Gumbel Manifold

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Abstract

Following Rao's idea to use the Fisher information matrix (FIM) as a Riemannian metric, we show that the family of Gumbel distributions determines a two dimensional Riemannian manifold. In this paper we illustrates the information geometry of the Gumbel space, and derive he geometry as; connections, curvature tensor, Ricci curvature with its eigenvalues and eigenvectors, scalar curvature, sectional curvature and mean curvature, where we show that Gumbel space has a negative constant scalar curvature. Moreover, we prove that log-Gumbel manifold is an isometric isomorphic of the origin manifold, which is important in stochastic process since Gumbel distributions are related to exponential distributions.

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1 Introduction

The origin work on information geometry was due to Rao [1], who considered a family of probability distributions as a Riemannian manifold using the Fisher information matrix (FIM) as a Riemannian metric. In 1975, Efron [2] defined the curvature in statistical manifolds, and gave a statically interpretation for the curvature with application to second order efficiency. Then Amari [3] introduced a one-parameter family of affine connections (α-connection), where the 0-connection is correspond to the Levi-Civita connection. He further proposed a differential-geometrical framework for constructing a higher-order asymptotic theory of statistical inference.

Several researchers studied the information geometry and its applications for some families of distributions. Amari [3] showed that the family of univariate Gaussian distributions has a constant negative curvature, and Sato [4] obtained the geometrical structure of the parameter space of the two-dimensional normal distribution. Gamma manifold studied by many researcher eg [3], also Arwini and Dodson [5] proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, using an information theoretic metric topology. Abdel-All, Mahmoud and Add-Ellah [6] showed that the family of Pareto distributions is a space with constant positive curvature and they obtained the geodesics, and they showed the relation between the geodesic distance and the J-divergence.

The family of Gumbel distributions does not form an exponential family, hence in the present paper we derive the geometrical quantities, as connections and curvatures objects on the Gumbel manifold without using the concept of potential function.

2- Gumbel distributions

The Gumbel distribution, also known as the Extreme Value Type I distribution has event space $\Omega = R$. and the following probability density function (pdf).

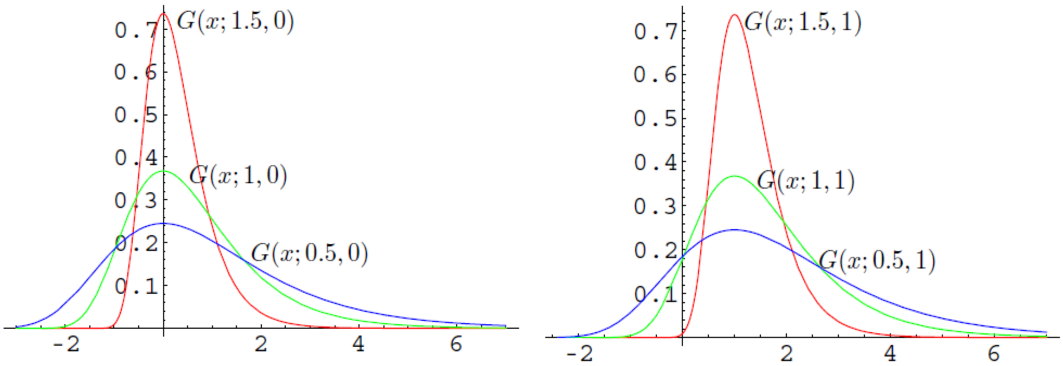


Figure 1 : In the left: Gumbel distributions where $\mu = 0$ and $\beta = 0.5, 1, 1.5$ for the range $x \in (-3,7)$. In the right: Gumbel distributions where $\mu = 1$ and $\beta = 0.5, 1, 1.5$ for the range $x \in (-3,7)$.

$$\mathbf{G(x;\ \beta,\mu) = \frac{1}{\beta} e^{-\left(\frac{x-\mu}{\beta}\right)} e^{-e^{-\left(\frac{x-\mu}{\beta}\right)}}\ for\ x \geq \mu} \qquad (\mathbf{2.1})$$

Where $\beta > 0$ is the scale parameter, and $\mu \in R$ is location parameter. In the case where $\beta = 1$ and $\mu = 0$ the Gumbel distribution has the standard form

$$\mathbf{G(x) = e^{-x} e^{-e^{-x}}}\ .$$

Figure 1 shows Gumbel distributions, in the cases where the location parameter $\mu = 0$ and $\mu = 1$ with different shape parameters $\beta = 0.5, 1, 1.5$ for the range $x \epsilon (-3,7)$. Note that the shape of the Gumbel distribution does not depend on the parameters.

