \qquad 1.14$$

$$e_z = \varepsilon_z + \alpha\, T = \frac{1}{E}\left[\sigma_z - v\left(\sigma_x + \sigma_y\right)\right] + \alpha\, T \qquad 1.15$$

- **Hooke's Law in Matrix Form [8]**

$$\boldsymbol{\varepsilon} = \boldsymbol{\acute{E}}\ \boldsymbol{\delta} \qquad 1.16$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -v & v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+v) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+v) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

$$\boldsymbol{e} = \boldsymbol{\varepsilon} + \boldsymbol{\alpha}\, \boldsymbol{T} \qquad 1.17$$

$$\begin{bmatrix} e_x \\ e_y \\ e_z \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -v & v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+v) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+v) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} + \begin{bmatrix} \alpha\, T \\ \alpha\, T \\ \alpha\, T \\ \alpha\, T \\ \alpha\, T \\ \alpha\, T \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{N}\ \boldsymbol{\delta} \qquad 1.18$$

Multiplying both sides by $\boldsymbol{N^{-1}}$

$$\boldsymbol{N^{-1}}\ \boldsymbol{\varepsilon} = \boldsymbol{N^{-1}}\ \ \boldsymbol{N}\ \boldsymbol{\delta}$$

$$\boldsymbol{N^{-1}}\ \boldsymbol{\varepsilon} = \boldsymbol{I}\ \boldsymbol{\delta}$$

Where $\boldsymbol{I}$ is unit matrix, then

$$\boldsymbol{\delta} = \boldsymbol{N}^{-1}\,\boldsymbol{\varepsilon}$$

$$\boldsymbol{D} = \boldsymbol{N}^{-1}$$

$$\boldsymbol{\delta} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$$

Where $\boldsymbol{D}$ is given as

$$\frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix}$$

- So we have

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \frac{E\,\alpha\,T}{(1-2v)}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And for 2D structures we may write [8],

- Plane stress $(\mathrm{x},\mathrm{y})$ , where $\sigma_x$ , $\sigma_y$ , $\tau_{xy} \neq 0$ and $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{E}{1-\upsilon^2}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \dfrac{1-\upsilon}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}-\frac{E\ \alpha\ T}{1-\upsilon}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

**Also**

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\upsilon & 0\\ -\upsilon & 1 & 0\\ 0 & 0 & 2(1+\upsilon)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}+\alpha\ T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

- Plane strain (x, y) where $\varepsilon_x, \varepsilon_y, \gamma_{xy} \neq 0$ and $\varepsilon_z, \gamma_{yz}, \gamma_{zx} = 0$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix}1-\upsilon & \upsilon & 0\\ \upsilon & 1-\upsilon & 0\\ 0 & 0 & 1-\upsilon\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}-\frac{E\ \alpha\ T}{1-\upsilon}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\frac{1+\upsilon}{2}\begin{bmatrix}1-\upsilon & -\upsilon & 0\\ -\upsilon & 1-\upsilon & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}+(1+\upsilon)\alpha\ T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

## 1.3.3 Equations of Equilibrium

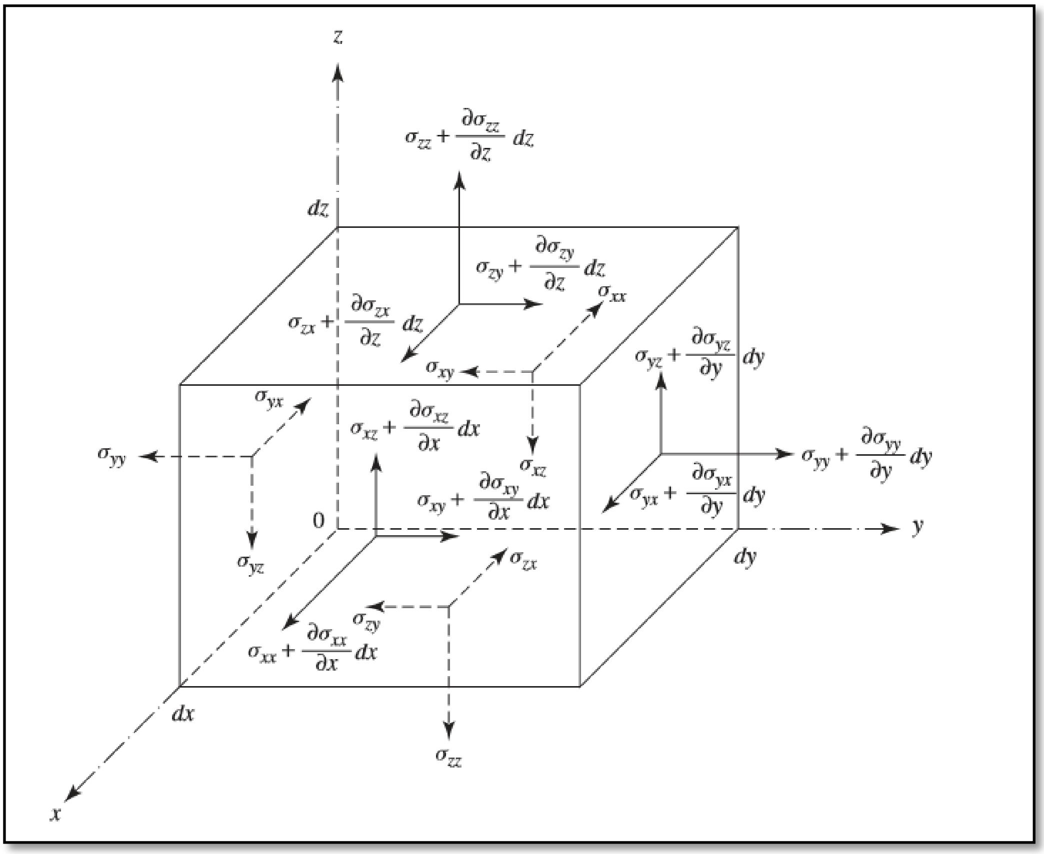



**Figure 1-3Stresses on unit volume [ 7 ]**

$$\frac{\partial \sigma_x}{\partial_x} + \frac{\partial \tau_{xy}}{\partial_y} + \frac{\partial \tau_{zx}}{\partial_z} + X = 0 \qquad 1.19$$

$$\frac{\partial \sigma_y}{\partial_y} + \frac{\partial \tau_{xy}}{\partial_x} + \frac{\partial \tau_{yz}}{\partial_z} + Y = 0 \qquad 1.20$$

$$\frac{\partial \sigma_z}{\partial_z} + \frac{\partial \tau_{xz}}{\partial_x} + \frac{\partial \tau_{zy}}{\partial_y} + Z = 0 \qquad 1.21$$

**Where**

$X, Y$ and $Z$ body forces

These 3 equations must be satisfied at all points on a deformed body for equilibrium

Indicating direction cosines: by $l$ , $m$ and $n$

**Where** $l = cos\theta_1$ , $m = cos\theta_2$ , $n = cos\theta_3$

Given surface forces: $\phi_x, \phi_y$ and $\phi_z$

$$\begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\phi_x = \sigma_x l + \tau_{xy} m + \tau_{xz} n \qquad (1.22)$$

$$\phi_y = \tau_{yx} l + \sigma_y m + \tau_{yz} n \qquad (1.23)$$

$$\phi_z = \tau_{zx} l + \tau_{zy} m + \sigma_z n \qquad (1.24)$$

Equilibrium between body forces $X, Y, Z$ and surface forces $\phi_x, \phi_y$ ,$\phi_z$ and external applied forces $F_i$ and moment $M$.

Must be satisfied

$$\int_s \phi_x \, ds + \int_v X \; dv + \sum F_{i_x} = 0 \qquad (1.24)$$

$$\int_s \phi_y \, ds + \int_v Y \; dv + \sum F_{i_y} = 0 \qquad (1.25)$$

$$\int_s \phi_z \, ds + \int_v Z \; dv + \sum F_{i_z} = 0 \qquad (1.26)$$

$$\int_s \left(\phi_y z - \phi_z y\right) ds + \int_v \left(Z_y - Y_z\right) \; dv + \sum M_x = 0 \qquad (1.27)$$

$$\int_s (\phi_x z - \phi_z x) \, ds + \int_v (X_z - Z_x) \; dv + \sum M_y = 0 \qquad (1.28)$$

$$\int_s \left(\phi_y x - \phi_x y\right) ds + \int_v \left(Y_x - X_y\right) \; dv + \sum M_z = 0 \qquad (1.29)$$

### 1.3.4 Equations of Compatibility of Strain

For solutions of 3D problems of elasticity normal strains $\varepsilon$ and shear strains $\gamma$ must satisfy the following equations:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} \qquad (1.30)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial z \, \partial y} \qquad (1.31)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial x \, \partial z} \qquad (1.32)$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \, \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xz}}{\partial z}\right] \qquad (1.33)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \, \partial x} = \frac{\partial}{\partial y}\left[-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z}\right] \qquad (1.34)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \, \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right] \qquad (1.35)$$

## 1.4 Linear and non Linear Structural Stability

In general, the literature for structural stability can be divided into two main topics **[4]:**

a. Linier study: which deals primarily with critical equilibrium status
b. Nonlinear study: This is concerned with the equilibrium path configuration in the vicinity of the equilibrium state.

In the study of linear stability of structures [9], systems are classified into conservative systems and non conservative systems. Conservative systems are systems subject to conservative forces.

A non-conservative system is a system which is subject to at least one non conservative force. see table (2)

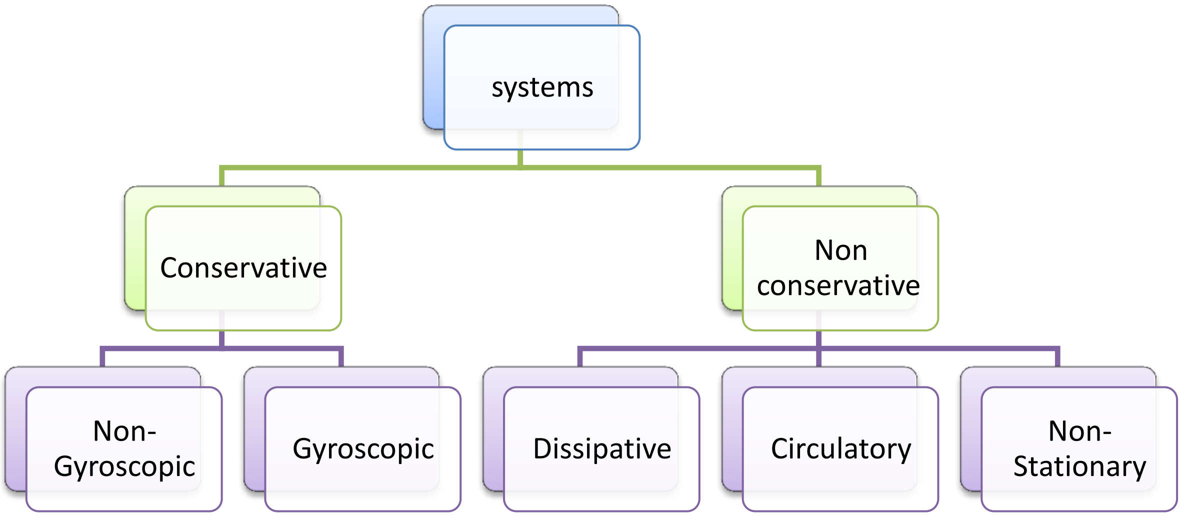



**Chart 1-1 Classification of Systems [ 4 ]**

Conservative forces are forces which when their point of application is displaced from a point A to a point B, the work done is only a function of the initial and final positions and does not depend on the displacement path.

Since forces are classified as reactions, and external loads. ,reactions can be considered conservative if they are nonworking (ex. Frictionless normal pressure).

Non conservative reactions are dissipative or doing negative work.(ex. Kinetic dry friction). (It's to be noted here, that nonworking reactions are considered as conservative, although they do not admit potential).

Loads are classified as follows: Stationary loads, which are independent of time but may be velocity dependent.

Non stationary loads, which depend explicitly on time and are non conservative (e.g. pulsating loads).

Velocity dependent stationary loads include those which do zero work (e.g. gyroscopic) and are conservative and those which do negative work i.e. are dissipative (e.g. air drag) and are non conservative.

Stationary velocity independent loads also fall into both categories. Those which can be derived from a potential (e.g. gravitational forces) and are conservative are called non circulatory whilst all others are non conservative and known as circulatory.

Examples of non conservative systems are those which have a dissipative reaction, or dissipative load, circulatory or non stationary type.

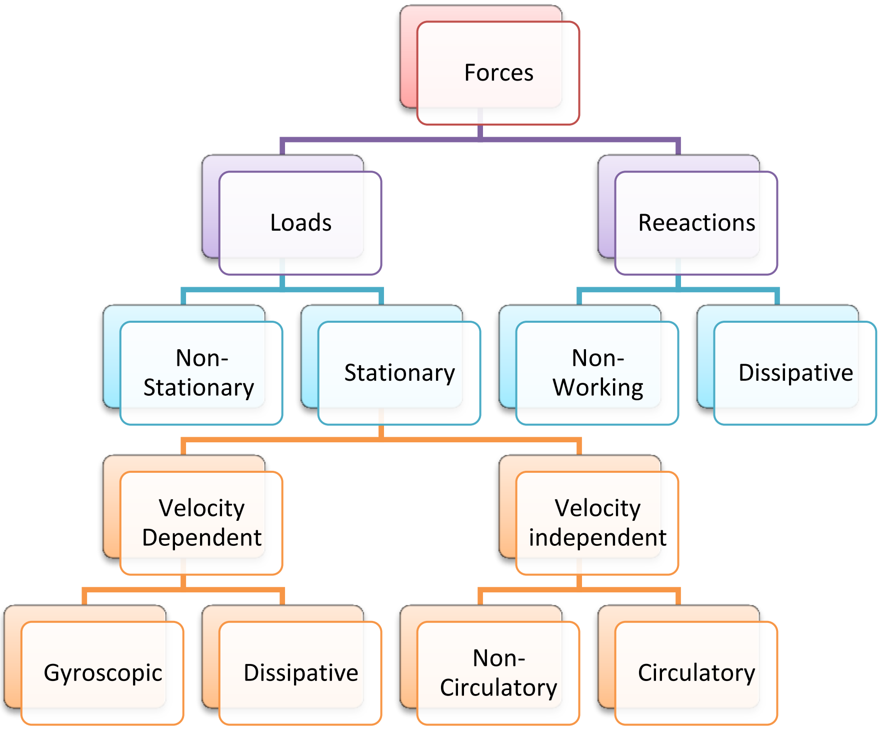



**Chart 1-2 Classification of Forces [4]**

In the analysis which follows, systems which will be treated are those which are classified as non-gyroscopic, in which reactions are nonworking, and loads are stationary, and non-circulatory.

