Contents

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Summary

Difference equations appear as natural descriptions of observed evolution, phenomena because most measurements of time evolving variables are discrete and as such those equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. This is especially true in the case of Lyapunov theory of stability. Nonetheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example; a simple difference equation resulting from a first order differential may have a phenomena often called appearance of "ghost" solutions or existence of chaotic orbits that can only happen for higher order differential equations and the theory of difference equations is interesting in itself. The aim of this thesis is to study the qualitative behavior of solution of some nonlinear difference equations of different order. We discussed, in detail the following:

- Finding the equilibrium points for some (systems) difference equations;
- Investigating the local stability character of the solutions of some (systems) difference equations;
- Finding conditions which insure that the solutions of the equations are bounded;
- Investigating the global asymptotic stability of the solutions of some difference equations;
- Finding conditions which gurartee that the solutions of the equations are periodic with prime period two or more;
- Finding conditions for oscillation of the solutions.

This thesis contains illustrative examples as applications of our results. The thesis consists of Introduction and there four chapters:

Introduction. This chapter is an introductory chapter and contains some basic definitions, elementary results that will be used throughout the next chapters.

Chapter 1. In this chapter we investigate the local stability, the boundedness, the global attractor, the periodicity character, and for the solutions of the nonautonomous difference equations:

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \quad \text{for } n \geq 0.$$

Chapter 2. This chapter discusses local stability, the boundedness, global stability, and semicycle for the solutions of higher order difference equations:

$$x_{n+1} = ax_n + \frac{bx_n}{1 + x_{n-1}^p}, \quad n = 0, 1, ...,$$

and

$$x_{n+1} = ax_{n-1} + \frac{b}{1 + x_n^p}, \quad n = 0, 1, ...$$

Chapter 3. Here we investigate the local stability, boundedness, and the global stability for the solutions of the difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{Ax_{n-k}^p + Bx_{n-m}^q}, \quad n \geq 0.$$

Chapter 4. In this chapter we study the local stability, global stability, oscillatory, and the periodicity character for the solutions of the following system of difference equations:

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^p w_{n-1}^{p_1}}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^q u_{n-1}^{q_1}}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^r v_{n-1}^{r_1}}, \text{ for } n \geq 0.$$

Our results generalize and complement some of the previous results in the literature (as described in the introduction of chapter). Moreover, some examples are given to illustrate the main results.

Introduction

In the world, important progress has been made during recent years in the theory of nonlinear difference equations. There is a set of nonlinear difference equation, known as the rational difference equations. Lately, there has been huge attention in discussion rational difference equations and of the purpose for this exigency for some methods whose can be used examining equations arising in mathematical modules desecrating real life statuses. Moreover, difference equations have given much attention from scientists from multiple disciplines. Possibility, this is to a great extent because of happening of PCs where differential equations are explained by utilizing their estimated distinction condition details. Also, computer has assisted to study behavior solutions of difference equations by the easy way. Although, all observations and prediction got using the computer has to been proven from the analytical point of view. Accordingly, to take consideration a rich topic of research and want to be investigated in the details.

The main role of this thesis is to study the behavior of some difference equations where difference equations have gotten much consideration from researchers from various disciplines. Perhaps this is to a great extent because of the appearance of computers where differential equations are solved by using their approximate difference equation formulations. With the utilize of PC one can without much of a stretch explore different avenues regarding difference equations and one can one can undoubtedly find that such conditions have intriguing properties with a great deal of structure and typicality. Obviously, all PC perceptions and expectations should likewise be demonstrated logically. In this way this a prolific territory of research, still in its earliest stages, with thorough and essential outcomes.

In spite of the fact that difference equations show themselves as scientific models portraying genuine circumstances in likelihood hypothesis, statistical problems, electrical networks, number hypothesis, geometry, electrical systems etc see [9], [29], [28], [31], [35], [13], [18], and [41].

The investigation of dynamics is the study of how things change after some time. Discrete dynamics is the examination of amounts that change at discrete focuses in time, for example the size of a population from one year to the next, or the change in the genetic make-up of a population from one generation to the next see book [12]. In general, we concurrently develop a model some situation and the mathematical theory necessary to analyze that module. As we develop our mathematical theory, we will be able add more components to our model.

The mean for studying change is to discover a connection between, what is happening now and what will be happened in the near future that is, cause and effect. By analyzing this relationship, we can often predict what will be happened in the distant future. The distant future is sometimes a given point in time but more often is a limit as time goes to infinity. In doing our analysis, we will use many algebraic and calculus topics such as, factoring, exponential and logarithms, solving systems of equations, and derivatives. We should also be able to apply discrete dynamics to any field in which things change, which is the most fields. The goal is to not only learn mathematics, but to get develop a differently way of thinking about the world.

