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The Geometrical Structures of Bivariate Gamma Exponential Distributions

Khadiga Ali Arwini

Mathematics Department, Tripoli University, Tripoli, Libya

E-mail address: Kalrawini@yahoo.com

ABSTRACT

This paper is devoted to the information geometry of the family of bivariate gamma exponential distributions, which have gamma and Pareto marginals, and discuss some of its applications. We begin by considering the parameter bivariate gamma exponential manifold as a Riemannian 3-manifold; by following Rao's idea to use the Fisher information matrix (FIM), and derive the α-geometry as: α-connections, α-curvature tensor, α-Ricci curvature with its eigenvalues and eigenvectors, and α-scalar curvature. Where here the 0-geometry corresponds to the geometry induced by the Levi-Civita connection, and we show that this space has a non-constant negative scalar curvature. In addition, we consider four submanifolds as special cases, and discuss their geometrical structures, and we prove that one of these submanifolds is an isometric isomorph of the univariate gamma manifold. Then we introduce log-bivariate gamma exponential distributions, which have log-gamma and log-Pareto marginals, and we show that this family of distributions determines a Riemannian 3-manifold which is isometric with the origin manifold. We give an analytical solution for the geodesic equations, and obtain the explicit expressions for Kullback-Leibler distance, J-divergence and Bhattacharyya distance. Finally, we prove that the bivariate gamma exponential manifold can be realized in R^4, using information theoretic immersions, and we give explicit information geometric tubular neighbourhoods for some special cases.

***Keywords*:** information geometry, statistical manifold, gamma distribution, Pareto distribution, bivariate distributions, bivariate gamma exponential distribution

AMS Subject Classification (2000): 53B20, 60E05

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1. INTRODUCTION

Statistical manifolds are representations of families of probability distributions that allow differential geometric methods to be applied to problems in stochastic processes, mathematical statistics and information theory. The origin work was due to Rao [10], who considered a family of probability distributions as a Riemannian manifold using the Fisher information matrix (FIM) as a Riemannian metric. In 1975, Efron [6] defined the curvature in statistical manifolds, and gave a statistical interpretation for the curvature with application to second order efficiency. Then Amari [2] introduced a one-parameter family of affine connections (α-connection), which contain the Levi-Civita connection of the Fisher metric when $\alpha = 0$, and turn out to have importance and are part of larger systems of connections for which stability results follow [5]. He further proposed a differential-geometrical framework for constructing a higher-order asymptotic theory of statistical inference. Amari defining the α-curvature of a submanifold, pointed out important roles of the exponential and mixture curvatures and their duality in statistical inference.

Several researchers studied the information geometry and its applications for some families of univariate and multivariate distributions. Amari [2] showed that the family of univariate Gaussian distributions has a constant negative curvature. Oller [9] studied the geometry of the extreme value distributions as, Gumbel, Weibull and Frethet distributions, and he showed that all these spaces are with constant negative curvatures, and he obtained the geodesic distances in each case. Gamma manifold studied by many researcher eg [2], also Arwini and Dodson [3] proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, using an information theoretic metric topology. Abdel-All, Mahmoud and Add-Ellah [1] showed that the Pareto family is a space with constant positive curvature and they obtained the geodesics, and they showed the relation between the geodesic distance and the J-divergence.

For bivariate and multivariate distributions [15-18], the Fisher information metrics or the geometrical structures for most of these bivariate and multivariate distributions has been found to be intractable; the only exceptions have been such families as, the bivariate and multivariate Gaussian distributions, the Mckay bivariate gamma distributions, and the Freund bivariate exponential distributions. For a summary of bivariate and multivariate distributions see Kotz, Balakrishnan and Johnson [7]. Sato, Sugawa and Kawaguchi [12] proved that the 5-dimensional space of bivariate Gaussian distributions has a negative constant curvature and if the covariance is zero, the space becomes an Einstein space. After that, Skovgaard [13] in 1984, extended this work to the family of multivariate Gaussian models, and provided the geodesic distances.

The geometry for the Mckay 3-manifold where the variables are positively correlated, was obtained by Arwini and Dodson [4], who showed that this space can provide a metrization of departures from randomness and independence for bivariate processes, also they used the geometry of Mckay manifold in some applications concerning tomographic images of soil samples in hydrology surveys, see [3]. After that, the geometry for the Freund 4-manifold was investigated by Arwini and Dodosn [3], and they showed the importance of this family because exponential distributions represent intervals between events for Poisson processes and Freund distributions can model bivariate processes with positive and negative covariance.

In this paper we are interested to study the Becker and Roux's and Steel and le Roux's bivariate gamma distribution, with gamma marginals, which are based on plausible physical models, and they also include the Freund distribution as a special case, see [7], but unfortunately the Fisher information matrix is found to be intractable. However, Steel and Roux in 1989 [14] studied compound distributions of this bivariate gamma, and they obtain the bivariate gamma exponential distribution with gamma and Pareto marginals, hence we consider this family of distribution as a Riemannian 3-manifold and discuss its geometry.

2. BIVARIATE GAMMA EXPONENTIAL DISTRIBUTIONS

The bivariate gamma exponential distribution has event space $\Omega = \mathbb{R}^+ \times \mathbb{R}^+$and probability density function (pdf) given by

$$f(x, y; a, b, c) = \frac{a^b c}{\Gamma(b)} x^b e^{-(a\,x + c\,x\,y)} \quad \text{for } a, b, c \in \mathbb{R}^+ \tag{2.1}$$

where Γ is the gamma function which has the formula

$$\Gamma(z) = \int_0^\infty \mathrm{t}^{z-1}\, \mathrm{e}^{-\mathrm{t}} \mathrm{dt}\,.$$

Figure 1 shows surface and contour plots of bivariate gamma exponential distribution *f*(*x*,*y*;*a*,*b*,*c*) for the range $x \in [0,5]$ and $y \in [0,3]$, with unit parameters $a = b = c = 1$.

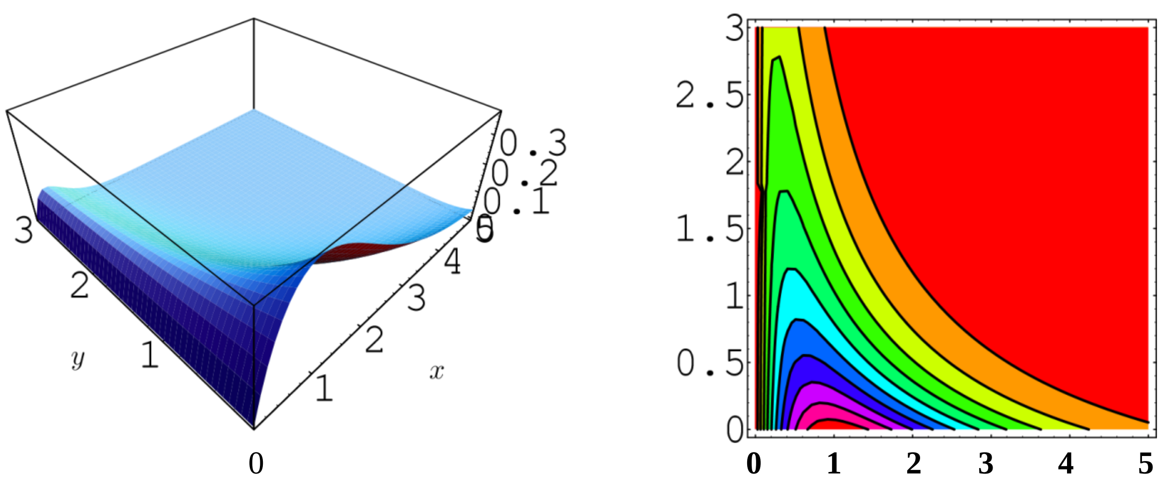

Figure 1. The bivariate gamma exponential family of densities with parameters $a = b = c = 1$ as a surface and as a contour plot, shown for the range $x \in [0,5]$ and $y \in [0,3]$.