This category can be treated using the Energy Approaches EA, the Finite element method FEM, and the **Finite strip method** FSM which will be adopted throughout.

For axially loaded structures, the buckling load, [4] , is defined as the load at which small disturbances of the position of equilibrium of the structure will lead to large deformations which exceed the allowable limits specified under certain working conditions and working time.

In aircraft structures, for example, large use is made of slender columns and thin plates which are very likely to fail by buckling and thus the determination of buckling stress is a problem of extreme importance, .since in practical structures the object of design is to retain a well-defined shape, the critical buckling study will define the load at which this well-defined shape is lost.

## 1.5 Methods for Buckling Analysis

In this section a survey of methods for buckling analysis will carried out, the methods which will be reviewed are as follow:

1. Euler Method (The Equilibrium Method)
2. Principle of Total Potential Energy
3. Ritz Method
4. Conservation of Energy Method
5. Galerkin's Method
6. Finite Difference Method
7. Finite Element Method
8. Finite Strip Method

**These methods will be presented in brief as follow:**

### 1 Euler Method (The Equilibrium Method)[4]:

This method is mostly encountered in the analysis of simple structural forms, for example, the axially loaded bar and the rectangular plate carrying in-plane edge loadings, and it's based on the concept of the condition of neutral equilibrium.

This means that there is no loss of equilibrium when the bar or plate is displaced slightly from its initial straight or plane form.

The bent position is used to establish equilibrium between the external applied forces and the internal forces, and the analysis can be outlined as follows:

a. The basic equilibrium equation between internal forces and external applied forces is set up,
b. This equation is transformed in terms of stresses,

c. From Hooke's Law the basic equation is transformed into an equation in terms of strain,
d. By expressing the strains in terms of the displacement, the equilibrium equation will become the differential equation which governs the deflection of the structure,
e. Since it is a stability problem, the integration of the differential equation will lead to a set of Eigenvalues and corresponding eigenvectors and the critical load will be the smallest of the Eigenvalues.

## 2 Principle of Total Potential Energy Method:

It is based on body in equilibrium total potential energy, which can be stated as follow [4]:

" An elastic body is in equilibrium if no change occurs in the total potential energy of the system for any small arbitrary displacement ".

This can be expressed as follow:

$$\delta(U_E + V) = 0 \qquad (1.36)$$

**Where**

$U_E$ is the strain energy of the elastic structure, and V is the potential energy of the applied external forces and their sum ( $U_E + V$) is known as the total potential energy $\pi$.

In the determination of the buckling load this method can be used in either one of the following procedure:

- Assumed Deflection Shape:

This procedure is attributed to Navier (1820) and applied to several examples in [10]

The analysis is carried out as follows:

a. The analytical expression of the total potential energy is found.
b. A suitable displacement function is chosen in terms of unknown coefficients.

   The chosen displacement function must satisfy both the geometric boundary conditions represented by slope and deflection and the natural boundary conditions represented by shear and bending moment.

   If it is not possible to satisfy both, at least the geometric boundary condition must be satisfied.

c. The total potential energy is then calculated in terms of the unknown parameters.
d. The expression of minimum total potential energy

$$\delta(U_E + V) = 0$$

Is then obtained by equating to zero the partial differentials of this energy with respect to each one of those unknown coefficients.

e. The critical load is found by elimination of the unknown parameters.

This procedure depends in the first place on the choice of the displacement function, if it is the exact one then the buckling load will be exact, if the assumed displacement function is not exact then the buckling load determined will be approximate and the degree of approximation depends on how closely the true buckling shape is represented.

- Variational Approach :

This procedure first appeared in 1891 in a paper by G.H. Bryan [11]. In the previous procedure, an important feature is the choice of buckling shape function which must, at least, satisfy the geometric boundary conditions, this second procedure is principally suitable for the study of the case when the assumed deflection shape cannot be easily found.

In this procedure, the variation calculus is used to find the conditions represented by the differential equation and the natural boundary conditions which the exact deflection shape must satisfy so that the total potential energy of the system becomes stationary$(\delta(U_E + V) = 0)$ . It leads only to the governing equations of the problem.

An application can be found in the above reference where a circular plate was studied and in [9], in which this procedure was applied to the simple case of axially loaded simply supported column.

## 3 Ritz Method:

This method is referred to by W. Ritz and first appeared in the literature in 1909. In the Ritz method, an assumed shape is used to represent the deformation of the system. This reduces the number of degree of freedom and the critical load is found by using only ordinary calculus [12].

**This method procedure can be as follows:**

1. The analytical expression of the total potential energy is obtained.
2. The deflection surface is expressed in expanded form as the sum of an infinite set of functions having undetermined coefficients.
3. The total energy of the structure is computed for the deflection surface and then minimised with respect to the undetermined coefficients.
4. This minimisation procedure leads to a set of linear homogeneous equations in the undetermined coefficients.

5. These equations have non vanishing solutions only if their coefficients vanish. The vanishing of this stability determinant provides the equation that may be solved for the buckling load.
6. If the set of functions is complete and capable of representing slope, deflection, shape and curvature of any possible deformation, the solution is exact.

For the exact analysis the order of determinant will be infinite, and if only a reduced number of terms are used, an approximate buckling load will be obtained, which will be higher than the exact value.

An application of this method can be found in [13] where the lateral buckling of deep beam was analysed.

## 4 Conservation of Energy Method

This method referred to as the Timoshenko method having been developed by Timoshenko, S. [14], 1910, and used extensively in [15] for the solution of a variety of instability problems.

This method is based on the energy concept which can best be explained by taking the axially loaded column as a particular example. This column when subjected to gradually increasing load remains stable in the straight form as long as the load is lower than the buckling load.

During this stage, the load does work by virtue of the shortening of the bar.

When the buckling load is reached, the straight form of equilibrium is no longer unique and another equilibrium condition represented by a bent form of the bar appears.

The work done by the load is now associated with a displacement arising in part from the bending of the bar and in part from the axial deformation of the bar.

Because of the bending, the strain energy of the bar is also increased by an amount due to the bending deformation and so it can be said that at the instant prior to buckling, the increase in external work is equal to the increase in strain energy.

- In mathematical term, before buckling the system is stable and we have

$$U_E - W_e > 0 \qquad (1.37)$$

**Where**

$U_E$ is the strain energy

$W_e$ is the external work

- After buckling stability is lost and $U_E - W_e < 0$

- Whilst at transition $U_E - W_e = 0$

This last condition is characteristic of the incipient state of buckling, and can be stated in the following form:

" A conservative system is in equilibrium, if the strain energy stored is equal to the work performed by the external loads. "

Analysis procedure can be as follow:

1. Expressions for the strain energy and the external work are obtained.

In general, the deformation of the structure prior to buckling is neglected, because it is considered to be much smaller than the bending deformation.

2. An assumed displacement function is chosen which must satisfy as many as possible of the boundary conditions. Generally, the function will involve n coefficients $a_i$ or parameters.

3. From the equality $U_E = W_E$ this equation is derived

$$P = \frac{U_E(a_1, a_2, \dots \dots \dots \dots, a_n)}{\overline{W}_e(a_1, a_2, \dots \dots \dots \dots, a_n)} \tag{1.38}$$

The parameters are then adjusted until $P$ becomes a minimum; this leads to n equations of this type

$$\delta P/\delta a_i = (1/\overline{W}_e^2)\left[\left(\frac{\delta U_E}{\delta a_i}\right)\overline{W}_e - \left(\frac{\delta \overline{W}_e}{\delta a_i}\right)U_E\right] = 0 \tag{1.39}$$

Which become

$$\frac{\delta U_E}{\delta a_i} - P\left(\frac{\delta \overline{W}_e}{\delta a_i}\right) = 0 \qquad (i = 1, n) \tag{1.40}$$

This final set of equations gives the n constants $a_i$ and the buckling load.

It can be seen that there is similarity of the procedures between the various energy approaches:

a. All of them lead to a buckling load higher than the exact one if limited number terms are considered.
b. All lead to the exact buckling load if the assumed displacement function represents exactly the true buckled shape and satisfies the boundary condition.

## 5 Galerkin's Method:

This method is attributed to B.G. Galerkin, and first appeared in 1915,[16].

In the previous method a trial function was assumed for the deflection and then substituted in the analytical expression of total potential energy to find the buckling load.

This method also uses a trial function to represent the deflection shape, the trial functions satisfying term by term the geometric and natural boundary conditions.

The undetermined constants in which the trial function is expressed are then determined by considering each term of the trial function in turn to be a weighing function, and then setting the weighted average of the residual function to zero.

The physical meaning of this condition is that if the trial function is representing exactly the buckled shape then the residual become zero. If the trial function does not represent exactly the buckled shape then the weighted averages when set up equal to zero will provide the conditions which the trial function must satisfy so that the error of approximation becomes a minimum.

Another method of obtaining Galerkin's conditions which is based on the principle of least square can be found in [17].

Once the Galerkin's conditions are obtained, which are in number equal to the number of the unknowns in the assumed trial function, a set of the necessary number of equations is obtained.

The determinant of the coefficients of the undetermined constants will give the buckling load.

This method can be summarised as follows:

- The differential equation is obtained from equilibrium or the principal of total potential energy.

Suppose this to be

$$L(w) = 0 \qquad (1.41)$$

- A trial function which satisfies geometric and natural boundary conditions is selected

$$\theta = \sum_{i=1}^{n} a_i \, \varphi_i \qquad (1.42)$$

- The equation residual is then found from

$$R = L(\theta) = \sum_{i=1}^{n} a_i L(\varphi_i) \qquad (1.43)$$

- In general, if W(x) is a weighing function, the weighted average of a function f(x) in an interval a < x < b is

$$\frac{\int_a^b W \, f \, dx}{\int_a^b W \, dx} \qquad (1.44)$$

In this case, setting the weighted average of the residual $R$ equal to zero equivalent to

$$\int_D \varphi_i \, L(\theta) = 0 \qquad i = 1, \dots \dots \dots \dots, n \qquad (1.45)$$

Where $D$ is the domain of the differential equations.

- This last step provides a system of n equations foe the n undetermined coefficients. The determinant of the coefficients of these unknowns provides the buckling load.

An application of this method to the buckling of a column and to the buckling of a plate in shear can be found in [9]

## 6 Finite Difference Method [18]:

In the previous article, the solution to a buckling problem is found by approximating the deformation shape by properly selecting a displacement function.

In addition to the problem of setting up the differential equation itself, there is also the problem of finding the right function to be used for the approximation, which must satisfy both geometric and natural boundary conditions. Unless a great deal of physical intuition is exercised the solution may differ greatly from the exact.

The finite difference method overcomes the second part of the problem associated with the differential equation by replacing the differential equation and the boundary condition by their finite difference approximations [18].

The result is that instead of dealing with the differential equation the solution to the problem is found by analysing a set of equivalent algebraic equations.

The basis of this method is that a derivative of a function at a point can be replaced by an algebraic expression formed by the value of the function at that point and several nearby points.

Instead analysing a system of infinite degree of freedom an idealised structure formed by discrete element is used, even with a relatively large mesh a good approximation to the exact solution can be obtained.

In buckling analysis, by using the finite difference expressions to replace the derivatives at the number of points chosen, and introducing the boundary conditions, we form a system of homogeneous equations. The determinant of the coefficients of these equations will be transformed into a characteristic equation and the smallest root will define the buckling load.

For the function $f(x)$, the derivatives can be replaced as follows:

$$\left[\frac{df}{dx}\right]_{x=i} = \frac{f_{i+h} - f_i}{h} \quad (1.46)$$

Where$f$, $f_{i+h}$ are values of $f(x)$ at $x = i$ and at $x = i + h$, and $h$ is the distance between those two points.

$$\left[\frac{d^2 f}{dx^2}\right]_{x=i} = \frac{f_{i+h} - 2f_i + f_{i-h}}{h^2} \quad (1.47)$$

$$\left[\frac{d^4 f}{dx^4}\right]_{x=i} = \frac{f_{i+2h} - 4f_{i+h} + 6f_i - 4f_{i-h} + f_{i-2h}}{h^4} \quad (1.48)$$

For the function $f(x, y)$, the derivatives in the governing differential equation assume the following expressions:

Relative to $x$ at point $j, k$

$$\left[\frac{d^2 w}{dx^2}\right]_{j,k} = \frac{w_{j+h,k} - 2w_i + w_{j-h,k}}{h^2} \quad (1.49)$$

And

$$\left[\frac{d^4 w}{dx^4}\right]_{j,k} = \frac{w_{j+2h,k} - 4w_{j+h,k} + 6w_{j,k} - 4w_{j-h,k} + w_{j-2h,k}}{h^4} \quad (1.50)$$

Relative to $y$ at point $j, k$

$$\left[\frac{d^2w}{dy^2}\right]_{j,k} = \frac{w_{j+h,k} - 2w_i + w_{j-h,k}}{h^2} \quad (1.51)$$

**And**

$$\left[\frac{d^4w}{dy^4}\right]_{j,k} = \frac{w_{j+2h,k} - 4w_{j+h,k} + 6w_{j,k} - 4w_{j-h,k} + w_{j-2h,k}}{h^4} \quad (1.52)$$

Application of this method can be found in ref [18]

## 7 Finite Element Method [4]

In the previous article, the finite difference method overcome the difficulty of finding a solution to the differential equation of a structure by replacing the governing differential equation by a set of equivalent algebraic equations that are usually easier to solve.

When an adequate computer is available, method has the advantage of generality of application.

The finite element method transforms the structure into an assembly of finite elements connected at nodes, and, using the elastic stiffness matrix of each single elements, the elastic stiffness of the complete structure is built up according to equilibrium and compatibility conditions dictated by the theory of elasticity and under the assumption of small displacements.

For the elastic structure, the following equation which relates applied forces on the complete structure to the displacements can be given:

$$\boldsymbol{P} = \boldsymbol{K}\,\boldsymbol{U}$$

For buckling analysis, in the finite element (displacement approach), the stiffness of the structure is no longer constant but is a function of the axial or in-plane load of the various elements and can be considered to be formed of two parts, one the constant elastic stiffness and the other the geometric stiffness, which is a function of the axial or in-plane loads.