The Gumbel distribution has mean $e(x) = \gamma\beta + \mu$ where $\gamma = 0.577$ is the Euler gamma constant, variance $\mathrm{var}(x) = \frac{1}{6}\pi^2\beta^2$, and standard deviation $\mathrm{std.dev}(x) = 1.28255\ \beta$.

2.1 Log-likelihood function and Shannon's entropy

The log-likelihood function for the Gumbel distribution (2.1) is

$$\mathbf{l(x;\ \beta,\mu) = \log\big(G(x;\ \beta,\mu)\big) = -\log(\beta) - \left(\frac{x-\mu}{\beta}\right) - e^{-\left(\frac{x-\mu}{\beta}\right)}}.$$

By direct calculation Shannon's information theoretic entropy for the Gumbel distribution, which is the negative of the expectation of the log-likelihood function, is given by

$$\mathbf{S_G(\beta,\mu) = -\int_{-\infty}^{\infty} l(x;\beta,\mu)\ G\ (x;\ \beta,\mu)dx}$$

$$\mathbf{= 1 + \gamma + \log(\beta) \qquad (2.2)}$$

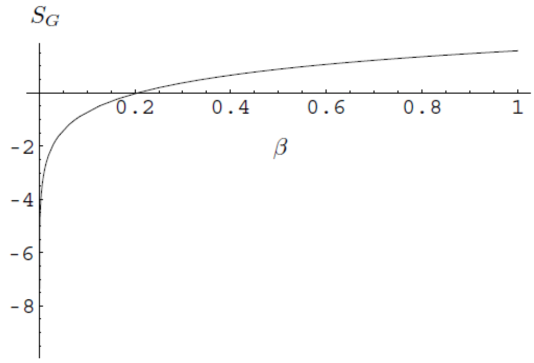


Figure 2: A surface and a contour plot for the Shannon's information entropy S_G, for bivariate gamma exponential distributions in the domain b ∈ (0,3).

2.2 Fisher information matrix FIM

The Fisher Information (FIM) is given by the expectation of the covariance of partial derivatives of the log-likelihood function. Here the Ffisher information metric components of the family of Gumbel distributions M with coordinate system $(\theta) = (\theta_1, \theta_2) = (\beta, \mu)$ are give by

$$\mathbf{g_{ij} = \int_{-\infty}^{\infty} \frac{\partial^2 l(x,\theta)}{\partial\theta_i \partial\theta_j}\ G(x)\ dx.}$$

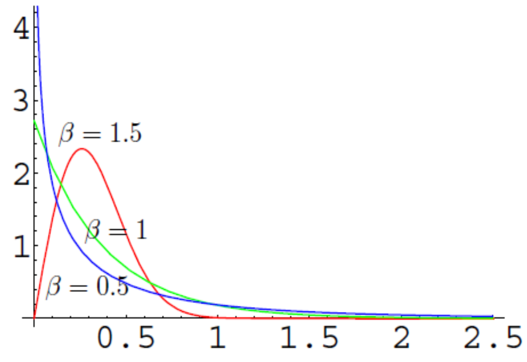
$$\mathbf{Hence\ , g = [g_{ij}] = \begin{bmatrix} \frac{6(\gamma-1)^2+\pi^2}{6\beta^2} & \frac{\gamma-1}{\beta^2} \\ \frac{\gamma-1}{\beta^2} & \frac{1}{\beta^2} \end{bmatrix} \qquad (2.3)}$$

and the variance covariance matrix is

$$\mathbf{g^{-1} = [g^{ij}] \begin{bmatrix} \frac{6\beta^2}{\pi^2} & \frac{-6(\gamma-1)\beta^2}{\pi^2} \\ \frac{-6(\gamma-1)\beta^2}{\pi^2} & \frac{(6(\gamma-1)^2+\pi^2)\beta^2}{\pi^2} \end{bmatrix} \qquad (2.4)}$$

