Smarandache idempotent in the ring of integers modulo *n*

Shaban.A.Traina
Department of mathematics
University of Garyounis
E-mail: Shabantraina@yahoo.com

Abir.K.Salibi
Department of mathematics
University of Omar-El Mukhtar
E-mail: Abir_sss@yahoo.com

Abstract.

In this paper we analyze and study the Smarandache idempotents (S-idempotents) in the ring of integers modulo *n*, $\mathbb{Z}_n$. We have shown in general that an idempotent element in a ring *R* may not be an S-idempotent. Also we have establish the existence of S-idempotents in $\mathbb{Z}_n$ for a specific value n. We have proved that $\mathbb{Z}_n$ has an S-idempotents with n is a perfect number, and *n* is of the form $2^i p$, (where *p* be an odd prime), or $3^i p$ (*p* a prime greater than 3), or in general when $n = p_1^i p_2$ (p_1 and p_2 are distinct odd primes). We provide many interesting properties and illustrate them with several examples.

Keywords : idempotent, Smarandache idempotent.

1. Introduction.

This section is devoted to the introduction of the basic notions concerning the Smarandache ring. The concept of Smarandache idempotents and Smarandache co- idempotents inis introduced. We recall only those results and definitions, which $\mathbb{Z}_n$ are very basically needed in this paper.

Definition 1.1 [2] : Let *R* be a ring, an element $a \in R \setminus \{0\}$ is called an *idempotent* in *R* if $a^2 = a$. If 1 is in *R* then 1 is called a *trivial idempotent*.

Definition 1.2 [3]: Let R be a ring with unit. An element $a \in R \setminus \{0,1\}$ is a *Smarandache idempotent* (*S-idempotent*) of R if,

1. $a^2 = a$
2. There exists $b \in R \setminus \{0,1,a\}$ such that:

   i) $b^2 = a$ and

   ii) $ab = a$ $(ba = a)$ or $ab = b$ $(ba = b)$.

Definition 1.3 [3]: Let $a \in R \setminus \{0,1\}$ be a Smarandache idempotent i.e. $a^2 = a$ and there exists $b \in R \setminus \{0,1,a\}$ such that $b^2 = a$ and $ab = a$ $(ba = a)$ or $ab = b$ $(ba = b)$. We call b the *Smarandache co-idempotent* (*S-co-idempotent*) and denote the pair by (a,b).

Example 1.4 : Let $\mathbb{Z}_{10} = \{0,1,2,...,9\}$ be the ring of integers modulo 10, then $6 \in \mathbb{Z}_{10}$ is an S-idempotent of $\mathbb{Z}_{10}$ for $6^2 \equiv 6$ (mod 10) and $4 \in \mathbb{Z}_{10}$ is an S-co-idempotent such that $4^2 \equiv 6$ (mod10) and $6.4 \equiv 4$ (mod 10).

Definition 1.5 [3] : Let R be a commutative ring with unit and G be a group. The group ring RG of the group G over the ring R consists of all finite formal sums of the form $\sum_i \alpha_i g_i$ (*i*-runs over a finite number), where $\alpha_i, \beta_i \in R$ and $g_i \in G$ satisfying the following conditions:

$$i)\quad \sum_i \alpha_i g_i = \sum_i \beta_i g_i \Leftrightarrow \alpha_i = \beta_i .$$

$$ii)\quad \left(\sum_i \alpha_i g_i\right) + \left(\sum_i \beta_i g_i\right) = \sum_i (\alpha_i + \beta_i) g_i .$$

$$iii)\quad \left(\sum_i \alpha_i g_i\right)\left(\sum_j \beta_j g_j\right) = \sum_k \gamma_k g_k \text{ where } \gamma_k = \sum \alpha_i \beta_j ,\ g_k = g_i g_j .$$

$$iv)\quad \alpha_i g_i = g_i \alpha_i .$$

$$v)\quad \alpha \sum_i \alpha_i g_i = \sum_i (\alpha \alpha_i) g_i \quad \text{for } \alpha, \alpha_i \in R \text{ and } \sum \alpha_i g_i \in RG .$$

RG is a ring with $0 \in R$ as its additive identity.