**Solution Procedure** [4]:

a. Each element stiffness is evaluated and each element of each stiffness matrix is given double subscript i and j, the first refers to force component and the second to the displacement component.

b. An array of order equal to the number of degree of freedom for the complete structure is prepared and named the assembly array.

c. A relation must be established between the element stiffness matrix labels i, j and the labels of the single components of the assembly array l, m.

d. Each element from the element stiffness matrix is sent to a place in the assembly stiffness matrix and summed up to the previous value of the assembly stiffness element.

e. This procedure is repeated for all the elements forming the idealised structure.

f. If different types of elements are involved, to satisfy the rules of matrices summation the stiffness of these element have to be of the same order and individual sub-matrices to be add have therefore to be built up of same number of individual components of force or displacement.

g. Having built up the assembly matrix, the constraint condition which is equivalent to zero displacement at certain nodes can be taken into account by eliminating the columns of stiffness coefficients multiplying this degree of freedom.

h. The result from the previous step is more equations than unknowns; the rows corresponding to the eliminated columns are set aside for later use to find support reactions.

i. The set of equations remaining after the elimination of rows and columns relative to the support conditions are solved for the displacements.

j. The initial force system is found by back substitution of the displacements found in the previous step into the force displacement relationships for the elements.

k. If it is necessary, the transformation from the global coordinate system to local coordinate must be performed before the stress computation is accomplished.

Summarising, for the displacement direct approach the following matrix operations must be performed:

$$\boldsymbol{P} = \boldsymbol{K}\,\boldsymbol{U}$$

By partitioning:

$$\begin{bmatrix} P_f \\ P_c \end{bmatrix} = \begin{bmatrix} K_{ff} & K_{fc} \\ K_{cf} & K_{cc} \end{bmatrix} \begin{bmatrix} U_f \\ U_c \end{bmatrix}$$

**Where**

Subscripts $c$ denote degree of freedom relative to support condition, $f$ for the unsupported.

Then for $U_c = 0$

$$P_f = K_{ff}\, U_f$$

$$P_c = K_{cf}\, U_f$$

The solution for the displacement

$$U_f = {K_{ff}}^{-1}\, P_f$$

And the support reactions

$$P_c = K_{cf}\, {K_{ff}}^{-1}\, P_f$$

Application of this method can be found in ref [4].

## 8 Finite Strip Method

There are three different methods based on finite strip idealisation and these are:

The exact finite strip (W. Wittrick, 1968), [1]

The finite strip for local instability (J.S. Przemieniecki, 1973) , [2]

The approximation finite strip (Y. Cheung, 1968) [3]

These methods have advantages of:

1. Great saving in computer time.
2. Less storage space because of the very narrow band width.
3. A relatively large range of application, since many structures have geometric and material properties which do not vary along one direction.
4. Less input data because of the lower number of nodes involved.

In this project finite strip based on Prezemienieki approach will be used .

**Solution Procedure:**

This procedure has the following outline:

From a preliminary study it was found that the relation between buckling stresses and associated half wave length is of a parabolic form.

For a particular structure, the minimum point of this parabola represents the buckling stress and the critical wave length.

The suggested a simple iterative procedure in which the minimum of the parabola is found using Cramer's rule for a series of values of $\sigma$ and $L$:

1. The relation between $\sigma$ and $L$ is represented by this equation

$$\sigma = aL^2 + bL + c$$

2. A typical plate component width is chosen, and three fractions of it, say $L$, $0.8L$, $0.6L$, are used to find three corresponding stresses.

   By substitution of these three pairs of values in the parabolic equation given above, a system of three equations in three unknowns, a, b and c are found.

3. Cramer's rule is used to find the values of these unknowns.

4. Substitution of these values for a, b and c in the parabolic equation gives the actual parabolic equation for the structure.

5. From

$$\frac{\partial \sigma}{\partial L} = 0$$

A critical wave length is found.

6. Using this critical wave length another buckling stress is determined.

7. Among these four pairs of values of stresses and the associated critical wave lengths a comparison is made and the pair of highest stress is neglected and the remainder used again in the parabolic equation to find new values of a, b and c.

8. The procedure is repeated until convergence is obtained; only six steps were found to lead to very accurate results using program.

## 1.6 Conclusion

1. Finite strip is established method for local stability.
2. The finite strip method (FSM) is a variant of the finite element method that has been put to highly effective use in the study of the stability of thin-walled structures. For any thin-walled structure which may effectively be modelled as "extruded" FSM provides an incredibly powerful simplification to FEM.
3. In this project the finite strip method will be used for buckling analysis of plate type structure, computer code is specially developed for handling a

variety of buckling problems of thin walled structure and comparison of solutions with existing other methods will be shown.

# Chapter II
# Finite Strip Method for Isotropic Plate

# 2 Finite Strip Method for Isotropic Plates

## 2.1 Introduction

In this chapter, method of analysis of buckling of Axially Loaded Plate by Principle of Total Potential Energy Method (P.T.P.E) and by Finite Strip Method (FSM).

## 2.2 Principle of Minimum Total Potential Energy Method

The theorem of minimum total potential energy [17]:

" The total potential energy of an elastic system, i.e. the internal energy (in terms of displacements) plus the potential energy of the external forces, has stationary value for all small displacements when system is in equilibrium, and the stationary value is a minimum when the equilibrium is stable "

$$d(U+V)=0 \tag{2.1}$$

Total potential energy of plate [17]:

We can now express the total potential energy, i.e. the internal energy plus the potential energy of the external forces, first in terms of the three moments $M_x$ , $M_y$ and $M_{xy}$ , and then, by virtue of below equation, in terms of the displacement $w$ .

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right) \tag{2.2}$$

$$M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right) \tag{2.3}$$

$$M_{xy}=M_t=D(1-v)\frac{\partial^2 w}{\partial x\,\partial y} \tag{2.4}$$

Where:

D is Flexural Rigidity of Plate

$v$ is Poison's Ratio

Then, the internal energy of plate

$$(dU)_x = \frac{1}{2} M_x \, dy \, d\theta \qquad (2.5)$$

**Where**

$$d\theta = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) dx = -\frac{\partial^2 w}{\partial x^2} \, dx$$

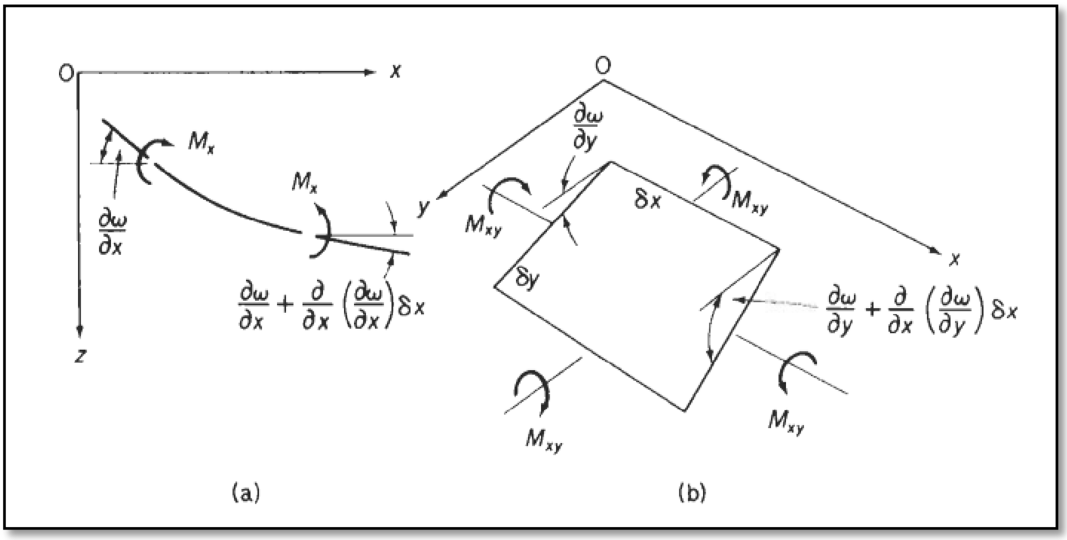



**Figure 2-1a) Strain energy of element due to bending, b) Strain energy due to twisting [17]**

From

$$(dU)_x = -\frac{1}{2} M_x \, \frac{\partial^2 w}{\partial x^2} \, dxdy$$

$$(dU)_y = -\frac{1}{2} M_y \, \frac{\partial^2 w}{\partial x^2} \, dxdy$$

$$(dU)_{xy} = \frac{1}{2} M_{xy} \ \frac{\partial^2 w}{\partial x \, \partial y} \ dxdy$$

$$(dU)_{yx} = -\frac{1}{2} M_{yx} \ \frac{\partial^2 w}{\partial x \, \partial y} \ dxdy = \frac{1}{2} M_{xy} \ \frac{\partial^2 w}{\partial x \, \partial y} \ dxdy$$

since

$$M_{xy} = -M_{yx}$$

The internal energy $dU_t$ due to the twisting moments is the sum of $(dU)_{xy}$ and $(dU)_{yx}$ and so

$$dU_t = M_{xy} \ \frac{\partial^2 w}{\partial x \, \partial y} \ dxdy \qquad (2.6)$$

Adding to this the work done by $M_x$ and $M_y$ , we obtain the internal energy for the plate element $dxdy$ in the form

$$dU = -\frac{1}{2}\left(M_x \ \frac{\partial^2 w}{\partial x^2} + M_y \ \frac{\partial^2 w}{\partial y^2}\right) dxdy + M_{xy} \ \frac{\partial^2 w}{\partial x \, \partial y} \ dxdy \qquad (2.7)$$

Substituting the values of $M_x$ ,$M_y$and $M_{xy}$

$$dU = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2v \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] dxdy + D(1-v)\left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 dxdy$$

$$dU = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-v)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dxdy$$

(2.8)

The total internal energy is obtained by integrating this expression over the whole plate area to give the standard formula [17]:

$$U = \frac{D}{2}\int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-v)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right] dxdy$$

(2.9)

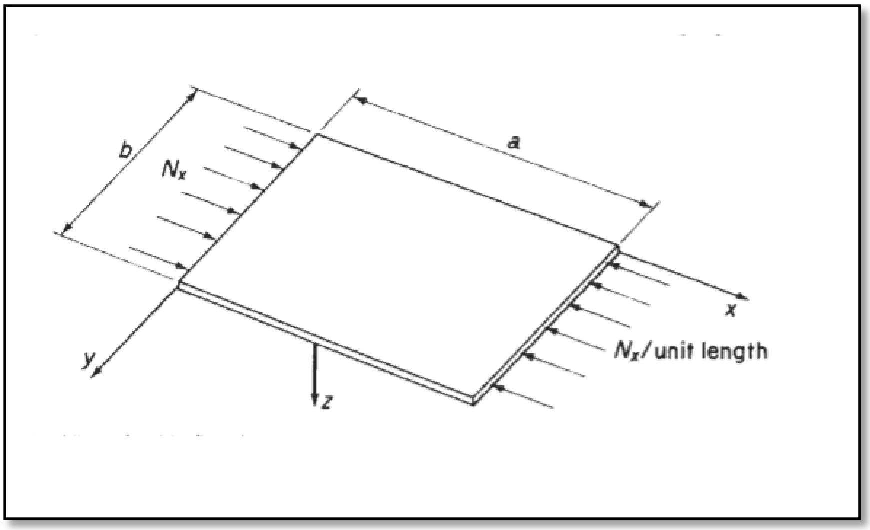



**Figure 2-2Buckling of thin plate**

Potential energy of external forces

$$V = -\frac{N_x}{2}\int_0^b \int_0^a \left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{2.10}$$

The displacement $\boldsymbol{w}$ normal to plane of plate

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2.11}$$

The boundary conditions are seen to be inherently satisfied by every term of the series, since each term gives

$$w = 0 \text{ at } \begin{cases} x = 0, & x = a \\ y = 0, & y = b \end{cases}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \; at \;\; x = 0, \quad x = a$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \; at \; y = 0, \quad y = b$$

Thus giving zero deflection and zero bending moment along all four edges of the plate.

Energies $U$ and $V$ expressed in terms of the unknown coefficients that define $w$ in above equation, the total potential energy is given

$$U + V = \pi^4 \frac{abD}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{\pi^2 b}{8a} N_x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 a_{mn}^2 \qquad (2.12)$$

Then; the minimum of (U+V):

$$\frac{\partial}{\partial a_{mn}} (U + V) = \pi^4 \frac{abD}{4} a_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{\pi^2 b}{4a} N_x m^2 a_{mn} = 0 \qquad (2.13)$$

Thus, either $a_{mn} = 0 - a$ trivial solution since it requires that the plate should remain flat or, by eliminating the unknown coefficient $a_{mn}, a$ relation is obtained between the load $N_x$ and physical characteristic of the plate. Thus

$$N_x = \pi^2 a^2 D \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \qquad (2.14)$$

Which is the load required.

It only remain therefore to find the combination of $m$ and $n$ (i.e the number of half- waves in the $x$ and $y$ directions respectively into which the plate deforms) that gives the lowest value of the critical load $N_x$ .

Clearly in the $y$ direction $n = 1$ gives a minimum value, so that whatever may be the values of the length $a$ and the width $b$ of the plate the deformation across the width invariably takes the form of a single half-sine wave as would be expected from simple physical reasoning.

The formula (2.14) takes therefore the simplified form

$$N_x = \pi^2 a^2 D \frac{1}{m^2}\left(\frac{m^2}{a^2} + \frac{1}{b^2}\right)^2 \tag{2.15}$$

$$N_x = \pi^2 a^2 D \frac{1}{m^2}\left(m + \frac{1}{m}.\frac{a^2}{b^2}\right)^2 \tag{2.16}$$

To explore the variation of $N_x$ for different value of $a$ and $b$, we consider certain special cases:

I. Length $a$ less than width $b$. as $m$ must be an integer and as in this case $a/b < 1$ , we see eq. (2.16) that, for any particular values of $a$ and $b$, the bracketed term takes the form of

$$m + (fraction < 1)$$

Which clearly is least when $m = 1$. Thus, if $a$ less than $b$, the plate deforms into a simple half-wave in each direction.

II. Length a = width b. That m=1 gives a lower critical load for this case than any other value of m, so that

$$N_x = \frac{\pi^2 D}{a^2}\left(1 + \frac{a^2}{b^2}\right)^2 = \frac{\pi^2 D}{b^2}\left(\frac{b}{a} + \frac{a}{b}\right)^2 \tag{2.17}$$

Moreover, so long as $m = 1$, we can prove at once that, for any given width $b$, the lowest value of $N_x$ is obtained by making $a/b = 1$, when from eq.(2.17)

$$N_x = 4\frac{\pi^2 D}{b^2} \tag{2.18}$$

III. Still keeping m=1 and b=const. We see from eq.(2.17) that

$$\frac{N_x}{\pi^2 D/_{b^2}} = k \tag{2.19}$$

We can plot it as shown in figure (2.3).