2. 1. Marginals and correlation

The marginal distributions, of X and Y are gamma distribution and Pareto distribution, respectively:

$$f_X(x) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-a\,x}, \quad x > 0 \tag{2.2}$$

$$f_Y(y) = \frac{b\, c\, a^b}{(a+c\, y)^{b+1}}, \quad y > 0 \tag{2.3}$$

While the conditional marginal function $f_{Y|X}$ given x is an exponential distribution, with density function:

$$f_{Y|X}(y \mid x) = (c\, x)\, e^{-(c\, x)\, y}, \qquad y > 0. \tag{2.4}$$

The covariance and correlation coefficient of X and Y are given by:

$$Cov(X,Y) = \frac{1}{c(b-1)}, \quad b > 1$$

$$\rho(X,Y) = -\frac{\Gamma(b)\sqrt{b-2}}{\sqrt{\Gamma(b+1)}\sqrt{\Gamma(b+2) - b\,\Gamma(b+1)}}, \quad b > 2$$

Note that the bivariate gamma exponential distribution does not contain the independent case, but has negative correlation which depends on only the parameter b. The correlation ρ decreases from 0 to −0.35 where $2 < b < 4$, and increases from −0.35 to 0 where $b > 4$. See Figure 2.

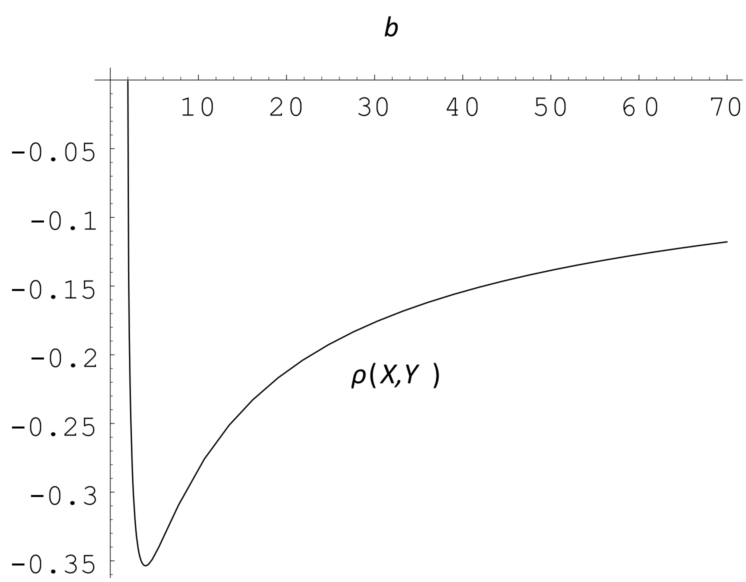

Figure 2. The correlation coefficient $\rho(X,Y)$ between the variables X and Y as a function of the parameter b, for the range $b \in (2,70)$. Note that the minimum value of ρ is -0.35 and occurs at $b = 4$.

2. 2. Log-likelihood function and Shannon's entropy

The log-likelihood function for the bivariate gamma exponential distribution (2.1) is

$$\begin{aligned} l(x,y;a,b,c) &= \log\big(\mathrm{f}(x,y;a,b,c)\big) \\ &= -x(a+cy)+\log(c)+b(\log(a)+\log(x))-\log\big(\Gamma(b)\big). \end{aligned}$$

By direct calculation Shannon's information theoretic entropy for the bivariate gamma exponential distribution, which is the negative of the expectation of the log-likelihood function, is given by

$$S_{\mathrm{f}}(a,b,c) = -\int_0^\infty \int_0^\infty l(x,y;a,b,c)\,\mathrm{f}(x,y;a,b,c)dxdy,$$

$$= 1+b-\log(c)+\log\big(\Gamma(b)\big)-b\,\psi(b) \qquad (2.5)$$

where $\psi(b)=\frac{\Gamma'(b)}{\Gamma(b)}$ is the digamma function. Note that $S_{\mathrm{f}}(a,b,c)$ is independent of the parameter a. In the case when b = 1 which is the random case for the marginal function f_X the Shannon's entropy is:

$$S_{\mathrm{f}}(a,b,c) = 2-\log(c)+\gamma$$

where γ is the Euler gamma, in this case S_f tends to zero when $c = e^{2+\gamma}$. When c = 1, Shannon's entropy is

$$S_{\mathrm{f}}(a,b,c) = 1+b-\log\big(\Gamma(b)\big)-b\,\psi(b).$$

In this case S_{f} tends to zero when $b \approx 126.5$. Figure 3 shows a plot of $S_{\mathrm{f}}(a,b,c)$ in the domain *b,c* $\in$ *(0,3)*, and Figure 4 show plots of $S_{\mathrm{f}}(a,b,c)$ for the cases when *b = 1* and *c = 1*.

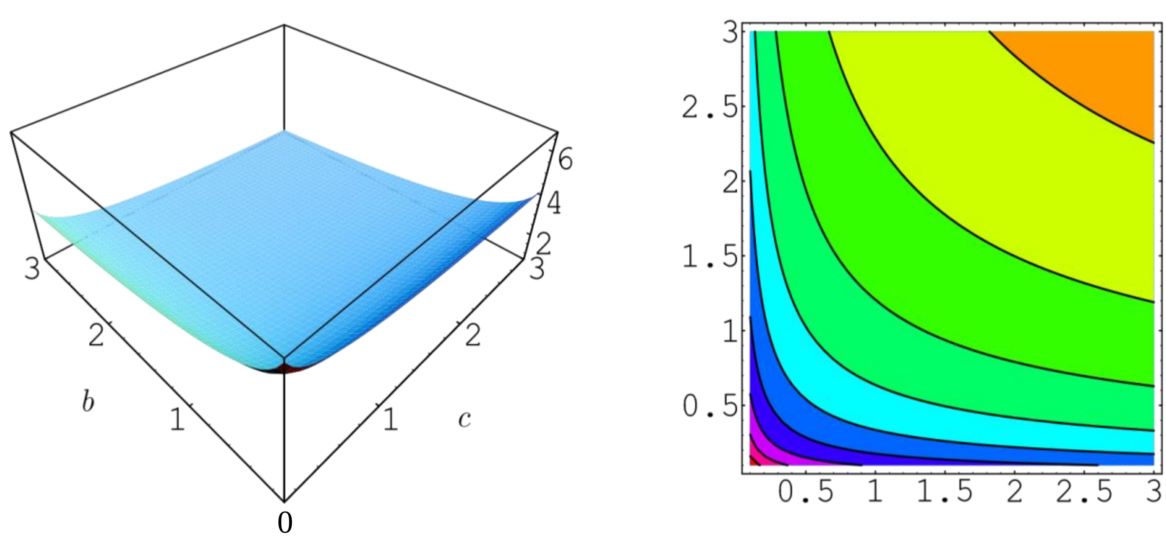

Figure 3. A surface and a contour plot for the Shannon's information entropy S_f, for bivariate gamma exponential distributions in the domain *b,c* $\in$ *(0,3)*.