The oscillation and global asymptotic behavior of the solutions are two such qualitative properties which are very important for applications in many areas such as control theory, mathematical biology, neural networks, etc see [6], [17], [23], [24], [25], and [37]. It is impossible to use computer based "numerical" techniques to study the oscillation or asymptotic behavior of all solutions of a given equation due to the global nature of these properties. Therefore, these properties have received the attention of several mathematicians, engineers and other scientists around the world.

Existence of the solutions of difference equations of deferent orders and the study of their qualitative properties such as locally, boundedness, global stability, the periodicity have been discussed by many authors, See, for examples [10], [11], [14], [15], [16], [19], [20], [22], [26], [27], [30], [36], [38], [39], [40], [42], [43], and [44].

Basic definitions and theorems

Here we recall some basic definitions and elementary results that will be used throughout the next chapters.

Let J be an interval real numbers and let $g : J^{k+1} \times J \rightarrow J$, where g is a continuously differentiable function. Consider the difference equation

$$y_{n+1} = g(y_n, y_{n-1}, ..., y_{n-k}), \quad n \geq 0, \tag{1}$$

where $y_{-k}, y_{-k+1}, ..., y_0 \in J$. Let $\overline{y}$ be the equilibrium point of Eq.(1). Any equilibrium point $\overline{y}$ of this equation is a point that satisfies the condition $\overline{y} = g(\overline{y}, \overline{y}, ..., \overline{y})$.

Definition : The sequence $\{y_n\}$ is called to be periodic with period p if

$$y_{n+p} = y_n, \quad for \;\; n = 0, 1, ...$$

Definition : Eq.(1) is called to be permanent and bounded if there exists number m and M, with $0 < m < M < \infty$ so for any initial condition $y_{-k}, y_{-k+1}, ..., y_0 \in (0,\infty)$ there exists a positive integer N which consist these initial conditions such that $m < y_n < M$, $n \geq N$.

The linearized equation of Eq.(1) about the equilibrium point $\overline{y}$ is

$$y_{n+1} = a_1 z_n + a_2 z_{n-1} + ... + a_{k+1} z_{n-k}, \tag{2}$$

where $a_i = \frac{\partial f}{\partial y_{n-i}}(\overline{y}, \overline{y}, ..., \overline{y})$, $i = 0, 1, ..., k$. The characteristic equation of $Eq.(2)$ is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} a_i \lambda^{k-i+1} = 0.$$

(i) The equilibrium point $\overline{y}$ of $Eq.(1)$ is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ so for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$. Where $|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + ... + |y_0 - \overline{y}| < \delta$, we have $|y_n - \overline{y}| < \epsilon$, $n \geq -k$.

(ii) The equilibrium point $\overline{y}$ of $Eq.(1)$ is globally asymptotically stable if $\overline{y}$ is locally stable and there exists $\lambda > 0$,such that for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$. With
$|y_{-k} - \overline{y}| + |y_{-k+1} - \overline{y}| + ... + |y_0 - \overline{y}| < \lambda$, we have $\lim_{n\to\infty} y_n = \overline{y}$.

(iii) The equilibrium point $\overline{y}$ of $Eq.(1)$ is global attractor if for all $y_{-k}, y_{-k+1}, ..., y_0 \in I$,
then $\lim_{n\to\infty} y_n = \overline{y}$.

(iv) The equilibrium point $\overline{y}$ of $Eq.(1)$ is globally asymptotically stable if $\overline{y}$ is locally stable, and $\overline{y}$ is a global attractor of $Eq.(1)$.

(iiv) The equilibrium point $\overline{y}$ of $Eq.(1)$ is unstable if $\overline{y}$ is not locally stable.

Definition : A positive semicycle of a solution $\{y_n\}$ of Eq.(1) consists of a "string"of terms $\{y_j, y_{j+1}, ..., y_n\}$, all greater than or equal to the equilibrium $\overline{y}$,with $j \geq -1$ and $n \leq \infty$ and such that

$$\text{either } j = -1, \quad \text{or} \quad j > -1 \quad \text{and} \quad y_{j-1} < \overline{y},$$

and

$$\text{either } n = \infty, \quad \text{or} \quad n < \infty \quad \text{and} \quad y_{n+1} < \overline{y}.$$