IV. Other values of $m$ with constant width $b$. For other values of $m$ and $a$

$$\frac{N_x}{\pi^2 D/_{b^2}} = k = \left(\frac{mb}{a} + \frac{a}{mb}\right)^2 \tag{2.19}$$

And we see that if we double m and also double the ratio $a/b$ , $k$ (and therefore $N_x$) remains unchanged. We need therefore only double the abscissa of the curve for $m = 1$ while keeping the ordinates $k$ the same to get the appropriate curve for $m = 2$. Similarly for $m = 3$ we treble the abscissa of every point on the curve $m = 1$, and so on, for the higher values of $m$, so getting a succession of curves as shown in fig.(2.3)

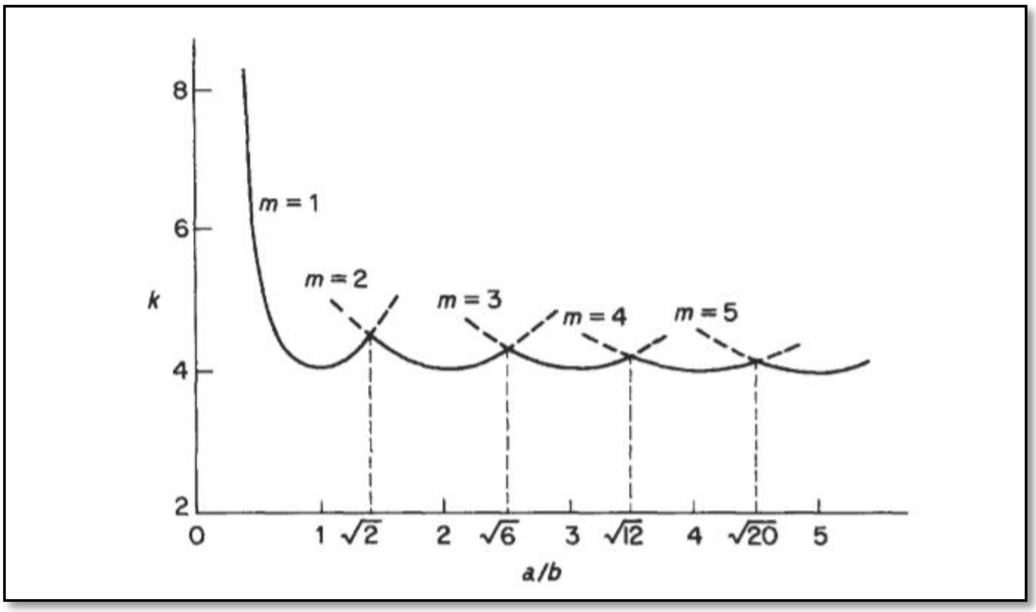



**Figure 2-3Buckling coefficient k for simple supported plates [17]**

Substituting on $D$ in eq. (2.19)

We can write the critical stress equation as follow:

$$\sigma_{cr} = \frac{k\pi^2 E}{12(1-v^2)} \cdot \frac{t^2}{b^2} \qquad (2.20)$$

## 2.3 Buckling of Axially Loaded Plate by Finite Strip Method (FSM) Procedure[4].

- **Assumption:**

1. Edge lines at junction between flat plate components remain fixed in space.
2. Component flat plates rotate about these edge lines.
3. Angles between elements of junctions before and after buckling remain the same.

4. Effect of side constraint is neglected.
5. Ends of structure are constrained to remain straight.

## 2.4 Elastic Stiffness Matrix for Strip

For the analysis, the structure is divided into a series of finite strips, each of length equivalents to half wave length L as shown in figure (2.4). A single strip with the relevant nodal displacements is represented in figure (2.5).

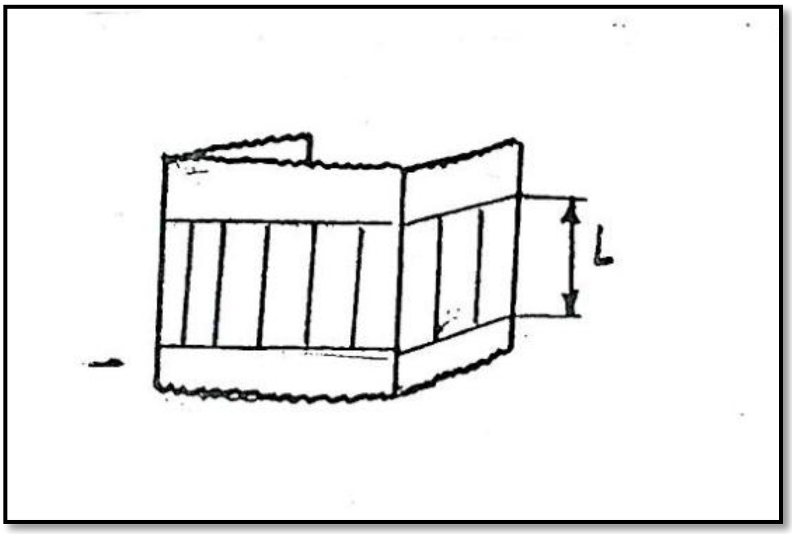



**Figure 2-4 finite strip for local stability idealisation**

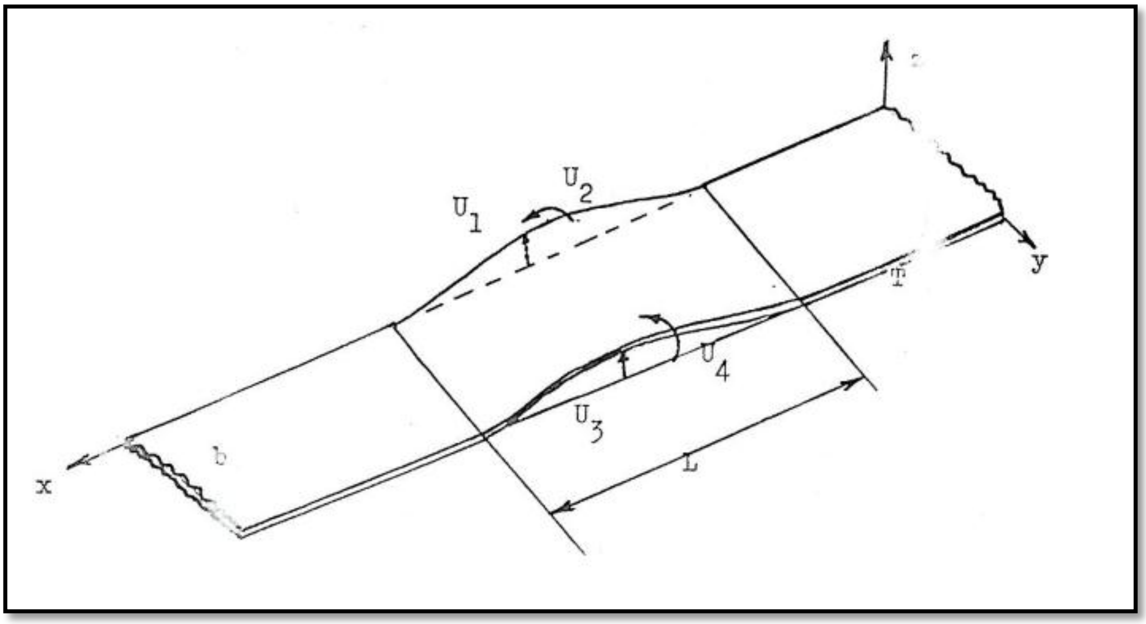



**Figure 2-5 single finite strip with nodal displacement**

The deflected shape for each strip is assumed to be given by:

$$w = f(Y)\sin\left(\frac{\pi\, x}{L}\right) \qquad , Y = \frac{y}{b} \qquad (2.21)$$

Where a cubic polynomial is used to represent the deflection in the transverse direction and the trigonometric term is to describe deflection in the longitudinal direction.

The deflected shape as a function of the nodal displacement will be

$$w = N_z\, U = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor\, s\ \{U_i\} \qquad (2.22)$$

Where

$$s = \sin\left(\frac{\pi\, x}{L}\right) \qquad , U_i = \lfloor U_1 \quad U_2 \quad U_3 \quad U_4 \rfloor$$

$$w = \lfloor (1-3y^2+2y^3) \quad (y-2y^2+y^3)b \quad (2y^2-2y^3) \quad (-y^2+ y^3)b \rfloor \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} s \qquad (2.23)$$

Then from the expression for strain a plate in bending

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{-z\delta^2 w}{\delta^2 x} \\ \frac{-z\delta^2 w}{\delta y^2} \\ -2\frac{z\delta^2 w}{\delta_{xy}} \end{bmatrix} = z \begin{bmatrix} \frac{\pi^2}{l^2}\ N\ sin\left(\pi\ ^{x}/_{L}\right) \\ -\frac{1}{b^2}\ N''\ sin\left(\pi\ ^{x}/_{L}\right) \\ -\frac{2\pi}{bl}\ N'\ sin\left(\pi\ ^{x}/_{L}\right) \end{bmatrix} U \qquad (2.24)$$

Where primes indicate differentiation relative to y, or in matrix from

$$\epsilon = b\ U \qquad (2.25)$$

Then from

$$K_E = \int_v b^T\ E\ b \quad dV \qquad (2.26)$$

With

$$E = \frac{E}{(1-v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{(1-v)}{2} \end{bmatrix}$$

Integrating, the elastic stiffness matrix dependent on half wave length will be

$$K_E = K_{e_1} + K_{e_2} + K_{e_3} \qquad (2.27)$$

Where

$$K_{e_1} = \frac{\pi^4\, E\, b\, t^3}{10080(1-v^2)L^3}\begin{bmatrix} 156 & & sym. & \\ 22b & 4b^2 & & \\ 54 & 12b & 156 & \\ -13b & -3b^2 & -22b & 4b^2 \end{bmatrix}$$

$$K_{e_2} = \frac{\pi^2\, E\, t^3}{360(1-v^2)b\, L}\begin{bmatrix} 36 & & sym. & \\ (3+15v)b & 4b^2 & & \\ -36 & -3b & 36 & \\ 3b & -b^2 & -(3+15v)b & 4b^2 \end{bmatrix}$$

$$K_{e_3} = \frac{E\, L\, t^3}{24(1-v^2)b^3}\begin{bmatrix} 12 & & sym. & \\ 6b & 4b^2 & & \\ -12 & -6b & 12 & \\ 6b & 2b^2 & -6b & 4b^2 \end{bmatrix}$$

## 2.5 Geometric Stiffness Matrix for Strip

The geometric stiffness matrix for the finite strip can be derived, by assuming middle plane constant stress $\sigma_x$ acting in x direction the geometric matrix will be given from

$$K_g = \int_0^L \int_0^b \int_{-t/2}^{t/2} \acute{N}^T\, \acute{N}\; dx\, dy\, dz \tag{2.28}$$

Where $\acute{N}$ is first derivative of $N$ relative to x and equal

$$\acute{N} = \frac{\pi}{L}\, N(y)\, \cos\left(\frac{\pi\, x}{L}\right)$$

The geometric matrix for single strip is given:

$$K_g = \frac{\sigma \, \pi^2 \, b \, t}{840 \, L} \begin{bmatrix} 156 & & sym. & \\ 22b & 4b^2 & & \\ 54 & 13b & 156 & \\ -13b & -3b^2 & -22b & 4b^2 \end{bmatrix}$$

## 2.6 The Determinant Equation

Once the elastic stiffness matrix and geometric stiffness matrix has been found for all single strips forming the structure, the elastic and the geometric stiffness matrices for the complete structure are assembled and the free body degrees of freedom are eliminated.

The equilibrium equation for the complete structure will be

$$(K_E + K_g)U = P$$

Introducing the constant $\lambda$ so that

$$P = \lambda \, P^*$$

And the geometric stiffness matrix becomes

$$K_g = \lambda \, {K_g}^*$$

Where ${K_g}^*$ is the geometric stiffness matrix for unit value of $\lambda$.

$$(K_e + \lambda \, {K_g}^*)U = \lambda \, P^*$$

And

$$U = (K_e + \lambda \, {K_g}^*)^{-1} \, \lambda \, P^*$$

Which means that the displacement tends to infinity when

$$\left|K_e + \lambda\, K_g{}^*\right| = 0 \tag{2.29}$$

$\left|K_e + \lambda\, K_g{}^*\right| = 0$ is the stability determinant.

## 2.7 The Buckling Load & The Buckling Mode

From eq. (2.29); the smallest value of its roots will be the buckling load for the structure.

And from $\left(K_e + \lambda\, K_g{}^*\right)U = 0$

Associated eigenvector will define the buckling shape. For the computer program, in this projector.

Since the buckling wave length is unknown, the following procedure is used to find the buckling stress and mode.

This procedure has the following outline:

From a preliminary study it was found that the relation between buckling stresses and associated half wave length is of a parabolic form.

For a particular structure, the minimum point of this parabola represents the buckling stress and the critical wave length.

The suggested a simple iterative procedure in which the minimum of the parabola is found using Cramer's rule for a series of values of $\sigma$ and $L$; with $L$ is half wave length of buckling :

9. The relation between $\sigma$ and $L$ is represented by this equation

$$\sigma = aL^2 + bL + c$$

10. A typical plate component width is chosen, and three fractions of it, say $L$, $0.8L$, $0.6L$, are used to find three corresponding stresses.

    By substitution of these three pairs of values in the parabolic equation given above, a system of three equations in three unknowns, a, b and c are found.

11. Cramer's rule is used to find the values of these unknowns.

12. Substitution of these values for a, band c in the parabolic equation gives the actual parabolic equation for the structure.

13. From

$$\frac{\partial \sigma}{\partial L} = 0$$

    A critical wave length is found.

14. Using this critical wave length another buckling stress is determined.

15. Among these four pairs of values of stresses and the associated critical wave lengths a comparison is made and the pair of highest stress is neglected and the remainder used again in the parabolic equation to find new values of a, b and c.

16. The procedure is repeated until convergence is obtained; only six steps were found to lead to very accurate results using program.

## 2.8 Conclusion

1. This method is very efficient for the study of a structure which fails by local buckling.
2. Band width is very small and computation time is, as can be see, shortest than the previous two methods.
3. It is based on the concept of the geometric stiffness matrix, and the solution procedure converges a six iterations to the exact value.