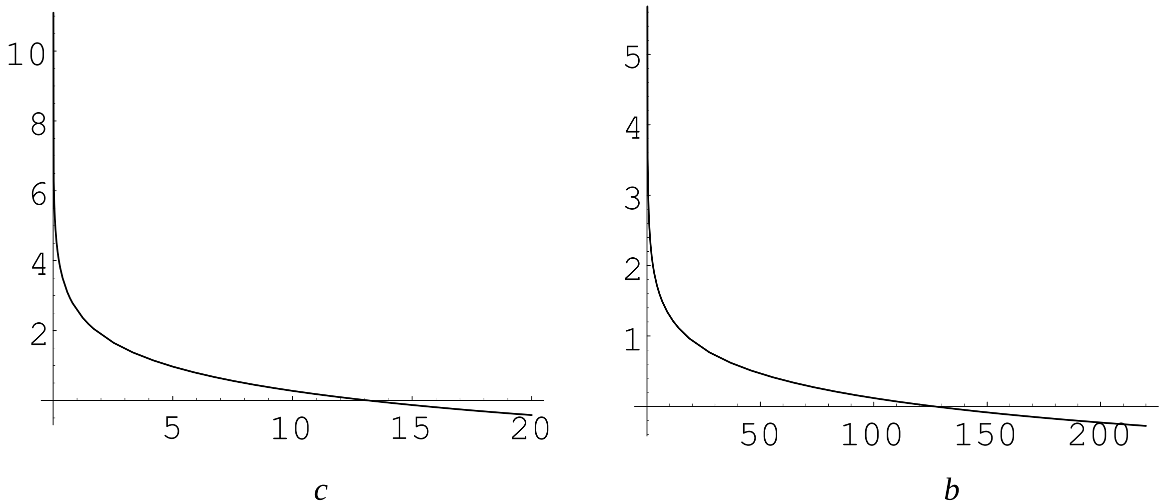

Figure 4. On the left, Shannon's information entropy S_f for bivariate gamma exponential distributions with unit parameter $\beta = 1$, in the domain $c \in (0,20)$.

Note that S_f tends to be zero when $c = e^{2+\gamma}$. On the right, Shannon's information entropy S_f for bivariate gamma exponential distributions with unit parameter $c = 1$, in the domain $b \in (0,250)$. Note that S_f tends to be zero when $b \approx 126.5$.

2. 3. Natural coordinate system and potential finction

We can rewrite the bivariate gamma exponential density function (2.1) as:

$$f(x, y; a, b, c) = \frac{a^b c}{\Gamma(b)} x^b e^{-(ax+cxy)}$$

$$= e^{a(-x)+c(-xy)+b(\log(x))-(\log(\Gamma(b))-b\log(a)-\log(c))} \tag{2.6}$$

Hence the family of bivariate gamma exponential distributions forms an exponential family, with natural coordinate system$(\Theta) = (\Theta_1, \Theta_2, \Theta_3) = (a, b, c)$, random variable $(x_1, x_2, x_3) = (F_1(x), F_2(x), F_3(x)) = (-x, -x\,y, \log(x))$, and potential function $\varphi(a, b, c) = \log(\Gamma(b)) - b\log(a) - \log(c)$.

2. 4. Fisher Information Matrix FIM

The Fisher Information (FIM) is given by the expectation of the covariance of partial derivatives of the log-likelihood function. Here the Fisher information metric components of the family of bivariate gamma exponential distributions M with coordinate system$(\Theta) = (\Theta_1, \Theta_2, \Theta_3) = (a, b, c)$, are given by:

$$g_{ij} = \int_0^\infty \int_0^\infty \frac{\partial^2 l(x, y, \Theta)}{\partial\Theta_i \partial\Theta_j} f(x, y)dydx = \frac{\partial^2 \varphi}{\partial\Theta_i \partial\Theta_j}$$

Hence:

$$g = [g_{ij}] = \begin{bmatrix} \frac{b}{a^2} & \frac{-1}{a} & 0 \\ \frac{-1}{a} & \psi'(b) & 0 \\ 0 & 0 & \frac{1}{c^2} \end{bmatrix} \tag{2.7}$$

and the variance covariance matrix is:

$$g^{-1} = [g^{ij}] = \begin{bmatrix} \frac{a^2\psi'(b)}{b\psi'(b)-1} & \frac{a}{b\psi'(b)-1} & 0 \\ \frac{a}{b\psi'(b)-1} & \frac{b}{b\psi'(b)-1} & 0 \\ 0 & 0 & c^2 \end{bmatrix} \tag{2.8}$$

3. GEOMETRY OF THE BIVARIATE GAMMA EXPONENTIAL MANIFOLD

3. 1. Bivariate gamma exponential manifold

Let M be the family of all bivariate gamma exponential distributions

$$M = \left\{ f(x,y;a,b,c) = \frac{a^b c}{\Gamma(b)} x^b e^{-(a\,x + c\,x\,y)} \middle| a,b,c \in \mathbb{R}^+ \right\}, x,y \in \mathbb{R}^+ \tag{3.9}$$

so the parameter space is $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and the random variables are $(x,y) \in \Omega = \mathbb{R}^+ \times \mathbb{R}^+$.

We can consider M as a Riemannian 3-manifold with coordinate system $(\Theta_1, \Theta_2, \Theta_3) = (a,b,c)$ and Fisher information metric g (2.7), which is

$$ds_g^2 = \frac{b}{a^2} da^2 + \psi'(b) db^2 + \frac{1}{c^2} dc^2 - \frac{1}{a} da\, db.$$

3. 2. *α-connection*

For each $\alpha \in$ R, the α (*or* ∇^α)-*connection* is the torsion-free affine connection with components

$$\Gamma_{ij,k}^{(\alpha)} = \int_0^\infty \int_0^\infty \left(\frac{\partial^2 \log f}{\partial\Theta_i\, \partial\Theta_j} \frac{\partial \log f}{\partial\Theta_k} + \frac{1-\alpha}{2} \frac{\partial \log f}{\partial\Theta_i} \frac{\partial \log f}{\partial\Theta_j} \frac{\partial \log f}{\partial\Theta_k} \right) f\, dy\, dx.$$

We have an affine connection ∇^α defined by:

$$\langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)}$$

where

$$\partial_i = \frac{\partial}{\partial\Theta_i}$$

So by solving the equations

$$\Gamma_{ij,k}^{(\alpha)} = \sum_{h=1}^{3} g_{kh}\, \Gamma_{ij}^{h(\alpha)} \quad ,(k = 1,2,3)$$

we obtain the components of $\nabla^{(\alpha)}$.

Here we give the analytic expressions for the α-connections with respect to coordinates $(\Theta_1, \Theta_2, \Theta_3) = (a, b, c)$.