Definition : A negative semicycle of a solution $\{y_n\}$ of $Eq.(1)$ consists of a "string"of terms $\{y_k, y_{k+1}, ..., y_n\}$, all less than to the equilibrium $\overline{y}$,with $k \geq -1$ and $n \leq \infty$ and such that

$$\text{either } k = -1, \quad \text{or} \quad k > -1 \quad \text{and} \quad y_{k-1} \geq \overline{y},$$

and

$$\text{either } n = \infty, \quad \text{or} \quad n < \infty \quad \text{and} \quad y_{n+1} \geq \overline{y}.$$

Definition [Oscillatory] :

(a) A sequence $\{y_n\}$ is called to oscillate about zero if the terms y_n are neither eventually all positive nor eventually all negative. Moreover, the sequence is called nonoscillatory. A sequence $\{y_n\}$ is called strictly oscillatory if for every $n_0 \geq 0$,there exists $n_1, n_2 \geq n_0$ such that $y_{n1}y_{n2} < 0$.

(b) A sequence $\{y_n\}$ is called to oscillate about $\overline{y}$ if the sequence $y_n - \overline{y}$ oscillates.

(c) A sequence $\{y_n\}$ is said strictly oscillatory about $\overline{y}$ if the sequence $y_n - \overline{y}$ is strictly oscillatory.

Let J be an interval real numbers and $g : J \times J \to J$, where g is a continuously differentiable function. Consider the difference equation

$$y_{n+1} = g(y_n, y_{n-1}), \qquad n = 0, 1, 2, \ldots . \tag{3}$$

The linearized equation of Eq.(3) is

$$z_{n+1} = a_1 z_n + a_2 z_{n-1}.$$

Theorem A [[33]] A (linearized stability).

(a) If both roots of the quadratic equation

$$\lambda^2 - a_1\lambda - a_2 = 0, \tag{4}$$

lie in the open unit disk, $|\lambda| < 1$, then the equilibrium point $\overline{y}$ of Eq.(3) is locally asymptotically stable.

(b) If at least of the roots of Eq.(4) has absolute value greater that one , then the equilibrium $\overline{y}$ of Eq.(3) is unstable.

(c) A necessary and sufficient condition for both roots of Eq.(4) to lie in the open unit disk $|\lambda| < 1$, is
$$|a_1| < 1 - a_2 < 2.$$

Here the locally asymptotically stable equilibrium $\overline{y}$ is also called a sink.

(d) A necessary and sufficient condition for one root of Eq.(4) to have absolute value great than one and for the other to have absolute values less than one is

$$|a_1| > |1 - a_2| \text{ and } a_1^2 + 4a_2 > 1.$$

In this case $\overline{y}$ is called a saddle point.

Theorem B [[33]] Let $[c, d]$ be an interval of real numbers and assume that

$$f : [c, d] \times [c, d] \to [c, d]$$

is a continuous function satisfying the following properties:

(a) $g(x, y)$ is non-decreasing in $x \in [c, d]$ for each $y \in [c, d]$, and $g(x, y)$ is non-increasing in $y \in [c, d]$ for each $x \in [c, d]$.

(b) If $(m, M) \in [c, d] \times [c, d]$ is a solution of the system

$$f(m, M) = m, \quad \text{and} \quad f(M, m) = M \ ,$$

then $m = M$. Then $Eq.(3)$ has a unique equilibrium $\overline{y} \in [c, d]$ and every solution of $Eq.(3)$ converges to $\overline{y}$.

Theorem C [[34]] Assume that $a_1, a_2, ..., a_{k+1} \in R$. Then

$$\sum_{i=1}^{k+1} | a_i |< 1$$

is a sufficient condition for the locally stability of Eq.(1).

Consider the difference equation

$$Y_{n+1} = O(Y_n), \quad n = 0, 1, ... \ . \tag{5}$$

where $Y_n \in R^n$ and $O \in C^1[R^{k+1}, R^{k+1}]$. Then the linearized equation associated with Eq.(5) is given by $Y_{n+1} = AY_N$, where A is the Jacobian matrix $DH(\overline{Y})$ of the function H evaluated at the equilibrium $\overline{Y}$.

Theorem D [[34]] Let $\overline{Y}$ be an equilibrium point of Eq.(5) and assume that O is a C^1 function in R^{k+1}. Then the following statements are true:

(a) If all the eigenvalues of the Jacobian matrix $DH(\overline{Y})$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{Y}$ of Eq.(5) is asymptotically stable.

(b) If at least one eigenvalues of the Jacobian matrix $DH(\overline{Y})$ has absolute value greater that one, then the equilibrium $\overline{Y}$ of Eq.(5) is unstable.