# Chapter III
# Computer Program

# 3 Computer Program

## 3.1 Introduction

In this chapter, layout procedure for the computer program is carried out, the program list is written in FORTRAN, based on the flow chart developed for the eigenvalue-eigenvector buckling problem of the elastic and geometric matrices for the assembled structure, the assembled structure is made of finite strips joined at the nodes.

## 3.2 Computer Program

The program process is as follow:

1. Following Run 1, geometric stiffness is computed for one strip element.

2. Assembly matrix is initiated and each element of geometric strip matrix is sent to the proper place in the assembly geometric matrix.

3. Boundary condition is introduced on the assembly matrix and reduced assembly geometric matrix $K_g$is formed.

4. Following Run 2, steps 1, 2, 3 are repeated for elastic stiffness matrix and reduced assembly elastic stiffness matrix $K_e$for full plate is derived.

5. To find Eigen-value and eigenvectors , and since subroutine require positive definite matrix to be introduced first, positions of matrices $K_e$ and $K_g$ are interchanged in characteristic determinant as follow:

$$\left| K_g + {}^{1}/_{\lambda}\ K_e \right| = 0$$

$$\left| K_g + \lambda'\ K_e \right| = 0$$

Where $\lambda'$ is the required Eigen-value.

A number of half wave lengths are used in the iteration procedure and the minimum of the curve will give the buckling load required

### 3.2.1 The Subroutines

For solving the characteristic equation for each step of the iteration process we proceed as follow [4]:

- Eigen-values and eigenvectors of the problem in the form $\boldsymbol{Ax} + \boldsymbol{Bx}$ should be found

- The second matrix $\boldsymbol{B}$ is decomposed into $L$ and $L^t$ , $\boldsymbol{B} = \boldsymbol{LL^t}$ . then using ( call CHOLDC {$K_e$ , N , n})

Note: $K_e = B$ and $K_g = A$

- The equation $\boldsymbol{Ax} = \boldsymbol{Bx}$ becomes

$$(L^{-1}\ A\ L^t)(L^t x) = (L^t x).$$

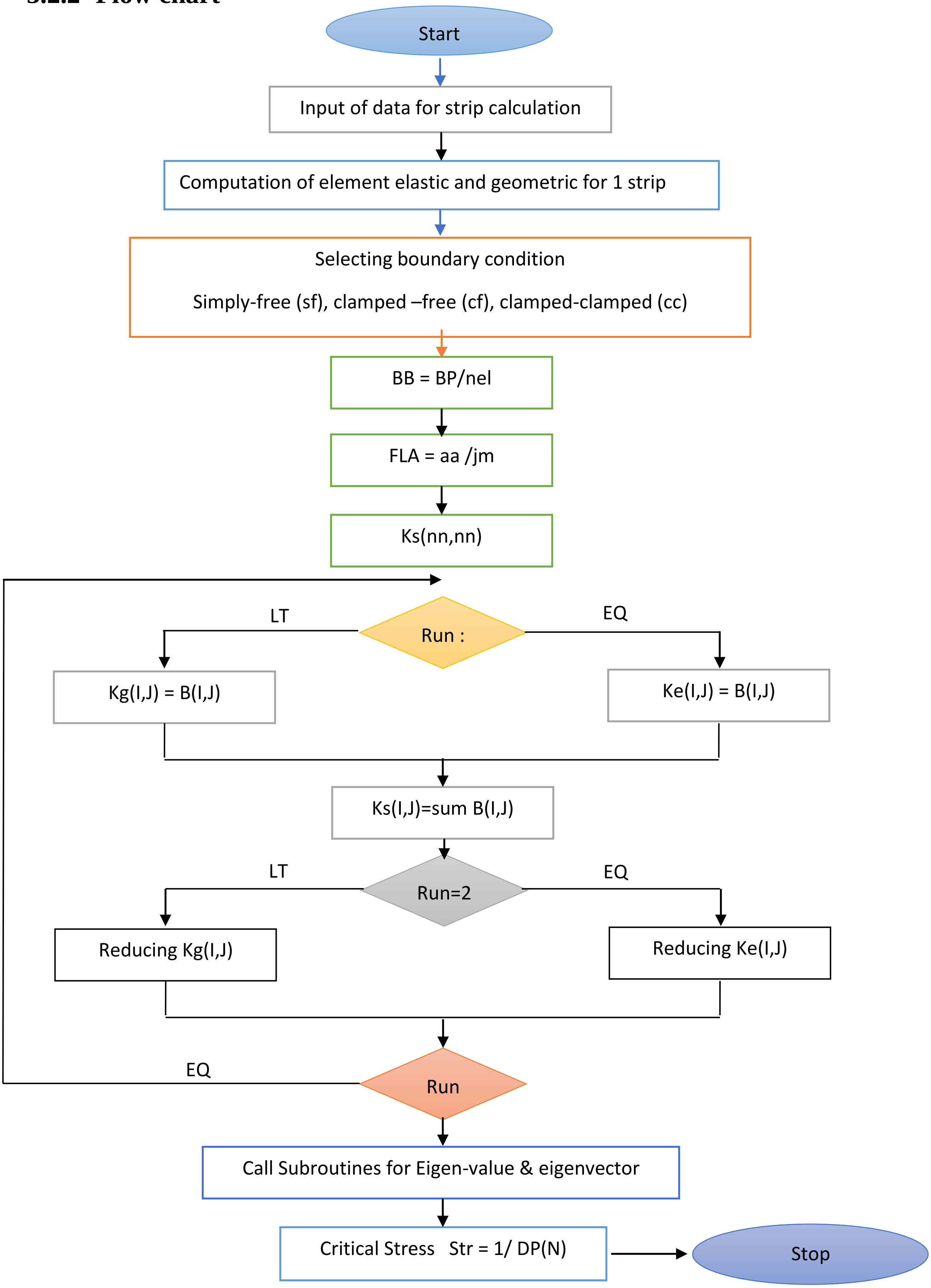
- Which can be written as $\boldsymbol{PY} = \boldsymbol{Y}$ where $\boldsymbol{P} = \boldsymbol{L^{-1}}\ \boldsymbol{A}\ \boldsymbol{L^{-t}}$ is the symmetric matrix (call PMAT{KEI, KG, KEIT, P, N}) .

- Householder's method is used to transform the matrix $\boldsymbol{P}$ into tridiagonal matrix. Using (call TRIDIAG{P, N, DP,EP})

- QL Algorithm is used to find eigenvalues and eigenvectors using (call tgli{dp, ep, n, N,z})

Note: the eigenvectors are related to the original $x$ by the relations$= L^t x$.

### 3.2.2 Flow chart



### 3.2.3 The Program Code List

The all details of program code list explain in the appendixes.

## 3.3 Case Study Ⅰ

The analysis is based on study of relatively long plate of steel with young's modulus E equal $210{,}000\, N/mm^2$, $\sigma_e$ equal $600\, N/mm^2$ with length $a = 457mm$ and width $b = 50.8mm$, thickness $t = 0.79\, mm$ .

**Three boundary conditions column matrices $NB\ (I)$ are as follow:**

- One side simply supported the other free $NB\ (1) = [1]$
- One side clamped the other free $NB\ (2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- Two sides clamped $NB\ (4) = \begin{bmatrix} 1 \\ 2 \\ 13 \\ 14 \end{bmatrix}$

- Simply Support – Free plate

**Table 3-1 Result of simply support – free plate**

| ***a/b*** | **Critical stress $N/mm^2$** | **Buckling coefficient K** |
|---|---|---|
| *0.449803* | 385.263 | 7.58598 |
| *0.478657* | 352.788 | 6.94652 |
| *0.510717* | 311.765 | 6.13877 |
| *0.546549* | 263.736 | 5.19307 |
| *0.58686* | 221.749 | 4.36633 |
| *0.632546* | 185.999 | 3.6624 |
| *0.684758* | 156.322 | 3.07805 |
| *0.745003* | 132.221 | 2.60349 |
| *0.815289* | 112.991 | 2.22484 |
| *0.898354* | 97.8796 | 1.92729 |

| | | |
|---|---|---|
| *0.998032* | 86.2259 | 1.69782 |
| *1.11986* | 77.5589 | 1.52716 |
| *1.27215* | 71.6685 | 1.41118 |
| *1.46794* | 68.6985 | 1.3527 |
| *1.729* | 63.8525 | 1.25728 |
| *2.09449* | 43.9169 | 0.823811 |
| *2.64272* | 35.2484 | 0.694054 |
| *3.55643* | 27.5739 | 0.54294 |
| *5.38386* | 22.8468 | 0.449863 |
| *10.8661* | 20.3187 | 0.400082 |

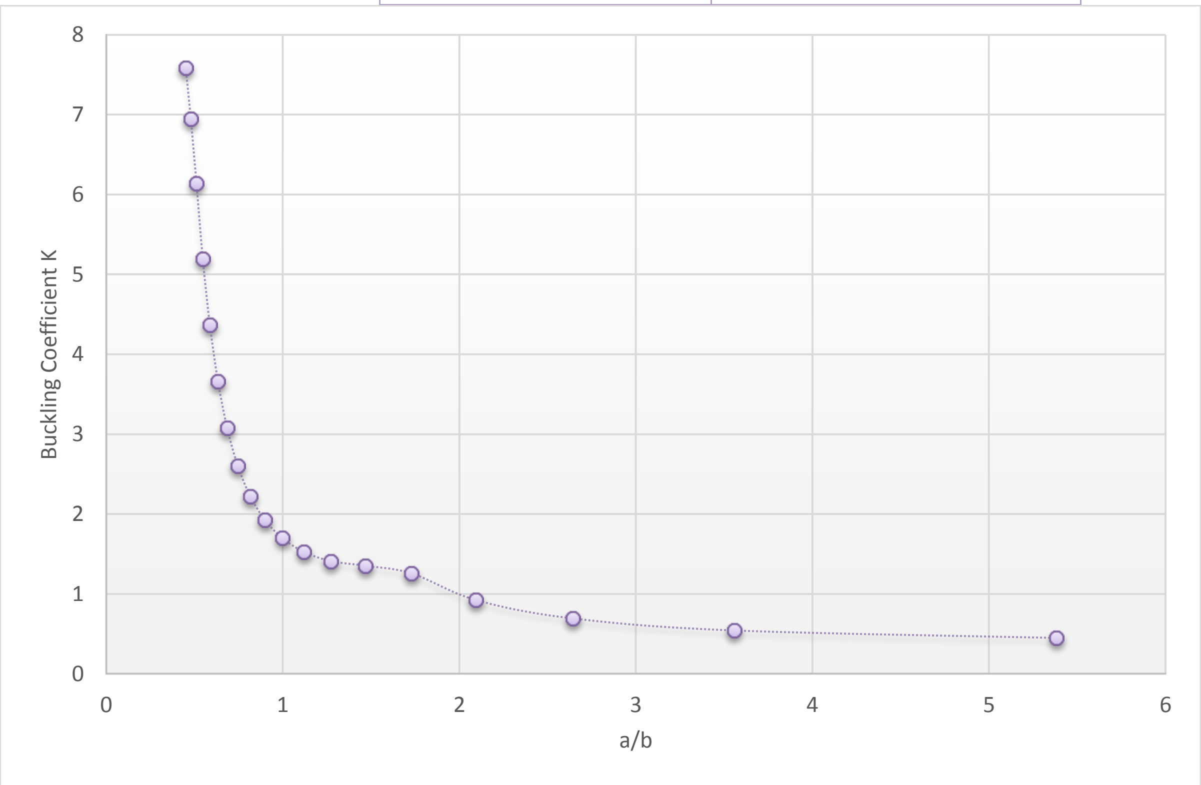



**Figure 3-1 Result of buckling coefficient K for simply support - free plate**

- Clamped – Free Plate

**Table 3-2 Result of clamped – free plate**

| ***a/b*** | **Critical stress** $N/mm^2$ | **Buckling coefficient K** |
|---|---|---|
| *0.449803* | 402.23 | 7.92007 |
| *0.478657* | 345.302 | 6.79911 |
| *0.510717* | 295.172 | 5.81204 |
| *0.546549* | 251.875 | 4.95952 |
| *0.58686* | 215.177 | 4.23691 |
| *0.632546* | 184.587 | 3.63459 |
| *0.684758* | 159.446 | 3.13955 |
| *0.745003* | 139.026 | 2.73748 |
| *0.815289* | 122.64 | 2.41483 |
| *0.898354* | 109.716 | 2.16035 |
| *0.998032* | 99.858 | 1.96624 |
| *1.11986* | 92.8926 | 1.82909 |
| *1.27215* | 88.9383 | 1.75123 |
| *1.46794* | 88.555 | 1.74368 |
| *1.729* | 93.1065 | 1.8333 |
| *2.09449* | 105.714 | 2.08156 |
| *2.64272* | 134.12 | 2.64087 |