The nonzero independent components $\Gamma_{ij,k}^{(\alpha)}$ are

$$\begin{gathered}
\Gamma_{11,1}^{(\alpha)} = \frac{b(\alpha-1)}{a^3}, \\
\Gamma_{11,2}^{(\alpha)} = \Gamma_{12,1}^{(\alpha)} = \Gamma_{21,1}^{(\alpha)} = \frac{-(\alpha-1)}{2a^2}, \\
\Gamma_{22,2}^{(\alpha)} = \frac{-(\alpha-1)\psi''(b)}{2}, \\
\Gamma_{33.3}^{(\alpha)} = \frac{(\alpha-1)}{c^3},
\end{gathered} \tag{3.10}$$

and the components $\Gamma_{jk}^{i(\alpha)}$ of the $\nabla^{(\alpha)}$-connections are given by

$$\Gamma^{(\alpha)1} = \left[\Gamma_{ij}^{(\alpha)1}\right] = \begin{bmatrix} \frac{(\alpha-1)(2b\psi'(b)-1)}{2a(b\psi'(b)-1)} & -\frac{(\alpha-1)\psi'(b)}{2b\psi'(b)-2} & 0 \\ -\frac{(\alpha-1)\psi'(b)}{2b\psi'(b)-2} & -\frac{a(\alpha-1)\psi''(b)}{2b\psi'(b)-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.11}$$

$$\Gamma^{(\alpha)2} = \left[\Gamma_{ij}^{(\alpha)2}\right] = \begin{bmatrix} \frac{b(\alpha-1)}{2a^2(b\psi'(b)-1)} & \frac{1-\alpha}{2ab\psi'(b)-2a} & 0 \\ \frac{1-\alpha}{2ab\psi'(b)-2a} & -\frac{b(\alpha-1)\psi''(b)}{2b\psi'(b)-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

$$\Gamma^{(\alpha)3} = \left[\Gamma_{ij}^{(\alpha)3}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha-1}{c} \end{bmatrix} \tag{3.13}$$

3. 3. α-curvatures

By direct calculation we provide various α-curvature objects of the bivariate gamma exponential manifold M, as: the α-curvature tensor, the α-Ricci curvature, the α-scalar curvature, the α-sectional curvature, and the α-mean curvature.

The α-curvature tensor components, which are defined as:

$$R_{ijkl}^{(\alpha)} = \sum_{h=1}^{2} g_{hl}\left(\partial_i \Gamma_{jk}^{h(\alpha)} - \partial_j \Gamma_{ik}^{h(\alpha)} + \sum_{m=1}^{2} \Gamma_{im}^{h(\alpha)}\Gamma_{jk}^{m(\alpha)} - \Gamma_{jm}^{h(\alpha)}\Gamma_{ik}^{m(\alpha)}\right) \quad ,(i,j,k,l = 1,2,3)$$

are given by

$$R_{1212}^{(\alpha)} = -R_{2112}^{(\alpha)} = \frac{(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a^2(b\psi'(b)-1)} \tag{3.14}$$

While the other independent components are zero.

Contracting $R_{ijkl}^{(\alpha)}$ with g^{il} we obtain the components $R_{jk}^{(\alpha)}$ of the α-Ricci tensor

$$R^{(\alpha)} = \left[R_{jk}^{(\alpha)}\right] = \begin{bmatrix} \frac{-b(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a^2(b\psi'(b)-1)^2} & \frac{(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a(b\psi'(b)-1)^2} & 0 \\ \frac{(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a(b\psi'(b)-1)^2} & \frac{-(\alpha^2-1)\psi'(b)\left(\psi'(b)+b\psi''(b)\right)}{4(b\psi'(b)-1)^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.15}$$

The eigenvalues and the eigenvectors of the α-Ricci tensor are given in the case when $\alpha = 0$, by

$$\begin{pmatrix} 0 \\ \frac{\left(b+a^2\psi'(b)\right)(b\psi'(b)-1)\left(\psi'(b)+b\psi''(b)\right)-\sqrt{(b\psi'(b)-1)^2(4a^2+b^2-2a^2b\psi'(b)+a^4\psi'(b)^2)\left(\psi'(b)+b\psi''(b)\right)^2}}{8a^2(b\psi'(b)-1)^3} \\ \frac{\left(b+a^2\psi'(b)\right)(b\psi'(b)-1)\left(\psi'(b)+b\psi''(b)\right)+\sqrt{(b\psi'(b)-1)^2(4a^2+b^2-2a^2b\psi'(b)+a^4\psi'(b)^2)\left(\psi'(b)+b\psi''(b)\right)^2}}{8a^2(b\psi'(b)-1)^3} \end{pmatrix} \tag{3.16}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{b}{2a} + \frac{a\psi'(b)}{2} - \frac{\sqrt{(b\psi'(b)-1)^2(4a^2+b^2-2a^2b\psi'(b)+a^4\psi'(b)^2)\left(\psi'(b)+b\psi''(b)\right)^2}}{2a(b\psi'(b)-1)\left(\psi'(b)+b\psi''(b)\right)} & 1 & 0 \\ \frac{b}{2a} + \frac{a\psi'(b)}{2} + \frac{\sqrt{(b\psi'(b)-1)^2(4a^2+b^2-2a^2b\psi'(b)+a^4\psi'(b)^2)\left(\psi'(b)+b\psi''(b)\right)^2}}{2a(b\psi'(b)-1)\left(\psi'(b)+b\psi''(b)\right)} & 1 & 0 \end{pmatrix} \tag{3.17}$$

By contracting the Ricci curvature components $R_{ij}^{(\alpha)}$ with the inverse metric components g^{ij} we obtain the scalar curvature $R^{(\alpha)}$:

$$R^{(\alpha)} = \frac{-(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{2(b\psi'(b)-1)^2} \tag{3.18}$$

Note that the α-scalar curvature $R^{(\alpha)}$ is a function of only the parameter b, and in the case when $\alpha = 0$ the scalar curvature $R^{(0)}$ is negative and its increases from $-\frac{1}{2}$ to 0.

Figure 5 shows the scalar curvature $R^{(\alpha)}$ in the range $\alpha \in [-2,2]$, $b \in (0,4)$, and the 0-scalar curvature $R^{(0)}$ in the range $b \in (0,4)$.

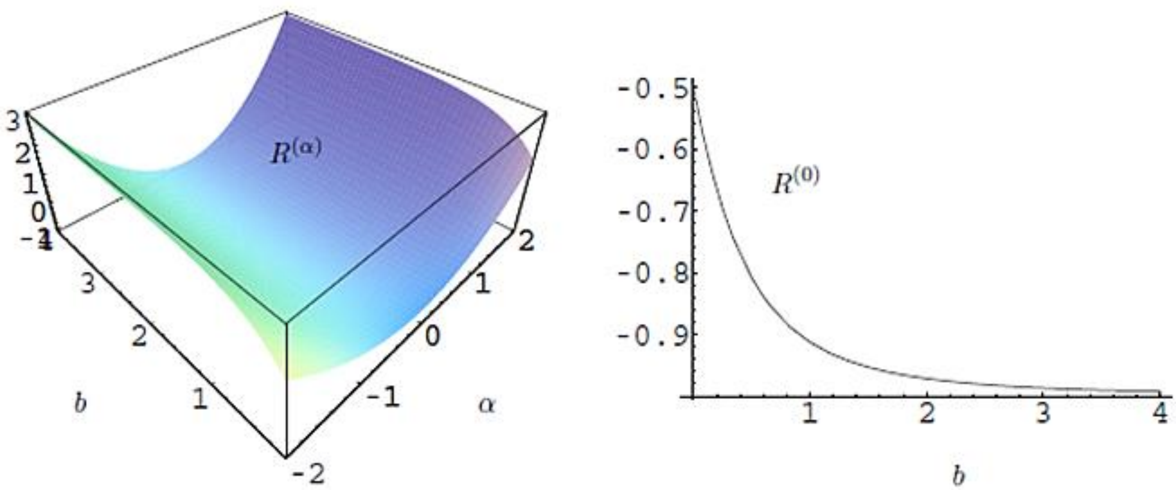

Figure 5. The scalar curvature $R^{(\alpha)}$ for bivariate gamma exponential manifold, in the range $\alpha \in [-2,2]$ and $b \in (0,4)$ on the left, and $\alpha = 0$ and $b \in (0,4)$ on the right. $R^{(0)}$ is negative increasing function of the parameter b from $-\frac{1}{2}$ to *0* as b tends to ∞.