Since $1 \in R$ we have $G = 1.G \subset RG$ and $R = R.e \subseteq RG$ where e is the identity element of G.

Note that definition 1.1.5 can be defined for a semigroup with a unit.

Example 1.6 : Let $\mathbb{Z}_3$ be the prime field of characteristic 3, $G = \{g \,/\, g^2 = 1\}$ be the cyclic group of order 2. $\mathbb{Z}_3 G = \{0, 1, 2, g, 2g, 1+g, 1+2g, 2+g, 2+2g\}$, clearly $2+2g$ is an S-idempotent of $\mathbb{Z}_3 G$ as $(2+2g)^2 = 2+2g$ and $(2+2g)(1+g) = 1+g$. Hence the claim.

Example 1.7 : Consider the group ring *QG* of the group *G* over *Q*, where $G = \{g \,/\, g^2 = 1\}$ be the cyclic group of order 2 and *Q* be the field of rationals. QG has S-idempotent.

Take $a = \frac{1}{2}(1+g) \in QG$, $b = \frac{-1}{2}(1+g) \in QG$.

Now we have $a^2 = a$, $b^2 = a$, $ab = b$. So *a* is an S-idempotent in *QG*.

Theorem 1.8 [3] : Let *F* be a field. *F* has no S-idempotent.

Note : Let $\mathbb{Z}_p$ be the ring of integers modulo *p*, *p* a prime, then $\mathbb{Z}_p$ has no idempotents. For example $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$ has no idempotents, so has no S-idempotents.

Now we have the following theorem.

Theorem 1.9 [3] : In a ring *R*, every S-idempotent is an idempotent and not vice versa.

Example 1.10 : In $\mathbb{Z}_{10}$, $5 \in \mathbb{Z}_{10}$ is an idempotent since $5^2 \equiv 5$ (mod 10) but it is not an S-idempotent.

Example 1.11 : Let $S_3 = \{1, p_1, p_2, p_3, p_4, p_5\}$, S_3 be the symmetric group of degree 3 where

$$1=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \quad p_1=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, \quad p_2=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix},$$

$$p_3=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}, \quad p_4=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} \text{ and } p_5=\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}.$$

$\mathbb{Z}_2=\{0,1\}$ be the prime field of characteristic two. Clearly the group ring $\mathbf{Z}_2S_3$ has idempotents which are S-idempotents.

Now $a=(1+p_4+p_5)$ is an S-idempotent in $\mathbf{Z}_2S_3$ since $(1+p_4+p_5)^2=1+p_4+p_5$, and $b_1=1+p_1+p_2\in\mathbf{Z}_2S_3$ is an S-co idempotent of a, such that $b_1^2=a$, $ab_1=a$.

One can see that the element $b_2=p_1+p_2+p_3$ is also an S-co-idempotent of a, such that $b_2^2=a$, $ab_2=a$.

This leads us to an interesting results that S-co-idempotent are not unique for a given S-idempotent.

Theorem 1.12 [3] : Let R be a ring. $a\in R$ be an S-idempotent. The S-co-idempotents of a in general is not unique.

Example 1.13 : In $\mathbb{Z}_{40}$, 25 is an S-idempotent has two S-co-idempotent 5 and 15.

Theorem 1.14 : Let $\mathbb{Z}_{p^n}$ be the set of integers modulo p^n, p a prime. Then $\mathbb{Z}_{p^n}$ has no non trivial idempotent element .

Proof : Let a be an idempotent element in $\mathbb{Z}_{p^n}$

Then $\quad a^2\equiv a \pmod{p^n}$

$$a(a-1)=kp^n \quad , \quad k\in\mathbb{Z}.$$

So we have $a\equiv 1 \pmod{p^n}$ or $a\equiv 0 \pmod{p^n}$.

Then $\mathbb{Z}_{p^n}$ has no non trivial idempotent.

Example 1.15 : Let $\mathbb{Z}_9 = \{0,1,2,...,9\}$ be the ring of integers modulo 9. It can easily be verified that $\mathbb{Z}_9$ has no idempotents and so has no S-idempotents.