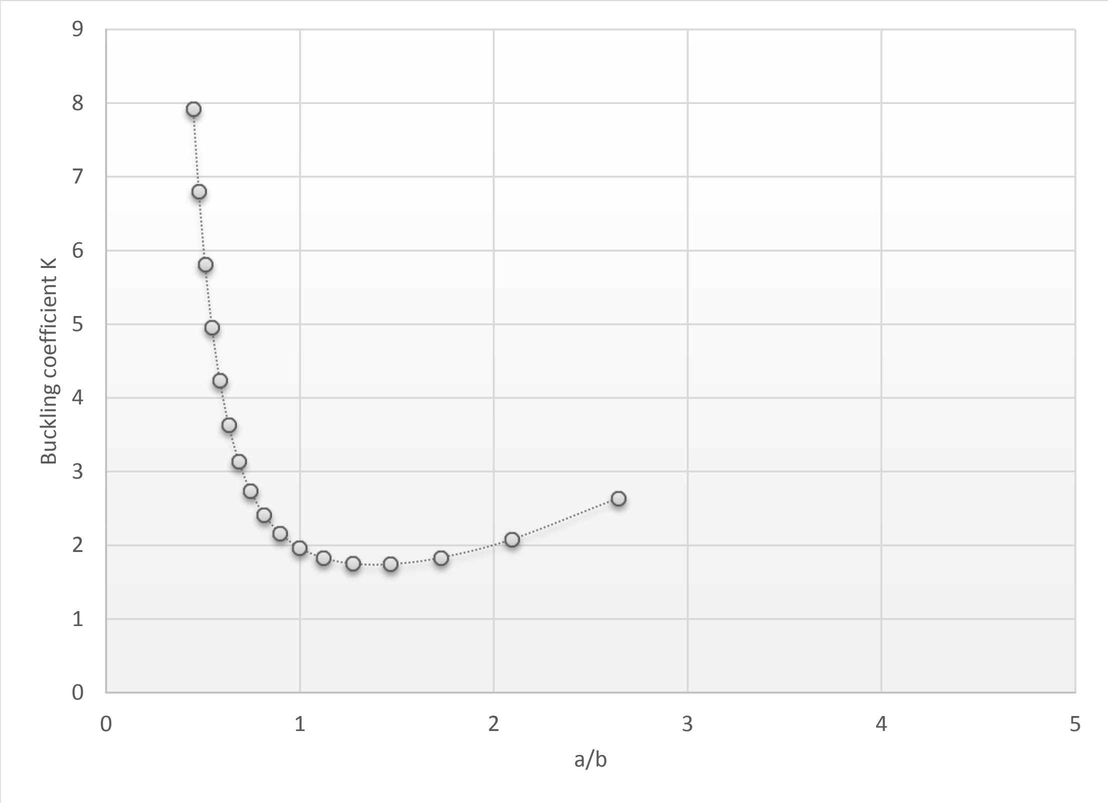



**Figure 3-2 Result of buckling coefficient K for clamped - free plate**

- Clamped – Clamped Plate

**Table 3-3 Result of clamped – clamped plate**

| ***a/b*** | **Critical stress $N/mm^2$** | **Buckling coefficient K** |
|---|---|---|
| *0.449803* | 611.022 | 12.0313 |
| *0.473477* | 589.275 | 11.6031 |
| *0.499781* | 572.405 | 11.2709 |
| *0.529180* | 560.755 | 11.0415 |
| *0.562254* | 554.828 | 10.9248 |
| *0.599738* | 555.360 | 10.9352 |
| *0.642576* | 548.035 | 10.7910 |
| *0.692005* | 541.122 | 10.6549 |
| *0.749672* | 542.458 | 10.6812 |
| *0.817824* | 554.427 | 11.9169 |

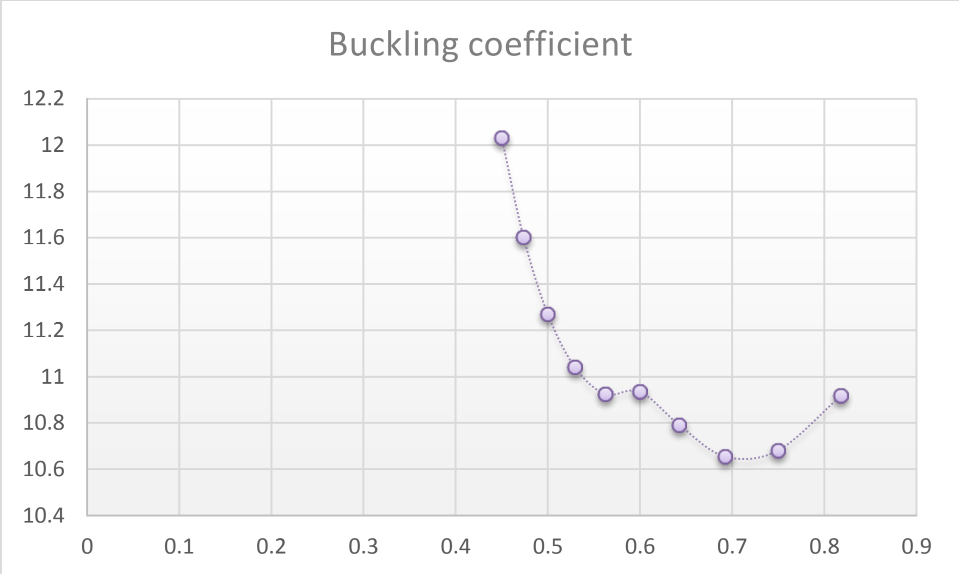



**Figure 3-3 Result of clamped – clamped plat**

## 3.4 Comparison

The results obtained from the program were compared using classical approach solution [20] the results found to be very closed as will be seen.

$$From \quad \sigma = k\,\frac{\pi^2 E}{12(1-v^2)}\left(\frac{t}{b}\right)^2 = k\,\frac{3.14^2 * 210{,}000}{12(1-0.3^2)} * \left(\frac{0.79}{50.8}\right)^2$$

At the simply support – free plate $a/b$ equal $2$ , buckling coefficient equal $0.85$ the buckling stress from the above calculation equation equals $38.979\ N/mm^2$ and from the program $a/b$ equal $2.09$ , buckling coefficient equal $0.82$ the buckling stress result $43.91\ N/mm^2$.

At the clamped – free plate $a/b$ equal $1$ , buckling coefficient equal 1.8 the buckling stress from the above calculation equation equals $82.53\ N/mm^2$ and from the program $a/b$ equal $1.2$ , buckling coefficient equal 1.75 the buckling stress result $88.93\ N/mm^2$.

At the clamped – clamped plate $a/b$ equal $0.8$ , buckling coefficient equal 12 the buckling stress from the above calculation equation equals $550.248\ N/mm^2$ and from the program $a/b$ equal $0.81$ , buckling coefficient equal 11.91 the buckling stress result $554.42\ N/mm^2$

Clamped – free plate comparison of results obtained by program with results obtained by classical theory, as shown in figure (3.4), the results are very close.

**Table 3-4 Results of clamped–free plate obtained by classical theory**

| ***a/b*** | **Critical stress** $N/mm^2$ | **Buckling coefficient K** |
|---|---|---|
| *0.5* | 275.124 | 6 |
| *0.6* | 174.2452 | 3.8 |
| *0.7* | 137.562 | 3 |
| *0.8* | 100.8788 | 2.2 |
| *0.9* | 87.1226 | 1.9 |
| *1* | 82.5372 | 1.8 |
| *1.25* | 77.9518 | 1.7 |
| *1.5* | 73.3664 | 1.6 |
| *1.75* | 80.2445 | 1.65 |

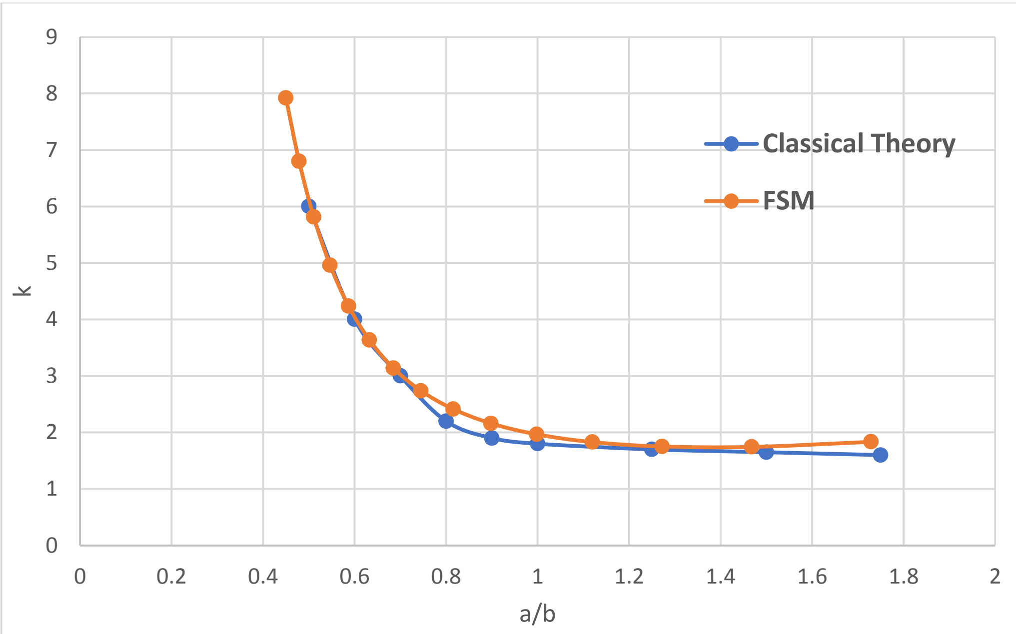



**Figure 3-4 Comparison of results for clamped – free plate**

# Chapter IV
# Conclusion and Recommendations

# 4 Conclusion and Recommendations

## 4.1 Conclusion

1. Finite strip is established method for local stability.
2. The finite strip method (FSM) is a variant of the finite element method that has been put to highly effective use in the study of the stability of thin-walled structures.
3. In this project the finite strip method was used for buckling analysis of plate type structure.
4. In this project a computer code is specially developed for handling buckling problems of thin pates with different types of side constraints and comparison of solutions with existing classical approach was carried out.
5. The comparison showed that the method is very efficient for the study of a structure which fails by local buckling.
6. Order of element elastic and geometric matrices is small and computation time is shorter than Finite Element Method.

7-The buckling mode and buckling load found from the formulation of the eigenvalue -eigenvector problem of the elastic and geometric assembly matrices of the structure and the solution is based on iteration procedure built in the program

## 4.2 Recommendation

This study led to the appearance of different ideas that could be the focus of a future research work using finite strip method such as:

1. Developing of elastic and geometric stiffness's matrices for orthotropic plates with different stiffness's in different directions.