The α-sectional curvatures $\varrho^{(\alpha)}(i,j)$, where $\varrho^{(\alpha)}(i,j) = \frac{R^{(\alpha)}_{ijij}}{g_{ii}\,g_{jj} - g_{ij}^2}$, $(i,j = 1,2,3)$, are

$$\varrho^{(\alpha)}(1,2) = \frac{-(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4(b\psi'(b)-1)^2}. \tag{3.19}$$

while the other components are zero.

The α-mean curvatures $\varrho^{(\alpha)}(i)(i = 1,2,3)$ where $\varrho^{(\alpha)}(i) = \sum_{j=1}^{3} \frac{1}{3}\ \varrho^{(\alpha)}(i,j)$, $(i = 1,2,3)$, are:

$$\varrho^{(\alpha)}(1) = \ \varrho^{(\alpha)}(2) = \frac{-(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{12(b\psi'(b)-1)^2}$$

$$\varrho^{(\alpha)}(3) = 0 \tag{3.20}$$

3. 4. Submanifolds

3. 4. 1. Submanifolds $M_1 \subset M$: $b = 1$

In the case when $b = 1$, the bivariate gamma exponential distribution (2.1) reduces to the form:

$$f(x,y;a,c) = a\, c\, x\, e^{-(a\,x+c\,xy)}, \quad x>0, y>0,\ a,c>0\ . \tag{3.21}$$

when the x and y marginals have the exponential distribution and Pareto distribution (with unit shape parameter), respectively:

$$f_X(x) = ae^{-a\,x}, x>0 \tag{3.22}$$

$$f_Y(y) = \frac{c\,a}{(a+c\,y)^2}, y>0 \tag{3.23}$$

The Fisher metric with respect to the coordinate system $(\Theta_1, \Theta_2) = (a, b)$ is

$$g = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{c^2} \end{bmatrix}. \tag{3.24}$$

Here we note that, the submanifold M_1 and the family of all independent bivariate exponential distributions, which are the direct product of two exponential distributions

$$ae^{-ax}.ce^{-cy}$$

Have the same Fisher metric. Hence the submanifold M_1 is an isometric of the manifold of independent bivariate exponential distributions. In this manifold all the curvatures are zero, while the nonzero independent components $\Gamma_{ij,k}^{(\alpha)}$ and $\Gamma_{jk}^{i(\alpha)}$ of the $\nabla^{(\alpha)}$-connections are:

$$\Gamma_{11,1}^{(\alpha)} = \frac{\alpha-1}{a^3}$$

$$\Gamma_{22,2}^{(\alpha)} = \frac{\alpha-1}{c^3}$$

$$\Gamma_{11}^{(\alpha)1} = \frac{\alpha-1}{a}$$

$$\Gamma_{22}^{(\alpha)2} = \frac{\alpha-1}{c} \tag{3.25}$$

3. 4. 2. Submanifold $M_2 \subset M$: c is constant

In the case when c is constant, the bivariate gamma exponential distribution (2.1) reduces to the form:

$$f(x,y;a,c) = \frac{a^b c}{\Gamma(b)}\, x^b e^{-(ax+cxy)}, \quad x>0, y>0,\ a,b>0\ . \tag{3.26}$$

The x and y marginals have the exponential and Pareto distributions, respectively:

$$f_X(x) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-ax}, \quad x > 0 \tag{3.27}$$

$$f_Y(y) = \frac{bca^b}{(a+cy)^{b+1}}, \quad y > 0 \tag{3.28}$$

where M_2 and M have the same correlation coefficient (2.1).

The Fisher metric with respect to the coordinate system $(\Theta_1, \Theta_2) = (a, b)$ is

$$g = \begin{bmatrix} \frac{b}{a^2} & -\frac{1}{a} \\ -\frac{1}{a} & \psi'(b) \end{bmatrix}. \tag{3.29}$$

Note that the submanifold M_2 and the gamma manifold, which is the family of all univariate gamma probability densities functions

$$\frac{a^b}{\Gamma(b)} x^{b-1} e^{-ax}, \quad x > 0, \qquad a, b > 0$$

have the same Fisher metric. Hence, the family of bivariate gamma exponential probability density functions with constant parameter *c*, is an isometric isomorphic of the gamma manifold with information theoretical metric topology.

Here we give the geometrical structures of the submanifold M_2. The nonzero independent components of the $\nabla^{(\alpha)}$-connections are given by:

$$\Gamma_{11,1}^{(\alpha)} = \frac{b(\alpha-1)}{a^3},$$

$$\Gamma_{11,2}^{(\alpha)} = \Gamma_{12,1}^{(\alpha)} = \Gamma_{21,1}^{(\alpha)} = \frac{-(\alpha-1)}{2a^2},$$

$$\Gamma_{22,2}^{(\alpha)} = \frac{-(\alpha-1)\psi''(b)}{2}, \tag{3.30}$$

$$\Gamma^{(\alpha)1} = \begin{bmatrix} \frac{(\alpha-1)(2b\psi'(b)-1)}{2a(b\psi'(b)-1)} & -\frac{(\alpha-1)\psi'(b)}{2b\psi'(b)-2} \\ -\frac{(\alpha-1)\psi'(b)}{2b\psi'(b)-2} & -\frac{a(\alpha-1)\psi''(b)}{2b\psi'(b)-2} \end{bmatrix}. \tag{3.31}$$

$$\Gamma^{(\alpha)2} = \begin{bmatrix} \frac{b(\alpha-1)}{2a^2(b\psi'(b)-1)} & \frac{1-\alpha}{2ab\psi'(b)-2a} \\ \frac{1-\alpha}{2ab\psi'(b)-2a} & -\frac{b(\alpha-1)\psi''(b)}{2b\psi'(b)-2} \end{bmatrix}. \tag{3.32}$$

The α-curvature tensor is given by

$$R_{1212}^{(\alpha)} = -R_{2112}^{(\alpha)} = \frac{(\alpha^2-1)(\psi'(b)+b\psi''(b))}{4a^2(b\psi'(b)-1)}. \tag{3.33}$$

while the other independent components are zero.

The α-Ricci curvature is

$$R^{(\alpha)} = \begin{bmatrix} \frac{-b(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a^2(b\psi'(b)-1)^2} & \frac{(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a(b\psi'(b)-1)^2} \\ \frac{(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{4a(b\psi'(b)-1)^2} & \frac{-(\alpha^2-1)\psi'(b)\left(\psi'(b)+b\psi''(b)\right)}{4(b\psi'(b)-1)^2} \end{bmatrix}. \tag{3.34}$$

The α-scalar curvature is

$$R^{(\alpha)} = \frac{-(\alpha^2-1)\left(\psi'(b)+b\psi''(b)\right)}{2(b\psi'(b)-1)^2}. \tag{3.35}$$

Note that M_2 and M have the same scalar curvature, which is a negative non constant in the case when $\alpha = 0$; and this has limiting value $-\frac{1}{2}$ as $b \to 0$ and 0 as $b \to \infty$. See Figure 5.