Theorem 1.16 : Let $\mathbb{Z}_n$ be the ring of integers modulo *n*, $n = 3p$, *6p* where *p* odd prime greater than 3, $n = 4p$ where *p* odd prime and $n = 5p$ where *p* a prime > 5. If *a* is an S-idempotent and *b* an S-co-idempotent of *a*, then $a + b \equiv n$ (mod *n*).

Proof : Since *a* is an idempotent, then $a^2 \equiv a$ (mod *n*) and $b \in \mathbb{Z}_n \setminus \{0,1,a\}$ such that $b^2 \equiv a$ (mod *n)* and $ab \equiv a$ (mod *n*) or $ab \equiv b$ (mod *n*).

i.e. $a^2 \equiv b^2$ (mod *n*).

So $a^2 - b^2 = kn$, $k \in \mathbb{Z}$,

i.e. $(a-b)(a+b) = kn$.

Since $a - b = k_1$, $k_1 \in \mathbb{Z}$, $k_1 < n$,

then $a + b = k_2 n$, $k_2 \in \mathbb{Z}$, with $k_1 . k_2 = k$.

So $a + b \equiv n$ (mod *n*).

Example 1.17 : In $\mathbb{Z}_{15}$; 10 is an S-idempotent, 5 is an S-co-idempotent and $10 + 5 = 15$.

Remark 1.18 : The last theorem is not true for all *n*. The case if $n = 2m^2$, $m^2 \neq 2^i$, $i \in \mathbb{Z}$, we have m^2 is an S-idempotent and *m* is an S-co idempotent , but $m^2 + m \not\equiv n$ (mod *n*). For example in $\mathbb{Z}_{18}$, $(18 = 2.3^2)$, $a = 9$ is an S-idempotent since $9^2 \equiv 9$ (mod 18) and $b = 3$ is an S-co-idempotent with $3^2 = 9$, $3.9 \equiv 9$ (mod 18), but $9 + 3 = 12 \not\equiv 18$ (mod 18).

Note : In $\mathbb{Z}_n$, if *a* is an S-idempotent which has two S-co-idempotent, then one of them must satisfies the condition $a + b \equiv n$ (mod *n*).

Theorem 1.19 : Let $\mathbb{Z}_n$ be the ring of integers modulo *n*, and let *a* be an idempotent element, then $(n+1) - a$ is also an idempotent .

Proof : If *a* is an idempotent, then

$a^2 - a = kn\,,\ k \in \mathbb{Z}$

$$[(n+1)-a]^2 - [(n+1)-a] = (n+1)^2 - 2a(n+1) + a^2 - (n+1) + a$$

$$= (n+1)^2 - 2na - (n+1) + a^2 - a$$

$$= n^2 + n - 2na + nk$$

$$= (n+1-2a+k\,)n$$

$$= hn\,,\ h = n+1-2a+k\ .$$

Then $(n+1)-a$ is an idempotent in $\mathbb{Z}_n$.

2. The existence of S-idempotent in $\mathbb{Z}_n$.

In this section we have proved the existence of an S-idempotent in the ring $\mathbb{Z}_n$ **,** when $n = 5p\ \ (p > 5),\ n = 8p\ \ (p$ odd prime) , and some more cases have been proved.

Theorem 2.20 : Let $\mathbb{Z}_{2p}$ be a ring of integers modulo $2p$, where p is an odd prime, then

i) $p\ \textit{and}\ p+1$ are two idempotents .

ii) $p+1$ is an S-idempotent .

Proof : *i)* Since $2 \backslash (p-1),$ then $2p \backslash p(p-1)$.

So $p^2 - p = 2pk\,,\ k \in \mathbb{Z}$, i.e. $p^2 \equiv p\ \ (\text{mod } 2p).$

Also $2 \backslash (p+1),$ then $p^2 + p = 2pk\,,\ k \in \mathbb{Z}$.