2.3 Log – Gumbel manifold

Here we introduce the log-Gumbel distribution, which arises from the Gumbel distribution (2.1) for non-negative random variable $y = e^{-x}$. So the log-Gumbel distribution, has probability density function



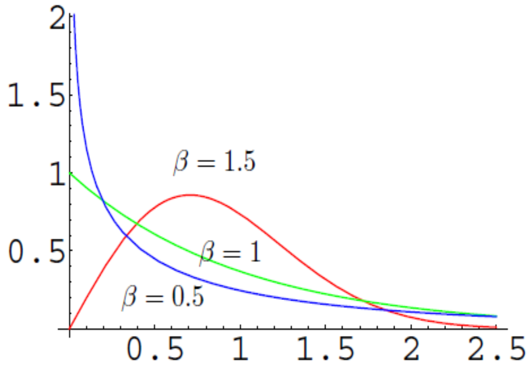


Figure 3: In the left: Log-Gumbel distributions where $\mu = 0$ and $\beta = 0.5, 1, 1.5$ for the range $x\epsilon(-3,7)$. In the right: Log-Gumbel distributions where $\mu = 1$ and $\beta = 0.5, 1, 1.5$ in the domain $y \epsilon (0.3)$.

$$\mathbf{LG}\,(\mathbf{y};\ \boldsymbol{\beta}, \boldsymbol{\mu}) = \frac{\mathbf{1}}{\boldsymbol{\beta}}\ \mathbf{y}^{\frac{1}{\beta}-1}\ \mathbf{e}^{\frac{\mu}{\beta}-e^{\frac{\mu}{\beta}}\mathbf{y}^{\frac{1}{\beta}}} \quad ,\mathbf{y}\ \in\ \mathbf{R}^{+} \qquad (\mathbf{2.5})$$

Where $\beta > 0, \mu \in R$. Figure 3 shows plos of the log-Gumbel family of densisties with central location parameters μ and $\beta = 0.5,\ 1,\ 1.5$ where $\mu > 0$ the graphics getting smaller, and where $\mu < 0$ the graphics getting bigger.

This family of densities determines a Riemannian 2-manifold which is isometric with the Gumbel 2-manifold M.

3 Geometry of the Gumbel manifold

3.1. Gumbel 2-manifold

Let M be the family of all Gumbel distributions

$$\mathbf{M} = \left\{\mathbf{G}(\mathbf{x};\ \boldsymbol{\beta}, \boldsymbol{\mu}) = \frac{\mathbf{1}}{\boldsymbol{\beta}}\mathbf{e}^{-\left(\frac{x-\mu}{\beta}\right)}\mathbf{e}^{-e^{-\left(\frac{x-\mu}{\beta}\right)}} | \boldsymbol{\beta} \in \mathbf{R}^{+}, \boldsymbol{\mu}\ \in \mathbf{R}\right\}, \mathbf{x} \in \mathbf{R} \quad (\mathbf{3.6})$$

So the parameter space is $R^{+} \times R$ and the random variables are $x \in \Omega = R$

We can consider M as a Riemannian 2-manifold with coordinate system $(\theta_1, \theta_2) = (\beta, \mu)$ and Fisher information metric g (2.3).

3.2 Connections

Here we give the analytic expressions for the connections with respect to coordinates $(\theta_1, \theta_2) = (\beta, \mu)$ the independent components Γ^{i}_{jk} are

$$\Gamma_{11}^{1} = \frac{6(\gamma - 1)^2 - \pi^2}{\pi^2 \beta}$$

$$\Gamma_{12}^{1} = \frac{6(\gamma - 1)}{\pi^2 \beta}$$

$$\Gamma_{22}^{1} = \frac{6}{\pi^2 \beta}$$

$$\Gamma_{11}^{2} = -\frac{6(\gamma - 1)^3 + (\gamma - 1)\pi^2}{\pi^2 \beta}$$

$$\Gamma_{12}^{2} = -\frac{6(\gamma - 1)^2 + \pi^2}{\pi^2 \beta}$$

$$\Gamma_{22}^{2} = \frac{6 - 6\gamma}{\pi^2 \beta} \qquad (3.7)$$

3.3 Curvatures

By direct calculation we provide various curvature objects of the bivariate gamma exponential Gumbel manifold, as: the curvature tensor, the Ricci curvature, the scalar curvature. The sectional curvature, and the mean curvature.