3. 4. 3. Submanifold $M_3 \subset M$: $a = 1$

In the case when $a = 1$, the bivariate gamma exponential distribution (2.1) reduces to the form:

$$f(x, y; a, c) = \frac{c}{\Gamma(b)}\, x^b e^{-(a\,x+c\,xy)}, \quad x > 0, y > 0,\ b, c > 0\ . \tag{3.36}$$

The x and y marginals have the gamma distribution (with scalar parameter = 1) and Pareto distribution (with scalar parameter = 1), respectively:

$$f_X(x) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}, \quad x > 0 \tag{3.37}$$

$$f_Y(y) = \frac{bc}{(1+c\,y)^{b+1}}, \quad y > 0 \tag{3.38}$$

The Fisher metric with respect to the coordinate system $(\Theta_1, \Theta_2) = (a, c)$ is

$$g = \begin{bmatrix} \psi'(b) & 0 \\ 0 & \frac{1}{c^2} \end{bmatrix}. \tag{3.39}$$

The nonzero independent components of the $\nabla^{(\alpha)}$-connections are given by:

$$\Gamma^{(\alpha)}_{11,1} = \frac{-\left((\alpha-1)\psi''(b)\right)}{2},$$

$$\Gamma^{(\alpha)}_{22,2} = \frac{\alpha-1}{c^3},$$

$$\Gamma_{11}^{(\alpha)1} = \frac{-\left((\alpha-1)\psi''(b)\right)}{2\psi'(b)},$$

$$\Gamma_{22}^{(\alpha)2} = \frac{\alpha-1}{c}, \tag{3.40}$$

While all the curvatures are zero.

3. 4. 4. Submanifold $M_4 \subset M$: $a = c$

In the case when $a = c$, the bivariate gamma exponential distribution (2.1) reduces to the form:

$$f(x,y;a,c) = \frac{a^{b+1}}{\Gamma(b)}\, x^b e^{-a(x+xy)}, \quad x > 0, y > 0,\; a,b > 0\,. \tag{3.41}$$

The x and y marginals have the gamma and Pareto distributions, respectively:

$$f_X(x) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-ax}, \quad x > 0 \tag{3.42}$$

$$f_Y(y) = \frac{b}{(1+y)^{b+1}}, \qquad y > 0 \tag{3.43}$$

and the correlation coefficient is given by:

$$\rho(X,Y) = \frac{\Gamma(b)^2\sqrt{(b-2)(b-1)^2}}{\left(\Gamma(b)-\Gamma(b+1)\right)\sqrt{\Gamma(b+1)\left(\Gamma(b+2)-b\Gamma(1+b)\right)}},\ b > 2. \tag{3.44}$$

With respect to the coordinate system $(\Theta_1, \Theta_2) = (a, b)$ the Fisher metric is:

$$g = \begin{bmatrix} \frac{b+1}{a^2} & -\frac{1}{a} \\ -\frac{1}{a} & \psi'(b) \end{bmatrix}. \tag{3.45}$$

The nonzero independent components of the $\nabla^{(\alpha)}$-connections are given by:

$$\Gamma_{11,1}^{(\alpha)} = \frac{(\alpha-1)(b+1)}{a^3},$$

$$\Gamma_{11,2}^{(\alpha)} = \Gamma_{12,1}^{(\alpha)} = \Gamma_{21,1}^{(\alpha)} = \frac{-(\alpha-1)}{2a^2},$$

$$\Gamma_{22,2}^{(\alpha)} = \frac{-(\alpha-1)\psi''(b)}{2}, \tag{3.46}$$

$$\Gamma^{(\alpha)1} = \begin{bmatrix} \frac{(\alpha-1)(2(1+b)\psi'(b)-1)}{2a((1+b)\psi'(b)-1)} & -\frac{(\alpha-1)\psi'(b)}{2(1+b)\psi'(b)-2} \\ -\frac{(\alpha-1)\psi'(b)}{2(1+b)\psi'(b)-2} & -\frac{a(\alpha-1)\psi''(b)}{2(1+b)\psi'(b)-2} \end{bmatrix}. \tag{3.47}$$

$$\Gamma^{(\alpha)2} = \begin{bmatrix} \frac{(1+b)(\alpha-1)}{2a^2((1+b)\psi'(b)-1)} & \frac{-(\alpha-1)}{2a((1+b)\psi'(b)-1)} \\ \frac{-(\alpha-1)}{2a((1+b)\psi'(b)-1)} & \frac{-(1+b)(\alpha-1)\psi''(b)}{2(1+b)\psi'(b)-2} \end{bmatrix}. \tag{3.48}$$

The α-curvature tensor is given by:

$$R^{(\alpha)}_{1212} = -R^{(\alpha)}_{2112} = \frac{(\alpha^2-1)(\psi'(b)+(1+b)\psi''(b))}{4a^2((1+b)\psi'(b)-1)}. \tag{3.49}$$

while the other independent components are zero.

α-Ricci curvature is:

$$R^{(\alpha)} = \begin{bmatrix} \frac{-(1+b)(\alpha^2-1)(\psi'(b)+(1+b)\psi''(b))}{4a^2((1+b)\psi'(b)-1)^2} & \frac{(\alpha^2-1)(\psi'(b)+(1+b)\psi''(b))}{4a((1+b)\psi'(b)-1)^2} \\ \frac{(\alpha^2-1)(\psi'(b)+(1+b)\psi''(b))}{4a((1+b)\psi'(b)-1)^2} & \frac{-(\alpha^2-1)\psi'(b)(\psi'(b)+(1+b)\psi''(b))}{4((1+b)\psi'(b)-1)^2} \end{bmatrix}. \tag{3.50}$$

The scalar curvature $R^{(0)}$is given by:

$$R^{(\alpha)} = \frac{-(\alpha^2-1)(\psi'(b)+(1+b)\psi''(b))}{2((1+b)\psi'(b)-1)^2}. \tag{3.51}$$

where the scalar curvature $R^{(0)}$ is negative non constant.

3. 5. Log-bivariate gamma exponential manifold

Here we introduce a log-bivariate gamma exponential distribution, which arises from the bivariate gamma exponential distribution (2.1) for non-negative random variables $n = e^{-x}$ and $m = e^{-y}$. So the log-bivariate gamma exponential distribution, has probability density function

$$g(n,m) = \frac{a^b c\, n^{a-1}(-\log(n))^b}{m\,\Gamma(b)} e^{-c\log(n)\,\log(m)},\ 0 < n < 1, 0 < m < 1. \tag{3.52}$$

where a, b, $c > 0$. Figure 6 shows plots of the log-bivariate gamma exponential family of densities with unit parameters a, b and c.

Note that the marginals of N and M are log-gamma distribution and log-Pareto distribution, respectively.

$$f_N(n) = \frac{a\, n^{a-1}(-a\log(n))^{b-1}}{\Gamma(b)}, \qquad 0 < n < 1$$

$$f_M(m) = \frac{(-1)^{2b} a^b\, b\, c}{m(a - c\log(m))^{b+1}}, \quad 0 < m < 1$$