Theorem E [[34]] Consider the difference equation

$$x_{n+1} = f(x_n, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{6}$$

where $f \in C[(0,\infty)^{k+1}, (0,\infty)]$ is increasing in each of its arguments, where the initial conditions $x_{-k}, ..., x_0$ are positive. Assume that Eq.(6) has a unique positive equilibrium $\overline{x}$, and suppose that the function h defined by

$$h(x) = f(x, x, ..., x), \ \ y \in (0, \infty),$$

satisfies

$$(h(x) - x)(x - \overline{x}) < 0, \text{ for } \ x \neq \overline{x}.$$

Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(6).

Theorem F [19] Let J be some interval of real numbers, $f \in C[J^{v+1}, J]$, and let $\{x_n\}_{n=-v}^{\infty}$ be a bounded solution of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \ \ n = 0, 1, ..., \tag{7}$$

with

$$I = \lim_{n\to\infty} \inf x_n, \ \ S = \lim_{n\to\infty} \sup x_n, \text{ with } I, S \in J.$$

Let Z denote the set of all integers $\{..., -1, 0, 1, ...\}$. Then there exist two solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \tag{8}$$

which satisfy the equation for all $n \in Z$, with

$$I_0 = I, \ S_0 = S, \ \text{ and } \ I_n, S_n \in [I, S], \ \text{ for all } n \in Z,$$

and such that for every $N \in Z$, I_N and S_N are limit points of $\{x_n\}_{n=-v}^{\infty}$. Therefore, for every $m \leq -v$ there exist two subsequences $\{x_{r_n}\}$ and $\{x_{l_n}\}$ of the solution $\{x_n\}_{n=-v}^{\infty}$ so the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \ \text{ and } \ \lim_{n\to\infty} x_{l_n+N} = S_N, \ N \geq m.$$

The solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(8) are called **Full limiting solutions** of Eq.(8) associated with the solution $\{x_n\}_{n=-v}^{\infty}$ of Eq.(7).

Chapter 1

Dynamical of a Non-Autonomous Difference Equation

1.1 Introduction

Our point in this chapter is to discuss the behavior of the positive solutions of the difference equations:

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \quad n \geq 0 \ . \tag{1.1}$$

where $\{a_n\}$ is a sequence of positive real numbers and the initial conditions x_{-1}, x_0, and p are arbitrary positive real numbers. In this survey we consider three cases of the sequence a_n.

This chapter is divided to two parts. **Part I** deals with the Eq.(1.1) when $p = 1$. **Part II** concerned with Eq.(1.1) when p is a positive real number.

Part I : Studying of Eq.(1.1) with P=1

Here our goal is to consider local stability, boundedness character, and the global asymptotic behavior of the positive solutions of the difference equation:

$$x_{n+1} = a_n + \frac{x_n}{x_{n-1}}, \quad n \geq 0, \tag{1.2}$$

where $\{a_n\}$ is a sequence of nonnegative real numbers and the initial condition x_{-1}, andx_0 are positive real numbers.

In the following we consider three cases of the sequence $\{a_n\}$.

1.2 Case 1. When $\lim_{n\to\infty} a_n = a$

1.2.1 Permanence of Eq.(1.2)

In this subsection we investigate the boundedness of Eq.(1.2).

Theorem 1.2.1 *Suppose that* $\lim_{n\to\infty} a_n = a \geq 1$, *at that point every positive solution of Eq.(1.2) is bounded and persists.*

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1.2). Then

$$x_n \geq a > 1, \; for\ all \;\; n \geq 1.$$

Let $\epsilon \in (0, a-1)$, we see from Eq.(1.2) that

$$x_n \geq a - \epsilon, \;\; \text{for all } n \geq -1.$$

Then we can find $L \in (a+\epsilon, a+\epsilon+1)$ such that

$$L - \epsilon \leq x_{-1}, x_0 \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Since $a > 1$, then we get

$$a \leq \frac{L-\epsilon-1}{L-\epsilon-a}.$$

Set

$$f(u,v) = a + \frac{u}{v}.$$

Then

$$f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = a + \frac{L-\epsilon}{\frac{L-\epsilon}{L-a-\epsilon}} = L - \epsilon,$$

and

$$f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) = a + \frac{\frac{L-a-\epsilon}{L-\epsilon}}{L-\epsilon} \leq a + \frac{1}{L-a-\epsilon} \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Now it follows from Eq.(1.2) that

$$x_1 = f(x_0, x_{-1}) \leq f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Again we see from Eq.(1.2) that

$$x_1 = f(x_0, x_{-1}) \geq f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = L-\epsilon.$$