So $(p+1)^2 \equiv p+1\ (\text{mod } 2p).$

Hence p and $p+1$ are idempotents in $\mathbb{Z}_{2p}$.

ii) Take $a = p+1 \in \mathbb{Z}_{2p}$ and $b = p-1 \in \mathbb{Z}_{2p}$

$$a^2 = (p+1)^2 \equiv p+1\ \ (\text{mod } 2p)$$

Therefore, $a^2 \equiv a\ \ (\text{mod } 2p).$

Also $b^2 = (p-1)^2 \equiv (p+1)\ \ (\text{mod } 2p)$

Therefore $b^2 \equiv a\ \ (\text{mod } 2p).$

And $ab = (p-1)(p+1)$

$$= p^2 - 1$$

$$\equiv p - 1 \pmod{2p}.$$

Therefore

$$ab \equiv b \pmod{2p}.$$

So $a = p + 1$ is an S-idempotent in $\mathbb{Z}_{2p}$.

Then $\mathbb{Z}_{2p}$ has idempotents which are not S-idempotent.

Example 2.21 : In $\mathbb{Z}_{14}$; 7 and 8 are two idempotents. On the other hand, only 8 is an S-idempotents since $6 \in \mathbb{Z}_{14}$, $6^2 \equiv 8 \pmod{14}$ and $6.8 \equiv 6 \pmod{14}$.

Theorem 2.22 : In the ring of integers modulo $5p$, with $p > 5$. If the first digit of p is

i) 1, then p is an idempotent .

ii) 9, then p^2 is an idempotent .

iii) 3,7 , then p^4 is an idempotent .

proof: *i)* If p starts with 1, then $5 \backslash (p-1)$. So p is an idempotent .

ii) If p starts with 9, then $5 \nmid p$ and $5 \nmid (p-1)$. Hence p is not an idempotent. But $5 \backslash (p+1)$. So p^2 is an idempotent.

iii) If p start with 3, 7 then $5 \nmid p$, $5 \nmid (p-1)$, and $5 \nmid (p+1)$, i.e. p and p^2 are not idempotents . On the other hand, $5 \backslash (p^2+1)$, i.e. p^4 is an idempotent.

A similar result can be obtain for the ring $\mathbb{Z}_{10p}$.

Example 2.23 : In $\mathbb{Z}_{55}$,$(55 = 5.11)$, 11 is an idempotent. In $\mathbb{Z}_{95}$,$(95 = 5.19)$, $(19)^2 \equiv 76 \pmod{95}$ is an idempotent. Also in $\mathbb{Z}_{65}$, $(65 = 5.13)$, $(13)^4 \equiv 26 \pmod{65}$ is an idempotent.

Theorem 2.24: Let $\mathbb{Z}_{8p}$ be the ring of integers modulo $8p$, p be an odd prime, then p^2 is an idempotent .

Proof : Let $p = 2r+1$, $r \in \mathbf{Z}$. Then $p(p-1)(p+1) = 4r(2r+1)(r+1)$

If r is odd $\Rightarrow r+1$ is even $\Rightarrow r+1 = 2l$, $l \in \mathbf{Z}$.

So we have $p(p-1)(p+1) = 8(2r+1)(r)(l)$.

Then $8 \setminus p(p-1)(p+1)$.

If r is even $\Rightarrow r = 2m$, $m \in \mathbf{Z}$.

So $8 \setminus p(p-1)(p+1)$.

Then p^2 is an idempotent.

Example 2.25 : In $\mathbb{Z}_{24}$, $(24 = 8.3)$, $(3)^2 = 9$ is an idempotent which is an S-idempotent.

Similar result one can establish from the last theorem are given in the next two corollaries.

Corollary 2.26 : In $\mathbb{Z}_n$, $n = 3p,\ 4p,\ 12p,\ 24p$ where p is an odd prime, p^2 is an idempotent **.**

Clearly $(2p)^4$ and $(3p)^2$ is also an idempotent in $\mathbb{Z}_{24p}$.

Corollary 2.27 : In $\mathbb{Z}_n$ **,** $n = 5p,\ 10p,\ 16p,\ 48p$ **,** where p is an odd prime, $p > 5$, p^4 is an idempotent.