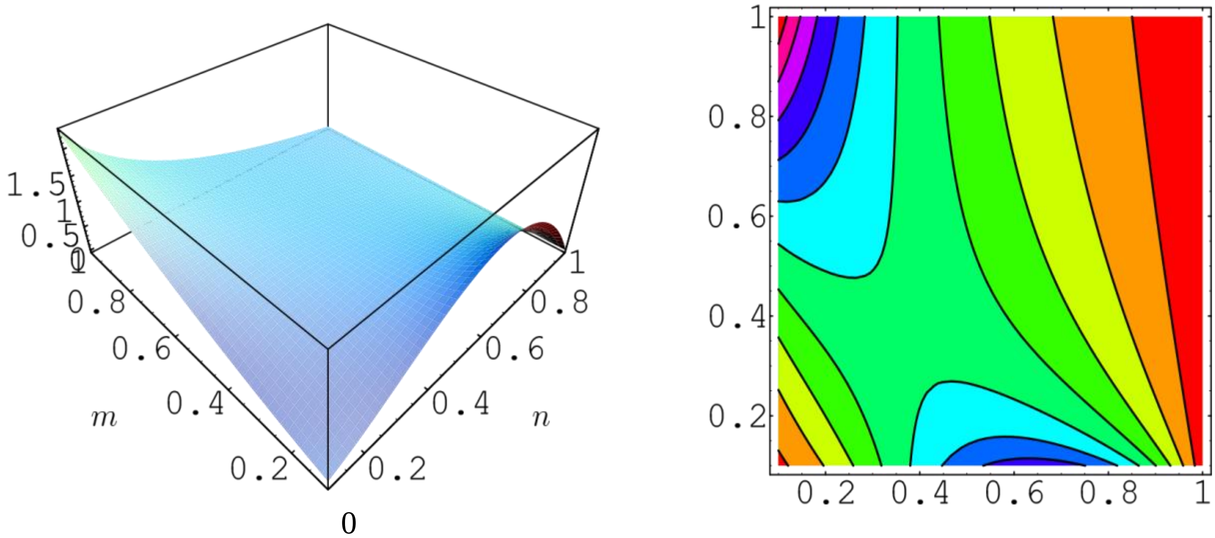

Figure 6. The log-bivariate gamma exponential family of densities with parameters $a = b = c = 1$ as a surface and as a contour plot, shown for the range $n,m \in [0,1]$

The covariance and correlation coefficient of the variables N and M are given by:

$$cov(N,M) = b\, e^{\frac{a}{c}} \left(\frac{a}{1+a}\right)^b \left(e^{\frac{1}{c}} E\left(1+b, \frac{1+a}{c}\right) - (-1)^{2b} E\left(1+b, \frac{a}{c}\right)\right)$$

$$\rho(N,M) = \frac{a^{-b}\, b \left(\frac{a}{1+a}\right)^b \left(e^{\frac{1}{c}} E\left(1+b, \frac{1+a}{c}\right) - (-1)^{2b} E\left(1+b, \frac{a}{c}\right)\right)}{\sqrt{\left((2+a)^{-b} - \frac{a^b}{(1+a)^{2b}}\right) b}\ \sqrt{\left(\frac{-1}{c}\right)^{2b} \left(2^b c^b \Gamma\left(-b, \frac{2a}{c}\right) - (-1)^{2b} a^b b\, \Gamma\left(-b, \frac{a}{c}\right)^2\right)}}.$$

where $E(n,z)$ is the exponential integral function, which is defined by $\int_1^\infty \frac{e^{-zt}}{t^n} dt$.

This family of densities determines a Riemannian 3-manifold which is isometric with the bivariate gamma exponential manifold M.

4. GEODESICS AND DISTANCES

4. 1. Geodesics

A curve $\gamma(t) = (\gamma_i(t))$ in a Riemannian manifold (M^n, g) is called geodesic if:

$$\frac{d^2\gamma_i}{dt^2} + \sum_{j,k=1}^{n} \Gamma_{jk}^{i} \frac{d\gamma_j}{dt}\frac{d\gamma_k}{dt} = 0 \; (i = 1,2,\dots,n)$$

where Γ_{jk}^{i} are the Christoffel symbols.

In the 3-manifold of bivariate gamma exponential distributions with Fisher metric g, the geodesic equations are given by:

$$a^{\cdot\cdot}(t) = \frac{1-2b\psi'(b)}{2a-2ab\psi'(b)}a^{\cdot}(t)^2 + \frac{2\psi'(b)}{1-b\psi'(b)}a^{\cdot}(t)b^{\cdot}(t) + \frac{a\psi''(b)}{2-2b\psi'(b)}b^{\cdot}(t)^2,$$

$$b^{\cdot\cdot}(t) = \frac{-b}{2a^2-2a^2b\psi'(b)}a^{\cdot}(t)^2 + \frac{1}{a-ab\psi'(b)}a^{\cdot}(t)b^{\cdot}(t) + \frac{b\psi''(b)}{2-2b\psi'(b)}b^{\cdot}(t)^2,$$

$$c^{\cdot\cdot}(t) = \tfrac{1}{c}c^{\cdot}(t)^2, \tag{4.53}$$

where $= \frac{d}{dt}$

Here we obtain the analytical solutions for the geodesic differential equations in the following case:

(1) If b is constant, then the geodesics are:

$a(t) = m_1$ and $c(t) = m_2 - c\log(t + c\, m_3)$, where m_i is constant $(i = 1,2,3)$.

(2) If a is constant, then the geodesics are:

$b(t) = m_1$ and $c(t) = m_2 - c\log(t + c\, m_3)$, where m_i is constant $(i = 1,2,3)$.

4. 2. Distances

Let M be the bivariate gamma exponential manifold, and let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two points in M where:

$$\mathfrak{f}(x,y;a_i,b_i,c_i) = \frac{a_i^{b_i}c_i}{\Gamma(b_i)}\, x^{b_i}e^{-(a_i x + c_i xy)}, \quad (i = 1,2).$$

Then the Kullback-Leibler distance, the J-divergence and the Battacharyya distance between bivariate gamma exponential distributions $\mathfrak{f}_1$ and $\mathfrak{f}_2$, are given by:

- Kullback-Leibler distance:

The KullbackLeibler distance or relative entropy is a non-symmetric measure of the difference between two probability distributions.

From f_1 to f_2 the Kullbback-distance $KL(f_1, f_2)$ is given by

$$KL(f_1, f_2) = \int_0^\infty \int_0^\infty f_1 \log\left(\frac{f_1}{f_2}\right) dx\, dy,$$

$$= \log\left(\frac{\Gamma(b_2)c_1}{\Gamma(b_1)c_2}\right) - \left(\log\left(\frac{a_2}{a_1}\right) + \psi(b_1)\right) b_2 + \left(\psi(b_1) + \frac{a_2}{a_1} - 1\right) b_1 + \frac{c_2}{c_1} - 1. \quad (4.54)$$

- J-divergence:

The J-divergence is a summarization of the Kullback-Leibler distance. Its given by this formula

$$J(f_1, f_2) = \int_0^\infty \int_0^\infty (f_1 - f_2) \log\left(\frac{f_1}{f_2}\right) dx\, dy, = KL(f_1, f_2) + KL(f_2, f_1),$$

$$= \left(\log\left(\frac{a_2}{a_1}\right) + \psi(b_1) - \psi(b_2) + \frac{a_2}{a_1} - 1\right) b_1 + \left(\log\left(\frac{a_1}{a_2}\right) - \psi(b_1) + \psi(b_2) + \frac{a_1}{a_2} - 1\right) b_2 + \frac{c_1}{c_2} + \frac{c_2}{c_1} - 2. \quad (4.55)$$

- Bhattacharyya distance:

The Bhattacharyya distance measures the similarity of two probability distributions. Between f_1 to f_2 the Bhattacharyya distance $B(f_1, f_2)$is given by

$$B(f_1, f_2) = -\log \int_0^\infty \int_0^\infty \sqrt{f_1 f_2}\, dx\, dy, = -\log\left(\frac{2^{\left(\frac{2+b_1+b_2}{2}\right)} \Gamma\left(\frac{b_1+b_2}{2}\right) \sqrt{\frac{a_1{}^{b_1} a_2{}^{b_2} c_1 c_2}{\psi(b_1)\psi(b_2)}}}{(a_1+a_2)^{\frac{b_1+b_2}{2}} (c_1+c_2)}\right). \quad (4.56)$$

5. AFFINE IMMERSIONS

If M is the 3-manifold of the bivariate gamma exponential distributions with the Fisher metric g and the exponential connection $\nabla^{(1)}$. Then bivariate gamma exponential 3-manifold M with the Fisher metric g and exponential connection $\nabla^{(1)}$, can be realized in $\mathbb{R}^4$ by the graph of a potential function, the affine immersion $\{h_M, \xi\}$:

$$h_M : M \to \mathbb{R}^4 : (\theta_i) \mapsto \begin{pmatrix} \theta_i \\ \varphi(\theta) \end{pmatrix}, \xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (5.57)$$

where θ is the natural coordinate system (2.7), and $\phi(\theta)$ is the potential function $\phi(\theta) = \log(\Gamma(b)) - b\log(a) - \log(c)$. This is a typical example of affine immersion, and called a graph immersion. More details can be found in [2, 11].