By induction we obtain that

$$L-\epsilon \leq x_n \leq \frac{L-\epsilon}{L-a-\epsilon}, \quad \text{for all} \;\; n = -1, 0, 1, \ldots.$$

Second assume that $a = 1$ and let $\epsilon \in (0, \delta)$ and $\delta \in (0, 1)$, it follows from Eq.(1.2) that

$$x_n \geq 1 - \epsilon + \delta \;, \;\; for \;\; n \geq 1.$$

Then one can find $L \in (1+\epsilon+\delta, 2+\epsilon+\delta)$ such that

$$L-\epsilon+\delta \leq x_{-1}, x_0 \leq \frac{L-\epsilon+\delta}{L-\epsilon-1+\delta}.$$

In this way whatever is left of the proof is like the above and it is overlooked. ■

1.2.2 Global Attractity of Eq.(1.2)

Here, we show that if $a > 1$, Therefore every positive solution of Eq.(1.2) converges to $(a+1)$.

Theorem 1.2.2 *Assume that $a \geq 1$. At that point each positive solution of Eq.(1.2) converges to the unique positive equilibrium point $\overline{x} = (a+1)$ of Eq.(1.2).*

Proof. Note, when $a \geq 1$, it was shown in Theorem 1.2.1 that each positive solution of Eq.(1.2) is bounded. Then we have the following

$$s = \lim_{n\to\infty} \inf x_n, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_n.$$

It is clear that $s \leq S$. We want to proof that $s \geq S$. Now it is easy to see from Eq.(1.2) that

$$s \geq a + \frac{s}{S}, \;\; and \;\; S \leq a + \frac{S}{s}.$$

Thus we have

$$sS \geq aS + s, \quad \text{and} \quad sS \leq as + S.$$

This implies that

$$aS + s \leq as + S.$$

Then we get

$$a(S-s) \leq (S-s),$$

or

$$(a-1)(S-s) \leq 0 \Leftrightarrow s \geq S.$$

Thus the proof is complete. ■

Example 1.2.3 *Figure (1) shows the global attractivity of the equilibrium point* $\overline{x} = 2$ *of Eq.(1.2) whenever* $x_{-1} = 1.21, x_0 = 1.32,$ *and* $a = 1.$

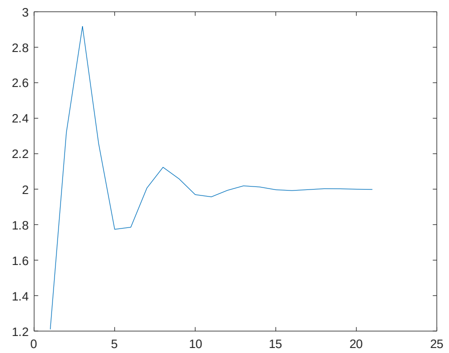


Figure (1)

Example 1.2.4 *Figure (2) shows the global attractivity of the equilibrium point* $\overline{x} = 6$ *of Eq.(1.2) whenever* $x_{-1} = 4,\ x_0 = 9,$ *and* $a = 5.$

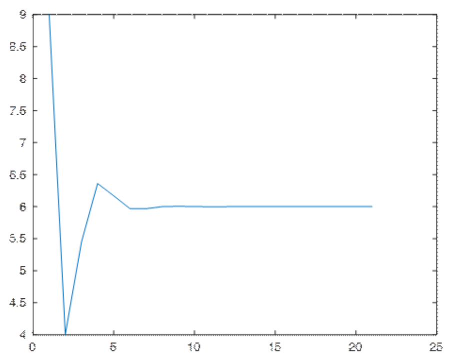


Figure (2)

1.3 Case 2. When a_n is periodic

In this subsection we research the periodicity character of the positive solutions of Eq.(1.2) whenever $\{a_n\}$ is a periodic sequence of period two of the form $\{\alpha, \beta, \alpha, \beta, ...\}, \alpha \neq \beta$. Assume that $a_{2n} = \alpha$, and $a_{2n+1} = \beta$. Then we have

$$x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}}, \tag{1.3}$$

and

$$x_{2n+2} = \beta + \frac{x_{2n+1}}{x_{2n}}, \quad n \geq 0. \tag{1.4}$$

1.3.1 Periodicity of the solutions

Here we investigate the periodic solutions of Eq.(1.2).