Example 2.28 : Consider $\mathbb{Z}_{120}$, $(120 = 24.5)$, it can be calculated that $p^2 \equiv 25$ (mod 120), $(2p)^4 \equiv 40$ (mod 120) and $(3p)^2 \equiv 105$ (mod 120) as non-trivial idempotents, and 96,81 and 16 are the other idempotents in $\mathbb{Z}_{120}$. One can easily verified that all these idempotents are S-idempotents. Now the S-co-idempotent for 25 is 95, for 40 is 80, 150 it is 15, for 96 the S-co-idempotent is 24, for 81 is 39, and for 16 the S-co-idempotent is 104.

Theorem 2.29 : In the ring of integers modulo $2pq$, p,q are odd primes, pq is an idempotent.

Proof : Since $2 \backslash (pq-1)$, then $pq(pq-1) = 2pqk$, $k \in \mathbb{Z}$. Hence *pq* is an idempotent.

Example 2.30 : Consider $\mathbb{Z}_{30}$, $(30 = 2.3.5)$, 15 is an idempotent which is not an S-idempotent.

Theorem 2.31: In the ring of integers modulo $2p^i q^j$, p, q are odd primes, $p^i q^j$ is an idempotent.

Proof : Since $2 \backslash (p^i q^j - 1)$, then $p^i q^j (p^i q^j - 1) = 2p^i q^j k$, $k \in \mathbb{Z}$.
Hence $p^i q^j$ is an idempotent.

Example 2.32 : Consider $\mathbb{Z}_{90}$, $(90 = 2.3^2.5)$, 45 is an idempotent which is not an S-idempotent.

Definition 2.33 [5] : A positive integer *n* is said to be a *perfect number* if *n* is equal to the sum of all its positive divisors, excluding *n* itself. e.g. 6 is a perfect number. As 6 = 1 + 2 + 3.

Theorem 2.34 : Let $\mathbb{Z}_n$ be the ring of integers modulo *n*, where *n* is an even perfect number of the form $n = 2^t(2^{t+1} - 1)$, where $2^{t+1} - 1$ is a prime, for some $t \geq 1$, then $a = 2^{t+1} \in \mathbb{Z}_n$ is an S-idempotent.

Proof : For $a = 2^{t+1} \in \mathbb{Z}_n$,

choose $b = (n - 2^{t+1}) \in \mathbb{Z}_n$.

Clearly
$$a^2 = (2^{t+1})^2$$
$$= 2^2.2^{2t} \quad (\because\ 2^t.2^{t+1} \equiv 2^t \pmod{n})$$
$$\equiv 2.2^t \quad (\bmod\ n)$$
$$= a.$$

Now
$$b^2 = (n - 2^{t+1})^2$$
$$\equiv 2.2^t \ (\bmod\ n)$$

$= a$.

Also $ab = 2^{t+1}(n - 2^{t+1})$

$\equiv b$ (mod n).

So we get $a^2 \equiv a$ (mod n), $b^2 \equiv a$ (mod n) and $ab \equiv b$ (mod n).

Therefore $\alpha = 2^{s+1}$ is an S- idempotent .

Example 2.35 : Take the ring $\mathbb{Z}_6$. Here 6 is an even perfect number . As $6 = 2.(2^2 - 1)$,

so $a = 2^2$ is an S-idempotent.

Since $a^2 \equiv a$ (mod 6).

For $b = 2$.

Then $a^2 \equiv a$ (mod 6), $b^2 \equiv a$ (mod 6) and $ab \equiv b$ (mod 6).

So $a = 4$ is an S-idempotent.

Theorem 2.36 : Let $\mathbb{Z}_{2^i p}$ be a ring of integers modulo $2^i p$,where p is an odd prime with $p \backslash (2^{t+1} - 1)$ for some $t \geq i$, then $a = 2^{t+1} \in \mathbb{Z}_{2^i p}$ is an S-idempotent.

Proof : Note that $p \backslash (2^{t+1} - 1)$ for some $t \geq i$.