**2.** Buckling analysis of orthotropic plates using finite strip procedure.

**3.** Comparison of solutions between finite strip method and experimental work. .

**4.** Developed of more general computer program to solve buckling problems of isotropic and orthotropic panels.

**5.** Use of Finite strip method for design of thin walled panel based on local buckling criteria.

# Chapter V

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# Chapter VI

# Appendices

## Appendices

## - Main program:

```
program finite_strip_for_buckling_of_isotropic_plate


  implicit none
  real(8),allocatable,Dimension(:,:) :: nd,ko,kg,ke,b,ks,KEI,KEIT,P,z
  real(8),allocatable,Dimension(:) :: NB,eP,dP,str
  real(8) PI,aa,bp,un,PHA,PHB,PHC,PHD,FLA,FLABP,E,BB,TT,x7
  integer :: i,j,k,nn,noc,m1,m2,run,nel,ll,ind,nod,bd,n,JM,jmm


  character(len=2) choice


! open(1,File='example.dat')
  open(2,File='finite-strip1.out')
  open(3,File='finite-strip_jm.out')
  open(4,File='finite-strip_jm.plt')
  open(44,File='finite-strip_strcr.out')
! open(14,File='finite-strip_1.plt')
! open(15,File='finite-strip_2.plt')


! Input Data


  e=210000       ! Young's modulus
  tt=0.79        ! Thickness of plate
  AA=12.7  !457  ! Length of plate
  BP=50.8        ! Width of plate
  un=0.3         ! Poison's ratio
  PI=3.141593    ! Pi
  nel=6          ! Number of elements
  ind=nel+1      ! Number of nodes
```

```
  ll=2          ! Degrees of freedom
  nn=ind*ll      ! Total number DOF (Size of assembly matrix)
  NOD=2          ! Number of nodes for element
  bd=ll*nod      ! Matrix b dimension

! Boundary Condition

  write(*,'("Enter boundary condition case "/"ss/free-->(sf)"/"clamped/free--
>(cf)"/"clamped/clamped-->(cc)")')
  read(*,*)choice
  select case(choice)
  case('sf','SF')
   noc=1          ! Number of constraints
   write(2,*)"Simply suported and free (sf)"
  case('cf','CF')
    noc=2          ! Number of constraints
    write(2,*)"Clamped and free (cf)"
  case('cc','CC')
    noc=4          ! Number of constraints
    write(2,*)"Clamped and clamped (cc)"
  case default
    write(2,*)"Error .. Enter valid boundary condition"
  end select
    n=nn-noc       ! size number of reduced matrix

  allocate(nd(ind,nod),KO(n,n),KG(n,n),KE(n,n),b(bd,bd),ks(nn,nn),&
     nb(noc),KEI(n,n),KEIT(n,n),P(n,n),eP(n),dP(n),z(n,n),str(60*n))

  select case(choice)
  case('sf','SF')
  write(2,*)"Simply supported and free (sf)"
```

```
  NB(1)=1
 case('cf','CF')
 write(2,*)"Clamped and free (cf)"
  NB(1)=1
  NB(2)=2
 case('cc','CC')
 write(2,*)"Clamped and clamped (cc)"
  NB(1)=1
  NB(2)=2
  NB(3)=nn-1
  NB(4)=nn
 case default
 write(2,*)"Error .. Enter valid boundary condition"
 end select

! Nodal numbering
 ND=0
 k=0
 do i=1,ind
  do j=1,nod
   k=k+1
   if   (i > 1 .and. j == 1)then
    k=k-1
    nd(i,j)=k
   else
    nd(i,j)=k
   endif
  enddo
enddo
 Write(2,'(//,2(2X,f5.2),/)')((ND(I,J),J=1,nod),I=1,ind)
```

```
  Write(2,'(//,2X,"AA=",g9.4,/,2X,"BP=",g9.4,/,2X,"E
=",g9.4,/,2X,"tt=",g9.4)')AA,BP,E,tt
  Write(3,'(5x,"a/b",10x,"Critical stress",8x,"Buckling coefficient")')
  Write(44,'(10x,"Critical stress",8x,"Buckling coefficient",5x,"QM")')

   Do jm=1,30
    Write(2,'(//," ************* jm=",i2," *************")')jm

     BB=BP/nel
     jmm=(31-jm)
     Fla=aa
     FLABP=aa/BP
!     FLABP=fla/BP  ! a/b

! Computation of Elastic Stiffness of Single Strip+ Goemetric Stiff Constant

  PHA=(PI**4)*E*BB*(TT**3)/(10080*(1-UN**2)*FLA**3)
  PHB=(PI**2)*E*(TT**3)/(360*(1-UN**2)*BB*FLA)
  PHC=E*FLA*(TT**3)/(24*(1-UN**2)*BB**3)
  PHD=(PI**2)*BB*TT/(840*FLA)

Write(2,'(//,2x,"PHA=",g9.4,/,2x,"PHB=",g9.4,/,2x,"PHC=",g9.4,/,2x,"PHD=",g9.4)')PH
A,PHB,PHC,PHD

  do run=1,2
    KS=0.0
    if (run == 1)then
!    Computation of geometric stiffness for single strip
      B(1,1)=PHD*156
      B(2,1)=PHD*22*BB
      B(3,1)=PHD*54
```

```
      B(4,1)=PHD*(-13)*BB
      B(2,2)=PHD*4*BB*BB
      B(3,2)=PHD*13*BB
      B(4,2)=PHD*(-3)*BB*BB
      B(3,3)=PHD*156
      B(4,3)=PHD*(-22)*BB
      B(4,4)=PHD*BB*BB*4

      do i=1,bd
        do j=1,bd
          b(i,j)=b(j,i)
        end do
      end do
      write(2,'(//,1X,"b=",/,6("---"))')
      do i=1,bd
        write(2,'(30(2X,es12.5))')(b(i,j),j=1,bd)
      enddo

! building of assembly matrix from single components for geometric matrix
       m1=1
       m2=0
       do i=1,nel
         ks(m1:bd+m2,m1:bd+m2)=ks(m1:bd+m2,m1:bd+m2)+b
         m1=m1+2
         m2=m2+2
      enddo
      write(2,'(//,1X,"ks=",/,6("---"))')
      do i=1,nn
      write(2,'(30(2X,g14.5))')(ks(i,j),j=1,nn)
      enddo
```

```
!   reduced geometric matrix
     !   reduced elastic matrix
      if(choice == 'sf')then
        write(2,'(//a)')"Simply supported and free (sf)"
        kg(1:n,1:n)=ks(ll:nn,ll:nn)
!        write(2,'(//,1x,"Kg=",/,6("---"),/,5(2X,f14.3))')kg
      elseif(choice == 'cf')then
        write(2,'(//a)')"Clamped and free (cf)"
        kg(1:n,1:n)=ks(ll+1:nn,ll+1:nn)
!        write(2,'(//,1x,"Kg=",/,6("---"),/,4(2X,f14.3))')kg
      elseif(choice == 'cc')then
        write(2,'(//a)')"Clamped and clamped (cc)"
        kg(1:n,1:n)=ks(ll+1:nn-2,ll+1:nn-2)
!        write(2,'(//,1x,"Kg=",/,6("---"),/,2(2X,f14.3))')kg
      else
        write(2,*)"Error .. Enter valid boundary condition"
      endif
      write(2,'(//,1X,"kg=",/,6("---"))')
      do i=1,n
      write(2,'(30(2X,g14.5))')(kg(i,j),j=1,n)
      enddo

    else
!  Elastic Stiffness matrix for single strip
      B(1,1)=PHA*156+PHB*36+PHC*12
      B(2,1)=PHA*22*BB+PHB*(3+15*UN)*BB+PHC*6*BB
      B(3,1)=PHA*54+PHB*(-36)+PHC*(-12)
      B(4,1)=PHA*(-13)*BB+PHB*(3)*BB+PHC*6*BB
      B(2,2)=4*BB*BB*(PHA+PHB+PHC)
```

```
      B(3,2)=PHA*13*BB+PHB*(-3)*BB+PHC*(-6)*BB
      B(4,2)=PHA*BB*BB*(-3)+PHB*BB*BB*(-1)+PHC*BB*BB*2
      B(3,3)=PHA*156+PHB*36+PHC*12
      B(4,3)=PHA*(-22)*BB+PHB*(-3-15*UN)*BB+PHC*(-6)*BB
      B(4,4)=4*BB*BB*(PHA+PHB+PHC)

      do i=1,bd
        do j=1,bd
          b(i,j)=b(j,i)
        end do
      end do
      write(2,'(//,1X,"b=",/,6("---"))')
      do i=1,bd
        write(2,'(30(2X,es12.5))')(b(i,j),j=1,bd)
      enddo

!  Building of assembly matrix from single components for elastic matrix
       m1=1
       m2=0
       do i=1,nel
         ks(m1:bd+m2,m1:bd+m2)=ks(m1:bd+m2,m1:bd+m2)+b
         m1=m1+2
         m2=m2+2
      enddo
      write(2,'(//,1X,"ks=",/,6("---"))')
      do i=1,nn
      write(2,'(30(2X,g14.5))')(ks(i,j),j=1,nn)
      enddo

 !   calculating of reduced elastic matrix for certain boundary conditions
```

```
      if(choice == 'sf')then
        write(2,'(//a)')"Simply supported and free (sf)"
        ke(1:n,1:n)=ks(ll:nn,ll:nn)
      elseif(choice == 'cf')then
        write(2,'(//a)')"Clamped and free (cf)"
        ke(1:n,1:n)=ks(ll+1:nn,ll+1:nn)
      elseif(choice == 'cc')then
        write(2,'(//a)')"Clamped and clamped (cc)"
        ke(1:n,1:n)=ks(ll+1:nn-2,ll+1:nn-2)
      else
        write(2,*)"Error .. Enter valid boundary condition"
      endif
      write(2,'(//,1X,"ke=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(ke(i,j),j=1,n)
      enddo
    endif


  enddo


     CALL CHOLDC(KE,N,n)
!================== PRINTING LOWER MATRIX OF A
========================
      write(2,'(//,1X,"LOWER MATRIX IS: ke=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(ke(i,j),j=1,n)
      enddo


      CALL LMI(KE,KEI,KEIT,N)
!=============== FINDING TRANSPOSE OF LOWER MATRIX
=====================
```

```
      write(2,'(//,1X,"INVERSE OF LOWER MAT: kei=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(kei(i,j),j=1,n)
      enddo
!================= FINDING A= L * L TRANSPOSE
========================
      write(2,'(//,1X,"TRANS OF INV MAT IS: keit=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(keit(i,j),j=1,n)
      enddo

      CALL PMAT(KEI,KG,KEIT,P,N)
      write(2,'(//,1X,"P MAT IS: p=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(p(i,j),j=1,n)
      enddo

      CALL TRIDIAG(P,N,DP,EP)
       Z=P
      write(2,'(//,1X,"Tridiagonal mat: p=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(p(i,j),j=1,n)
      enddo
      CALL tqli(dP,eP,n,n,z)
      write(2,'(//,1X,"Z MAT EIGENVECTORS IS: z=",/,6("---"))')
      do i=1,n
        write(2,'(30(2X,g14.5))')(z(i,j),j=1,n)
      enddo
      WRITE(2,'(//,1X,"EIGENVALUES:",/,6("---"),/,1(2X,F10.8))')(DP(I),I=1,N)
```

```
    str(jm)=1.0/maxval(dp)
    x7=str(jm)/(E*((TT/BP)**2))

    Write(3,'(2X,g14.6,5x,g14.6,5x,g14.6)')FLABP,str(jm),x7

    if(  jm > 20 .and. (str(jm)-str(jm-1)) > 0.0 )exit
    aa=aa+5.0

enddo  !jm

  call plot

  end program finite_strip_for_buckling_of_isotropic_plate
```

## Subroutines:

```
! Subroutines $ Functions

! Cholesky's Decomposition
! NUMERICAL RECIPES FORTRAN POWER STATION 4.0
!---------------------------------------------
! MODIFIED BY AHMAD AL-MAKHLUFI 21-3-2013

      SUBROUTINE choldc(a,n,np)
      INTEGER n,np
      DOUBLE PRECISION a(np,np),p(n),sum
      INTEGER i,j,k
   !  REAL sum
      do 13 i=1,n
        do 12 j=i,n
          sum=a(i,j)
          do 11 k=i-1,1,-1
```

```
          sum=sum-a(i,k)*a(j,k)
11        continue
        if(i.eq.j)then
          if(sum.le.0.)THEN
          WRITE(*,*)"choldc failed"
          STOP
          ENDIF
          p(i)=sqrt(sum)
          a(j,i)=p(i)
        else
          a(j,i)=sum/p(i)
        endif
12      continue
13    continue
      DO I=1,N
      DO J=1,N
      IF (J.GT.I)THEN
      A(I,J)=0.0
      ELSE
      A(I,J)=A(I,J)
      END IF
      END DO
      END DO
      return
      END




!     inversion of lower matrix


      SUBROUTINE LMI(A,X,Y,N)
      DIMENSION A(n,n),X(n,n),y(n,n)
```

```
      DOUBLE PRECISION A,X,Y,SUM
      INTEGER N,I,J,M,IN
      DO 10 I=1,N
      DO 10 J=1,N
10    X(I,J)=0.0
      Do 20 I=1,N
      J=I
20    X(I,J)=X(I,J)+1/A(I,I)
      DO 11 I=2,N
      IN=I-1
      DO 12 J=1,N
      DO 13 M=J,IN
      SUM=0.0
      SUM=SUM+A(I,M)*X(M,J)/A(I,I)
      X(I,J)=X(I,J)-SUM
13    CONTINUE
12    CONTINUE
11    CONTINUE
!       Transpose of the inverse of L=Y:
      Do 30 I=1,N
      Do 30 J=1,N
30    Y(I,J)=X(J,I)
      RETURN
      END

      SUBROUTINE PMAT(XLI,A,XLIT,Y,N)
      DIMENSION XLI(N,N),A(N,N),XLIT(N,N),Y1(N,N),Y(N,N)
      DOUBLE PRECISION XLI,A,XLIT,Y,Y1
      INTEGER N,I,J,K
      DO 10 I=1,N
      DO 10 J=1,N
      Y1(I,J)=0.0
```

```
      DO 10 K=1,N
10    Y1(I,J)=Y1(I,J)+XLI(I,K)*A(K,J)
      DO 20 I=1,N
      DO 20 J=1,N
      Y(I,J)=0.0
      DO 20 K=1,N
20    Y(I,J)=Y(I,J)+Y1(I,K)*XLIT(K,J)
      RETURN
      END


      SUBROUTINE TRIDIAG(A,N,D,SD)
      INTEGER N,I,J,K,KK,KKK,M,N1
      Real*8 SUM
      Real*8 A1,R1
      REAL*8,DIMENSION(N,N)::A,Z,P,PP,Z1,Z2,Z3
      REAL*8,DIMENSION(N,1)::X
      REAL*8,DIMENSION(1,N)::Y
      REAL*8,DIMENSION(N)::D,SD


!         Open(2,FILE='MAT.DAT')
!         Open(1,FILE='MAT.OUT')
!         DATA((A(I,J),J=1,4),I=1,4) &
!         /4.0,1.0,-2.0,2.0,1.0,2.0,0.0,1.0,-2.0,0.0,3.0,-2.0,2.0,1.0,-2.0,-1.0/
!   N =10
! write(*,*)"n=",n
!          read(2,*)((A(I,J),J=1,n),I=1,n)
!          write(1,'("A",/,7(3x,f10.8),/)')((A(I,J),J=1,N),I=1,N)
      KK=1
100   KKK=KK+1
      SUM=0.0
```

```
      DO I=KKK,N
      SUM=SUM+A(I,KK)*A(I,KK)
      END DO
      A1=-SQRT(SUM)
      R1=SQRT(0.5*(A1**2.-A(KKK,KK)*A1) )
      DO I=1,KK
      X(I,1)=0.0
      END DO
      X(KKK,1)=(A(KKK,KK)-A1)/(2*R1)
      M=KKK+1
      DO I=M,N
      X(I,1)=A(I,KK)/(2.*R1)
      END DO
!           write(1,'("x",/,(3x,f10.5),/)') ((X(I,J),J=1,1),I=1,N)
      Do 30 I=1,N
      Do 30 J=1,1
30    Y(1,I)=X(I,J)
!     write(1,'("y",/,7(3x,f10.5),/)') ((Y(I,J),J=1,N),I=1,1)
      DO I=1,N
      DO J=1,N
      Z(I,J)=X(I,1)*Y(1,J)
      END DO
      END DO
!           write(1,'("z",/,7(3x,f10.5),/)')  ((Z(I,J),J=1,N),I=1,N)
      DO I=1,N
      DO J=1,N
      Z1(I,J)=2.*Z(I,J)
      END DO
      END DO
!           write(1,'("z1",/,7(3x,f10.5),/)')  ((Z1(I,J),J=1,N),I=1,N)
      DO I=1,N
      DO J=1,N
```

```
      Z2(I,J)=0.0
      END DO
      END DO
      DO I=1,N
        J=I
      Z2(I,J)=Z2(I,J)+1.0
      END DO
!           write(1,'("z2",/,7(3x,f10.5),/)')  ((Z2(I,J),J=1,N),I=1,N)
      DO I=1,N
      DO J=1,N
      Z3(I,J)=Z2(I,J)-Z1(I,J)
      END DO
      END DO
!     write(1,'("Z3",/,7(3x,f10.5),/)')  ((Z3(I,J),J=1,N),I=1,N)
      DO I=1,N
      DO J=1,N
      P(I,J)=0
      DO K=1,N
      P(I,J)=P(I,J)+Z3(I,K)*A(K,J)
      END DO
      END DO
      END DO
!     write(1,'("P",/,7(3x,f10.5),/)')  ((P(I,J),J=1,N),I=1,N)
      DO I=1,N
      DO J=1,N
      pp(i,j)=0.0
      DO K=1,N
        PP(I,J)=PP(I,J)+P(I,K)*Z3(K,J)
      END DO
      END DO
      END DO
!           write(1,'("PP",/,7(3x,f10.8),/)')  ((PP(I,J),J=1,N),I=1,N)
```

```
      DO I=1,N
      DO J=1,N
      A(I,J)=PP(I,J)
      END DO
      END DO
      N1=N-2
      KK=KK+1
      IF(KK.LE.N1) GOTO 100
!======================================
      DO I=1,N
      DO J=1,N
      IF(I.EQ.J)THEN
       D(I)=A(I,J)
      ELSEIF(J.EQ.I+1)THEN
        SD(1)=0
        SD(J)=A(I,J)
      ENDIF
      ENDDO
      ENDDO
!======================================
      RETURN
!     STOP
      END

      !  tqli.for

      SUBROUTINE tqli(d,e,n,np,z)
      INTEGER n,np
      DOUBLEPRECISION d(np),e(np),z(np,np)
!     USES pythag
      INTEGER i,iter,k,l,m
      DOUBLEPRECISION b,c,dd,f,g,p,r,s
```

```
      DOUBLEPRECISION pythag

!           WRITE(2,'(2X,F10.5)')(D(I),I=1,NP)
!           WRITE(2,'(2X,F10.5)')(E(I),I=1,NP)

      do 11 i=2,n
        e(i-1)=e(i)
11    continue
      e(n)=0.
      do 15 l=1,n
        iter=0
1         do 12 m=l,n-1
          dd=abs(d(m))+abs(d(m+1))
          if (abs(e(m))+dd.eq.dd) goto 2
12        continue
        m=n
2         if(m.ne.l)then
        if(iter.eq.30) then
        write(*,*) 'too many iterations in tqli'
          iter=iter+1
          g=(d(l+1)-d(l))/(2.*e(l))
          r=pythag(g,1.D0)

          g=d(m)-d(l)+e(l)/(g+sign(r,g))
          s=1.
          c=1.
          p=0.
          do 14 i=m-1,l,-1
            f=s*e(i)
            b=c*e(i)
            r=pythag(f,g)
            e(i+1)=r
```

```
          if(r.eq.0.)then
            d(i+1)=d(i+1)-p
            e(m)=0.
            goto 1
          endif
          s=f/r
          c=g/r
          g=d(i+1)-p
          r=(d(i)-g)*s+2.*c*b
          p=s*r
          d(i+1)=g+p
          g=c*r-b

!     Omit lines from here ...
          do 13 k=1,n
            f=z(k,i+1)
            z(k,i+1)=s*z(k,i)+c*f
            z(k,i)=c*z(k,i)-s*f
13          continue
!     ... to here when finding only eigenvalues.
14        continue
        d(l)=d(l)-p
        e(l)=g
        e(m)=0.
        goto 1
        endif
      endif
15      continue
       return
       end
```

```
! pythag.for

      FUNCTION pythag(a,b)
      DOUBLEPRECISION pythag
      DOUBLEPRECISION a,b
      DOUBLEPRECISION absa,absb
      absa=abs(a)
      absb=abs(b)
      if(absa.gt.absb)then
        pythag=absa*sqrt(1.+(absb/absa)**2)
      else
        if(absb.eq.0.)then
          pythag=0.
        else
          pythag=absb*sqrt(1.+(absa/absb)**2)
        endif
      endif
      return
      END

!================================ Plotting the output by using gnuplot
================================
subroutine plot
  write(4,'(a/,a,a/a/a/a/a)')" set terminal wxt 0",                                        &
             &     "plot 'finite-strip_jm.out' using 1:2 with lp title 'Critical stress', ",          &
             &     "'finite-strip_jm.out' using 1:3 with lp title 'Buckling coefficient', ",          &
             &     " set title 'finite strip for buckling of isotropic plates '",                     &
             &     " set xlabel 'a/b'",                                                               &
             &     " set ylabel 'Critical stress & Buckling coefficient'   ",                         &
             &     " pause -1"
```