Theorem 1.3.1 *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then Eq.(1.2) has periodic solution of prime period two.*

Proof. Let $\{x_n\}$ be a solution of Eq.(1.2), with the initial values x_{-1}, and x_0 such that

$$x_{-1} = \frac{\alpha x_{-1} + x_0}{x_{-1}}, \quad and \quad x_0 = \frac{\beta x_0 + x_{-1}}{x_0}. \tag{1.5}$$

Let $x_{-1} = x$, and $x_0 = y$, then we obtain from (1.5)

$$x = \alpha + \frac{y}{x}, \quad \text{and} \quad y = \beta + \frac{x}{y}. \tag{1.6}$$

Now we want to prove that (1.6) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (1.6) we get

$$y = (x - \alpha)x. \tag{1.7}$$

From (1.7) and the second relation of (1.6) we obtain

$$x(x - \alpha) = \beta + \frac{x}{x(x - \alpha)},$$

or

$$x(x - \alpha)^2 - \beta(x - \alpha) - 1 = 0.$$

Now define the function

$$f(x) = x(x - \alpha)^2 - \beta(x - \alpha) - 1, \quad x > \alpha. \tag{1.8}$$

Then

$$\lim_{x \to \alpha^+} f(x) = -1, \quad and \quad \lim_{x \to \infty} f(x) = \infty.$$

Hence Eq.(1.8) has at least one solution $x > \alpha$. Then if $y = (x - \alpha)x$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. ■

Example 1.3.2 *Figure (3) shows that the solution of Eq.(1.2) is periodic solution of period two when* $x_{-1} = 1.34$, $x_0 = 3.210$*,* $\alpha = 1$*, and* $\beta = 0.1$.

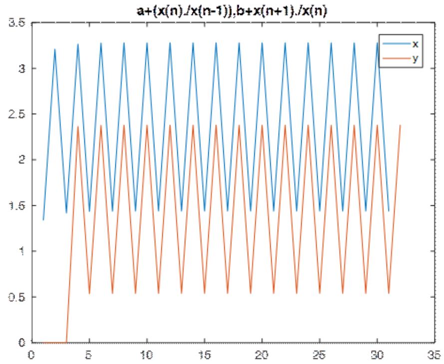


Figure (3)

1.3.2 Local Stability of the periodic solutions

Here we investigate the local stability character of Eq.(1.2).

Theorem 1.3.3 *Assume that* $\{x_n\}_{n=-1}^{\infty}$ *be a periodic solution of period two of Eq.(1.2) and consider Eq.(1.2) when the case* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$ *with* $\alpha \neq \beta$*. Suppose that*

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x}.$$

Then $\{x_n\}_{n=-1}^{\infty}$*is locally asymptotically stable.*

Proof. It was shown in Theorem 1.3.1 that there exist x, y such that

$$x = \alpha + \frac{y}{x}, \quad \text{and} \quad y = \beta + \frac{x}{y}. \tag{1.9}$$

Now Eq.(1.2) can be rewritten in the following form by splitting the even-indexed and odd-indexed terms:

$$\begin{aligned} u_{n+1} &= \alpha + \tfrac{v_n}{u_n}, \\ v_{n+1} &= \beta + \tfrac{\alpha u_n + v_n}{u_n v_n}. \end{aligned} \tag{1.10}$$

Now, we consider the map T on $[0, \infty) \times [0, \infty)$ such that

$$T(u, v) = \begin{bmatrix} T_1(u, v) \\ T_2(u, v) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{v}{u} \\ \beta + \frac{\alpha u + v}{uv} \end{bmatrix}.$$

Then we have

$$\frac{\partial T_1}{\partial u} = \frac{-v}{u^2}, \quad \text{and} \quad \frac{\partial T_1}{\partial v} = \frac{1}{u},$$

$$\frac{\partial T_2}{\partial u} = \frac{-v^2}{v^2u^2}, \quad \text{and} \quad \frac{\partial T_2}{\partial v} = \frac{-\alpha u^2}{u^2v^2},$$

Therefore the Jacobian matrix of T at (x, y) is

$$J_T(x,y) = \begin{bmatrix} \frac{-y}{x^2} & \frac{1}{x^2} \\ \frac{-1}{x^2} & \frac{-\alpha}{y^2} \end{bmatrix},$$

and its characteristic equation associated with (x, y) is

$$\lambda^2 + \lambda\left(\frac{\alpha}{y^2} + \frac{y}{x^2}\right) + \frac{\alpha}{x^2y} + \frac{1}{x^3} = 0. \tag{1.11}$$

It follows from (1.9) that $\frac{y}{x^2} = 1 - \frac{\alpha}{x}$ and since $x > \alpha$, and $y > \beta$ we have

$$\frac{\alpha}{y^2} + \frac{y}{x^2} + \frac{\alpha}{x^2y} + \frac{1}{x^3} < \frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} + 1 - \frac{\alpha}{x} < 1.$$