Therefore

$$2^{t+1} \equiv 1 (\text{ mod } p) \text{ for some } t \geq i .$$

$$\Leftrightarrow 2^t.2^{t+1} \equiv 2^t \ (\text{ mod } 2^i p) \quad \text{as gcd } (2^t, 2^i p) = 2^i \ , \ t \geq i .$$

Now take $a = 2^{t+1} \in \mathbb{Z}_{2^i p}$ and $b = (2^i p - 2^{t+1}) \in \mathbb{Z}_{2^i p}$.

Then it easy to see that :

$a^2 \equiv a$ (mod $2^i p$), $b^2 \equiv a$ (mod $2^i p$) and $ab \equiv b$ (mod $2^i p$).

Hence $a = 2^{t+1}$ is an idempotent.

Example 2.37 : Take the ring $\mathbb{Z}_{2^3.7}$. Here $7 \backslash (2^{5+1} - 1)$, so $t = 5$.

Take $a = 8$, $b = 48$.

Then it easy to see that $a^2 \equiv a$ (mod $2^3.7$), $b^2 \equiv a$ (mod $2^3.7$) and $ab \equiv b$ (mod $2^3.7$).

Theorem 2.38 : Let $\mathbb{Z}_{3^i p}$ be the ring of integers modulo $3^i p$, where p is an odd prime such that $p \backslash (2.3^t - 1)$ for some $t \geq i$, then $a = 2.3^t \in \mathbb{Z}_{3^i p}$ is an S-idempotent.

Proof : Suppose $p \backslash (2.3^t - 1)$ for some $t \geq i$.

Take $a = 2.3^t \in \mathbb{Z}_{3^i p}$ and $b = (3^i p - 2.3^t) \in \mathbb{Z}_{3^i p}$.

Then

$$
\begin{aligned}
a^2 &= (2.3^t)^2 \\
&= 2^2 .\, 3^{2t} \\
&\equiv 2.3^t \ (\bmod\ 3^i p) \\
&= a.
\end{aligned}
$$

As $2.3^t \equiv 1 \ (\bmod\, p)$ for some $t \geq i$

$\Leftrightarrow$ $2.3^t.3^t \equiv 3^t \ (\bmod\, 3^i p)$ as gcd $(3^t, 3^i p) = 3^i$, $t \geq i$.

Similarly $b^2 \equiv b \ (\bmod\ 3^i p)$ and $ab \equiv b \ (\bmod\ 3^i p)$.

So $a = 2.3^t$ is an S-idempotent.

Example 2.39 : Take the ring $\mathbb{Z}_{3^2.5}$. Here $5 \backslash 2.3^5 - 1$, so $t = 5$.

Take $\alpha = 2.3^5 \equiv 36 \ (\bmod\ 45)$ and $b = (3^2.5 - 2.3^5) \equiv 9 \ (\bmod\ 45)$.

$a^2 \equiv a \ (\bmod\ 45)$, $b^2 \equiv a \ (\bmod\ 45)$ and $ab \equiv b \ (\bmod\ 45)$.

We can generalize Theorem 2.36 , 2.38 as follows :

Theorem 2.40 : Let $\mathbb{Z}_{p^i q}$ be the ring of integers modulo $p^i q$,Where p, q are distinct odd primes and $q \backslash 2.p^t - 1$ for some $t \geq i$, then $a = 2p^t \in \mathbb{Z}_{p^i q}$ is an S-idempotent.

Proof : Suppose $q \backslash 2.p^t - 1$ for some $t \geq i$.

Take $a = 2p^t \in \mathbb{Z}_{p^i q}$ and $b = (p^i q - 2p^t) \in \mathbb{Z}_{p^i q}$.

Easily we can show that

$a^2 \equiv a \pmod{p^i q}$, $b^2 \equiv a \pmod{p^i q}$ and $ab \equiv b \pmod{p^i q}$.

So $a = 2p^t$ is an S- idempotent.

3. Conclusion .

One can see in the last section, we have establish the existence of at last one non-trivial S-idempotent. On the other hand, the existence of S-idempotent in the ring $\mathbb{Z}_n$ for every n has not been yet established.

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