The curvature tensor components, which are defined as :

$$R_{ijkl} = \sum_{h=1}^{2} g_{hl}\left(\partial_i \Gamma_{jk}^{h} - \partial_j \Gamma_{ik}^{h} + \sum_{m=1}^{2} \Gamma_{im}^{h}\Gamma_{jk}^{m} - \Gamma_{jm}^{h}\Gamma_{ik}^{m}\right), (i, j, k, l = 1, 2)$$

Are given by :

$$R_{2112} = -\frac{1}{\beta^4}$$

$$R_{2121} = \frac{1}{\beta^4}$$

$$R_{1212} = \frac{1}{\beta^4}$$

$$R_{1221} = -\frac{1}{\beta^4} \qquad (3.8)$$

While the other independent components are zero.

Contracting R_{ijkl} with g^{il} we obtain the components R_{jk} of the Ricci tensor

$$\mathbf{R} = [\mathbf{R_{jk}}] = \begin{bmatrix} -\dfrac{6(\gamma-1)^2+\pi^2}{\pi^2\beta^2} & \dfrac{6-6\gamma}{\pi^2\beta^2} \\ \dfrac{6-6\gamma}{\pi^2\beta^2} & -\dfrac{6}{\pi^2\beta^2} \end{bmatrix} \quad (3.9)$$

The eignevalues and the eigenvectors of the Ricci tensor are given by

$$\begin{pmatrix} \dfrac{-\left(12+6(\gamma-2)\gamma+\pi^2+\sqrt{36(2+(\gamma-2)\gamma)^2+12(\gamma-2)\gamma\,\pi^2+\pi^4}\right)}{2\pi^2\beta^2} \\ \dfrac{-\left(12+6(\gamma-2)\gamma+\pi^2-\sqrt{36(2+(\gamma-2)\gamma)^2+12(-\gamma-2)\gamma\,\pi^2+\pi^4}\right)}{2\pi^2\beta^2} \end{pmatrix} .. (3.10)$$

$$\begin{pmatrix} \dfrac{6(\gamma-2)\gamma+\pi^2+\sqrt{36(2+(\gamma-2)\gamma)^2+12(\gamma-2)\gamma\,\pi^2+\pi^4}}{12(\gamma-1)} & 1 \\ \dfrac{6(\gamma-2)\gamma+\pi^2-\sqrt{36(2+(\gamma-2)\gamma)^2+12(\gamma-2)\gamma\,\pi^2+\pi^4}}{12(\gamma-1)} & 1 \end{pmatrix} \quad (3.11)$$

By contracting the Ricci curvature components R_{ij} with the inverse components g^{ij} we obtain the scalar curvature R:

$$\mathbf{R} = -\frac{12}{\pi^2} \qquad (3.12)$$

Note that the scalar curvature R is negative constant.

The sectional curvatures Q(i,j) where Q(i,j) = $\frac{R_{ijij}}{g_{ii}g_{jj}-g_{ij}^2}$ $(i, j = 1,2)$, are

$$Q(\mathbf{1},\mathbf{2}) = -\frac{6}{\pi^2} \qquad (3.13)$$

The mean curvatures Q(i) where Q(i) = $\sum_{j=1}^{2}\frac{1}{3}\ Q(i,j)$, $(i = 1,2)$, are

$$\mathbf{Q(2)} = \mathbf{Q(1)} = -\frac{2}{\pi^2}\ . \qquad (3.14)$$

4 Conclusion

In this paper we derived the geometrical properities for the 2-manifold of the Gumbel distributions, using the Fisher information matrix (FIM) as a Riemannian metric. The connections and curvatures objects as: curvature tensor, Ricci curvature, scalar curvature, sectional curvature and mean curvature are obtained, where we showed that the Gumbel manifold has a negative constant scalar curvature. Moreover, we showed that the log-Gumbel manifold is an isometric isomorph of the Gumbel manifold.

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