Thus

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x} < 1.$$

Then all roots of Eq.(1.11) have modulus less than 1. Therefore by Theorem D that System (1.10) is asymptotically stable. The proof is complete. ■

Theorem 1.3.4 *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$.*Then every solution of Eq.(1.2) converges to a period two solution of Eq.(1.2).*

Proof. We know by Theorem 1.2.1 that every positive solution of Eq.(1.2) is bounded, it follows that there are some positive constants l, L, s, and S such that

$$l = \lim_{n\to\infty} \inf x_{2n+1}, \quad \text{and} \quad L = \lim_{n\to\infty} \sup x_{2n+1},$$

$$s = \lim_{n\to\infty} \inf x_{2n}, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_{2n}.$$

Then it is easy to see from Eq.(1.3) and Eq.(1.4) that

$$l \geq \alpha + \frac{s}{L}, \quad \text{and} \quad L \leq \alpha + \frac{S}{l},$$

and

$$s \geq \beta + \frac{l}{S}, \quad \text{and} \quad S \leq \beta + \frac{L}{s}.$$

Then we obtain

$$Ll \geq \alpha L + s, \quad \text{and} \quad Ll \leq \alpha l + S,$$

and

$$Ss \geq \beta S + l, \quad \text{and} \quad Ss \leq \beta s + L.$$

Thus we get

$$\alpha L + s \leq Ll \leq \alpha l + S, \quad \text{and} \quad \beta S + l \ \leq Ss \leq \beta s + L.$$

Thus we have

$$\alpha(L - l) \leq S - s, \quad \text{and} \quad \beta(S - s) \leq L - l. \tag{1.12}$$

Thus it is clear from (1.12) that $s = S$ and $l = L$. Now suppose $\lim_{n\to\infty} x_{2n+1} = S$, and $\lim_{n\to\infty} x_{2n} = L$. We want to proof that $S \neq L$. From Eq.(1.3) and Eq.(1.4) we get

$$S = \alpha + \frac{L}{S}, \quad \text{and} \quad L = \beta + \frac{S}{L}.$$

As that sake of contradiction assume that $L = S$, then

$$L = \alpha + 1, \quad \text{and} \quad S = \beta + 1$$

thus $\alpha = \beta$ which is a contradiction. So $\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}$. The proof is so complete. ■

Example 1.3.5 *Figure (3) shows that the solution of Eq.(1.2) is periodic solution of period two when* $x_{-1} = 2.3$, $x_0 = 1.3$, $\alpha = 0.73827543$, *and* $\beta = 0.6763772$.

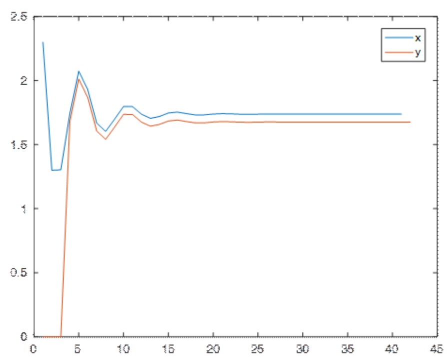


Figure (3)

Example 1.3.6 *Figure (4) shows that the solution of Eq.(1.2) is periodic solution of period two when* $x_{-1} = 15.30$, $x_0 = 10.30$, $\alpha = 6$, *and* $\beta = 1$.

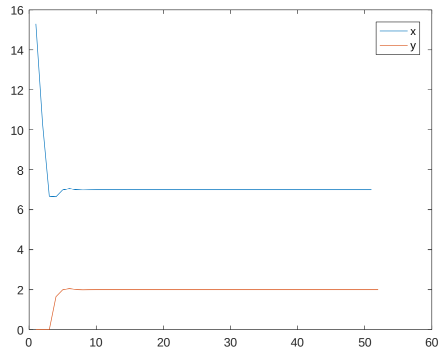


Figure (4)

1.4 Case 3. The autonomous case of Eq.(1.2)

Consider Eq.(1.2) with $a_n = a$, where $a \in (0, \infty)$ then Eq.(1.2) has the form

$$x_{n+1} = a + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ..., \tag{1.13}$$

where the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Clearly, the only equilibrium point of Eq.(1.13) is $\overline{x} = a + 1$.

The linearized equation of Eq.(1.13) about the equilibrium point $\overline{x} = a + 1$ is

$$y_{n+1} - \frac{1}{a+1} y_n + \frac{1}{a+1} y_{n-1} = 0.$$

1.4.1 Local Stability

In this subsection we deal the local stability of Eq.(1.13).