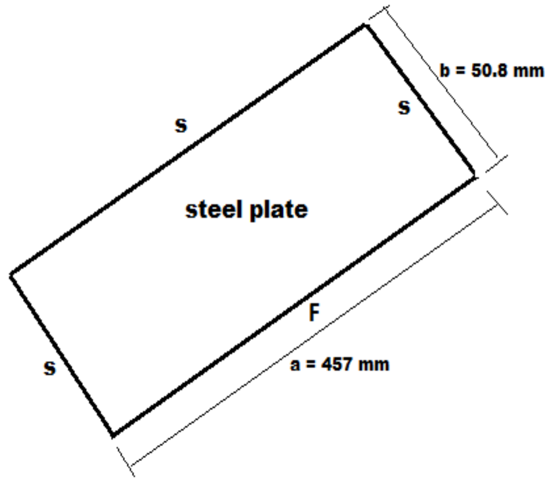
```
  call system('gnuplot finite-strip_jm.plt')
end subroutine plot
!===================================================================
===================================


!!================================= Plotting the output by using gnuplot
=================================
!subroutine plot
!  write(14,'(12(a/))')" set terminal win 0" ,                                   &
!          &     " plot 'finite-strip_jm.out' using 1:2 with lp title 'Critical stress' " ,     &
!          &     " set title 'finite strip for buckling of isotropic plates '" ,           &
!          &     " set xlabel ' a/b ' " ,                                  &
!          &     " set ylabel ' Critical stress '   ",                           &
!          &     "  ",                                                   &
!!  write(15,'(12(a/))')" set terminal wxt 0" ,                                 &
!          &     " set terminal win 1 ",                                   &
!          &     " plot 'finite-strip_jm.out' using 1:3 with lp title 'Buckling coefficient ' " ,   &
!          &     " set title 'finite strip for buckling of isotropic plates '" ,           &
!          &     " set xlabel ' a/b '" ,                                   &
!          &     " set ylabel ' Buckling coefficient '   " ,                       &
!          &     " pause -1 "
!  call system('gnuplot finite-strip_1.plt')
!!  call system('gnuplot finite-strip_2.plt')
!end subroutine plot
!!==================================================================
================================
```

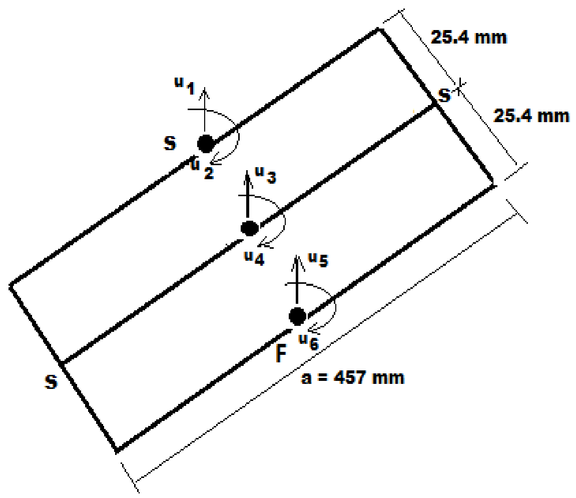
# Explain Solution Finite Strip Local Buckling Problem

## Givine:

**steel plate**

| | | | |
|---|---|---|---|
| t | 0.79 | mm | thickness |
| E | 210000 | $N/mm^2$ | young's modulus |
| δe | 600 | $N/mm^2$ | elastic stress |
| a | 457 | mm | lenght of plate |
| b | 50.8 | mm | width of plate |
| b/2 | 25.4 | mm | width for two strips |
| L | 101.6 | mm | half wave lengh |
| ν | 0.3 | | |
| π | 3.14 | | |




**for this case the boundary condition : simply support-free**

Assembly matrix Ks

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| Ks = | $k^1_{11}$ | $k^1_{12}$ | $k^1_{13}$ | $k^1_{14}$ | 0 | 0 |
| | $k^1_{21}$ | $k^1_{22}$ | $k^1_{23}$ | $k^1_{24}$ | 0 | 0 |
| | $k^1_{31}$ | $k^1_{32}$ | $k^1_{33} + k^2_{11}$ | $k^1_{34} + k^2_{12}$ | $k^2_{13}$ | $k^2_{14}$ |
| | $k^1_{41}$ | $k^1_{42}$ | $k^1_{43} + k^2_{21}$ | $k^1_{44} + k^2_{22}$ | $k^2_{23}$ | $k^2_{24}$ |
| | 0 | 0 | $k^2_{31}$ | $k^2_{32}$ | $k^2_{33}$ | $k^2_{34}$ |
| | 0 | 0 | $k^2_{41}$ | $k^2_{42}$ | $k^2_{43}$ | $k^2_{44}$ |

6*6

$k^1$ for strip one

$k^2$ for strip two

## Elastic stiffness matrix $K_E$

$$K_E = \phi_1 K_{e1} + \phi_2 K_{e2} + \phi_3 K_{e3}$$

**Where :**

| | |
|---|---|
| $\phi_1$ | 0.0266 |
| $\phi_2$ | 1.2 |
| $\phi_3$ | 29.39 |
| $\phi_4$ | 0.0023 |

$K_{e1}$ = $\phi_1$

| | | | |
|---|---|---|---|
| 156 | 22b | 54 | 13b- |
| 22b | 4b² | 13b | 3b²- |
| 54 | 13b | 156 | 22b- |
| 13b- | 3b²- | 22b- | 4b² |

4*4

$K_{e1}$ =

| | | | |
|---|---|---|---|
| 4.15 | 14.85 | 1.44 | -8.77 |
| 14.85 | 68.58 | 8.77 | -51.43 |
| 1.44 | 8.77 | 4.15 | -14.85 |
| -8.77 | -51.43 | -14.85 | 68.58 |

$K_{e2}$ = $\phi_2$

| | | | |
|---|---|---|---|
| 36 | 3+15v | -36 | 3b- |
| 3+15v | 4b² | 3b | b²- |
| -36 | 3b | 36 | (3+15v)b- |
| 3b- | b²- | (3+15v)b- | 4b² |

4*4

$K_{e2}$ =

| | | | |
|---|---|---|---|
| 43.47 | 230.03 | -43.47 | -92.01 |
| 230.03 | 3116.13 | 92.01 | 779.03 |
| -43.47 | 92.01 | 43.47 | -230.03 |
| -92.01 | 779.03 | -230.03 | 3116.13 |

$K_{e3}$ = $\phi_3$

| | | | |
|---|---|---|---|
| 12 | 6b | -12 | 6b- |
| 6b | 4b² | 6b- | 2b² |
| -12 | 6b- | 12 | 6b- |
| 6b- | 2b² | 6b- | 4b² |

4*4

$K_{e3}$ =

| | | | |
|---|---|---|---|
| 352.71 | 4479.46 | -352.71 | -4479.46 |
| 4479.46 | 75852.2 | -4479.46 | 37926.1 |
| -352.71 | -4479.5 | 352.71 | -4479.46 |
| -4479.5 | 37926.1 | -4479.46 | 75852.2 |

## Elastic matrix $K_e$

B=

| | | | |
|---|---|---|---|
| 400.33 | 4724.34 | -394.75 | -4580.24 |
| 4724.34 | 79036.87 | -4378.67 | 38653.68 |
| -394.75 | -4378.67 | 400.33 | -4724.34 |
| -4580.2 | 38653.68 | -4724.34 | 79036.87 |

$K_S$ =

| | | | | | |
|---|---|---|---|---|---|
| 400.33 | 4724.34 | -394.75 | -4580.24 | 0.00 | 0.00 |
| 4724.34 | 79036.9 | -4378.7 | 38653.68 | 0.00 | 0.00 |
| -394.75 | -4378.7 | 800.66 | 0.00 | -394.75 | -4580.24 |
| -4580.2 | 38653.7 | 0.00 | 158073.7 | -4378.7 | 38653.7 |
| 0.00 | 0.00 | -394.75 | -4378.67 | 400.33 | -4724.34 |
| 0.00 | 0.00 | -4580.2 | 38653.68 | -4724.3 | 79036.9 |

## Geometric stiffness matrix $K_g$

$K_g$ = $\phi_4$

| | | | |
|---|---|---|---|
| 156 | 22b | 54 | 13b- |
| 22b | $4b^2$ | 13b | $3b^2$- |
| 54 | 13b | 156 | 22b- |
| 13b- | $3b^2$- | 22b- | $4b^2$ |

4*4

$K_g$ =

| | | | |
|---|---|---|---|
| 0.36 | 1.30 | 0.13 | -0.77 |
| 1.30 | 5.98 | 0.77 | -4.49 |
| 0.13 | 0.77 | 0.36 | -1.30 |
| -0.77 | -4.49 | -1.30 | 5.98 |

## Geometric matrix

| B= | | | | |
|---|---|---|---|---|
| | 0.36 | 1.30 | 0.13 | -0.77 |
| | 1.30 | 5.98 | 0.77 | -4.49 |
| | 0.13 | 0.77 | 0.36 | -1.30 |
| | -0.77 | -4.49 | -1.30 | 5.98 |

| Ks = | | | | | | |
|---|---|---|---|---|---|---|
| | 0.36 | 1.30 | 0.13 | -0.77 | 0.00 | 0.00 |
| | 1.30 | 5.98 | 0.77 | -4.49 | 0.00 | 0.00 |
| | 0.13 | 0.77 | 0.72 | 0.00 | 0.13 | -0.77 |
| | -0.77 | -4.49 | 0.00 | 11.96 | 0.77 | -4.49 |
| | 0.00 | 0.00 | 0.13 | 0.77 | 0.36 | -1.30 |
| | 0.00 | 0.00 | -0.77 | -4.49 | -1.30 | 5.98 |

from boundary condition the reduced matrix Ko for elastic stiffness matrix KE

| Ke = | | | | | |
|---|---|---|---|---|---|
| | $k^1_{22}$ | $k^1_{23}$ | $k^1_{24}$ | 0 | 0 |
| | $k^1_{32}$ | $k^1_{33} + k^2_{11}$ | $k^1_{34} + k^2_{12}$ | $k^2_{13}$ | $k^2_{14}$ |
| | $k^1_{42}$ | $k^1_{43} + k^2_{21}$ | $k^1_{44} + k^2_{22}$ | $k^2_{23}$ | $k^2_{24}$ |
| | 0 | $k^2_{31}$ | $k^2_{32}$ | $k^2_{33}$ | $k^2_{34}$ |
| | 0 | $k^2_{41}$ | $k^2_{42}$ | $k^2_{43}$ | $k^2_{44}$ |

5*5

| Ko for elastic = | | | | | |
|---|---|---|---|---|---|
| | 79036.9 | -4378.7 | 38653.68 | 0 | 0 |
| | -4378.7 | 800.66 | 0.00 | -394.75 | -4580.2 |
| | 38653.7 | 0.00 | 158073.7 | -4378.67 | 38653.7 |
| | 0 | -394.75 | -4378.67 | 400.33 | -4724.3 |
| | 0 | -4580.2 | 38653.68 | -4724.34 | 79036.9 |

from boundary condition the redused matrix Ko for geometric stiffness matrix Kg

| Kg = | | | | | |
|---|---|---|---|---|---|
| | $k^1_{22}$ | $k^1_{23}$ | $k^1_{24}$ | 0 | 0 |
| | $k^1_{32}$ | $k^1_{33} + k^2_{11}$ | $k^1_{34} + k^2_{12}$ | $k^2_{13}$ | $k^2_{14}$ |
| | $k^1_{42}$ | $k^1_{43} + k^2_{21}$ | $k^1_{44} + k^2_{22}$ | $k^2_{23}$ | $k^2_{24}$ |
| | 0 | $k^2_{31}$ | $k^2_{32}$ | $k^2_{33}$ | $k^2_{34}$ |
| | 0 | $k^2_{41}$ | $k^2_{42}$ | $k^2_{43}$ | $k^2_{44}$ |

5*5

$$K_o \text{ for geometric} = \begin{vmatrix} 5.98 & 0.77 & -4.49 & 0 & 0 \\ 0.77 & 0.72 & 0.00 & 0.13 & -0.77 \\ -4.49 & 0.00 & 11.96 & 0.77 & -4.49 \\ 0 & 0.13 & 0.77 & 0.36 & -1.30 \\ 0 & -0.77 & -4.49 & -1.30 & 5.98 \end{vmatrix}$$

from

$$\left| K_E - \lambda K_g \right| = 0$$

**Then can be found Eigen-value and Eigen-vector.**