In the following two sections we are interested on the graph immersions for the subspaces M_1 and M_2.

5. 1. Agraph immersion for the space *M*₁

Here we provide an affine immersion in Euclidean R^3 for the space M_1, which consists bivariate gamma exponential distributions with parameter $c = 1$. The coordinates $(\theta_1, \theta_2) = (a, b)$ form a natural coordinate system for the manifold M_1, with potential function $\phi(\theta) = \log(\Gamma(b)) - b\log(a)$. Then M_1can be realized in Euclidean $\mathbb{R}^3$ by the graph of the affine immersion $\{h_{M_1}, \xi\}$ where ξ is the transversal vector field along h_{M_1}:

$$h_{M_1} : M_1 \to \mathbb{R}^3 : \binom{a}{b} \mapsto \begin{pmatrix} a \\ b \\ \log(\Gamma(b)) - b\log(a) \end{pmatrix}, \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

.

The submanifold of bivariate gamma exponential distributions where the marginal of X is exponential distributions is represented by the curve ($b = 1$):

$$(0, \infty) \to \mathbb{R}^3 : a \mapsto \{a, 1, -\log a\}$$

and a tubular neighbourhood of this curve will represents bivariate gamma exponential distributions having exponential and Pareto marginals, where Pareto has a unit shape parameter. Moreover, in section 3.4.1 we showed that the space M_1 is isometric with the 3-manifold of univariate gamma distributions, so the tubular neighbourhood of the curve $b = 1$ in the surface h_{M_1} also will contain all exponential distributions, which represents departures from randomness. Details on gamma manifold affine immersion can be found in chapter 5 in [3].

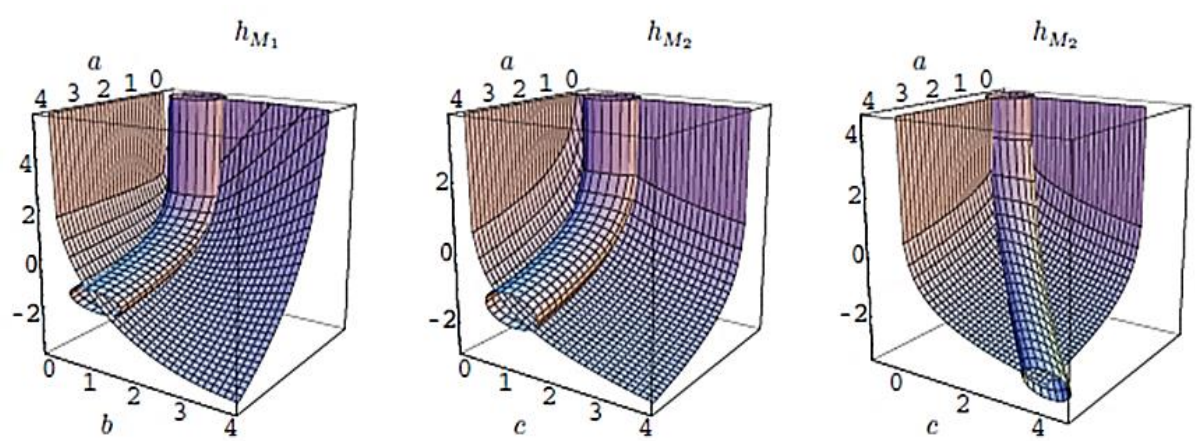

Figure 7. An affine embedding h_{M_1} for the space M_1 as a surface in $\mathbb{R}^3$ in the range $a, b \in (0,4)$, and an $\mathbb{R}^3$-tubular neighbourhood of the curve $b = 1$ which represents bivariate gamma exponential distributions having exponential and Pareto marginals. An affine embedding h_{M_2}

of the bivariate gamma exponential distributions with unit parameter b as a surface in $\mathbb{R}^3$ in the range a, $c \in (0,4)$, and an $\mathbb{R}^3$-tubular neighbourhoods of the curves $c = 1$ (in the middle) and $c = a$ (in the right) in the surface. These curves represent bivariate gamma exponential distributions having exponential and Pareto marginals.

Figure 7 shows an affine embedding h_{M_1} of the bivariate gamma exponential distributions with unit parameter c as a surface in R^3 in the range $a, b \in (0,4)$, and an $\mathbb{R}^3$-tubular neighbourhood of the curve $b = 1$ in the surface. This curve represents bivariate gamma exponential distributions having exponential and Pareto marginals.

5. 2. Agraph immersion for the space M_2

Here we provide an affine immersion in Euclidean R^3 for the space M_2 , which consists bivariate gamma exponential distributions with parameter $b = 1$. The coordinates $(\theta_1, \theta_2) = (a, c)$ form a natural coordinate system for the manifold M_2, with potential function $\varphi(\theta) = -\log(a) - \log(c)$. Then M_2 can be realized in Euclidean $\mathbb{R}^3$ by the graph of the affine immersion $\{h_{M_2}, \xi\}$ where ξ is the transversal vector field along h_{M_2}:

$$h_{M_2} : M_2 \to \mathbb{R}^3 : \begin{pmatrix} a \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \\ -\log(a) - \log(c) \end{pmatrix}, \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

This surface consists all bivariate gamma exponential distributions where the marginals are gamma and Pareto (with unit shape parameters).

The case when the marginal of X is exponential distribution is represented by the curve $(c = 1)$:

$$(0, \infty) \to \mathbb{R}^3 : a \mapsto \{a, 1, -\log(a)\}$$

and a tubular neighbourhood of this curve will represents bivariate gamma exponential distributions having exponential and Pareto marginals. Moreover, this tubular neighbourhood will contains the direct product of two exponential distributions.

While the curve $(c = a)$:

$$(0, \infty) \to \mathbb{R}^3 : a \mapsto \{a, 1, -2\log(a)\}$$

will represents bivariate gamma exponential distributions having exponential and Pareto marginals, where Pareto has a unit shape parameter. This curve will also represent the direct product of two identical exponential distributions.

In figure 7 we show an affine embedding h_{M_2} of the bivariate gamma exponential distributions with unit parameter b as a surface in R^3 in the range $a, c \in (0,4)$, and an $\mathbb{R}^3$-tubular neighbourhoods of the curves $c = 1$ and $c = a$ in the surface.

These curves represent bivariate gamma exponential distributions having exponential and Pareto marginals.

6. CONCLUSIONS

The central discussion for this paper is to derive the geometrical structures for the Riemannian 3-manifold of the bivariate gamma exponential distributions, equipped with the Fisher information matrix. The αconnections and α-curvatures are obtained including those on four submanifolds. The geodesic differential equations are given, and solved analytically in some cases. Moreover, we obtained explicit expressions for geodesic distance, Kullback-Leibler distance, J-divergence and Bhattacharyya distance. Finally, this work proved that the bivariate gamma exponential manifold can be realized in $\mathbb{R}^4$, using information theoretical immersions, and illustrations for tubular neighbourhoods are shown in some special cases of practical interest, which provided a metrization of departures from randomness for bivariate processes.

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