Lemma 1.4.1 *The following statements are true.*

1. *The equilibrium point* $\overline{x} = a+1$ *of Eq.(1.13) is locally asymptotically stable if* $a > 1$.

2. *The equilibrium point* $\overline{x} = a + 1$ *of Eq.(1.13) is unstable if* $0 \leq a \leq 1$.

Proof. The proof is followed directly by Theorem A and so will be omitted. ■

1.4.2 Boundedness

Here, we investigate the bounded character of Eq.(1.13).

Theorem 1.4.2 *Suppose that $a > 1$, then every positive solution of Eq.(1.13) is bounded.*

Proof. It follows from Eq.(1.13) that

$$x_{2n+1} = a + \tfrac{x_{2n}}{x_{2n-1}},$$
$$x_{2n} = a + \tfrac{x_{2n-1}}{x_{2n-2}}.$$

Therefore

$$x_{2n-1} > a, \quad \text{and} \quad x_{2n-2} > a, \quad \text{for every } n \geq 1.$$

Then

$$x_{2n+1} = a + \frac{x_{2n}}{x_{2n-1}} < a + \frac{x_{2n}}{a}, \quad \text{and} \quad x_{2n} = a + \frac{x_{2n-1}}{x_{2n-2}} < a + \frac{x_{2n-1}}{a}$$

Then it follows by induction that

$$x_{2n+1} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_{-1}}{a^n} = a + \frac{a}{a-1} + \frac{x_{-1}}{a^n},$$

and

$$x_{2n} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_0}{a^n} = a + \frac{a}{a-1} + \frac{x_0}{a^n}.$$

The result now follows. ∎

Theorem 1.4.3 *Assume that $a > 1$ then every solution of Eq.(1.13) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1.13), then

$$x_{n+1} = a + \frac{x_n}{x_{n-1}} > a, \quad \text{for all } n \geq 1. \tag{1.14}$$

Again it follows from Eq.(1.13) that

$$x_{n+1} = a + \frac{x_n}{x_{n-1}} \leq a + \frac{x_n}{a}.$$

Then

$$\limsup x_n \leq \frac{a}{1 - \frac{1}{a}} = \frac{a^2}{a-1}. \tag{1.15}$$

Then the result follows from (1.14) and (1.15). ∎

1.4.3 Global attractor

In the following Theorem, we establish sufficient conditions for global attractor of Eq.(1.13).

Theorem 1.4.4 *Assume that $a > 1$.Then the equilibrium point $\overline{x} = a + 1$ is a global attractor of Eq.(1.13).*

Proof. Let $f : [c, d]^2 \to [c, d]$ be a function defined by $f(u, v) = a + \frac{u}{v}$. Assume that (m, M) is a solution of the system

$$m = f(m, M), \quad \text{and} \quad M = f(M, m).$$

Then we get

$$(a - 1)(M - m) = 0,$$

Since $a > 1$, then we obtain

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1.13) and then the proof is complete. ■

Remark 1.4.5 *In case 3 this case has been treated by many others such as [Amleh]. Here we give an alternative proofs of our results.*

Part II : Studying of Eq.(1.1)

In this part we investigate the behavior of the positive solutions of the difference equation

$$x_{n+1} = a_n + \frac{x_n^p}{x_{n-1}^p}, \quad \text{for } n \geq 0,$$

where p is a positive real number, a_n is a positive sequence and the initial conditions x_{-1}, x_0 are positive real numbers.

1.5 Case 1. When $a_n = a \in R^+$

In this case Eq.(1.1) takes the form

$$x_{n+1} = a + \frac{x_n^p}{x_{n-1}^p}, \quad n = 0, 1, 2... . \tag{1.16}$$

1.5.1 Local Stability of the Equilibrium Points

At the present we discuss the local stability character of the solutions of Eq.(1.16).

It is easy to see that the only positive equilibrium point of Eq.(1.16) is given by $\overline{x} = a + 1$. Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(x, y) = a + \frac{x^p}{y^p}.$$

Therefore

$$\frac{\partial f(x,y)}{\partial x} = \frac{px^{p-1}}{y^p}, \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = -\frac{px^p}{y^{p+1}}.$$

We see that

$$\frac{\partial f(\overline{x},\overline{x})}{\partial x} = \frac{p}{a+1} = p_1, \quad \text{and} \quad \frac{\partial f(\overline{x},\overline{x})}{\partial y} = -\frac{p}{a+1} = p_2\ .$$

Then the linearized equation of Eq.(1.16) about $\overline{x}$ is

$$y_{n+1} - \frac{p}{a+1} y_n + \frac{p}{a+1} y_{n-1} = 0.$$

Theorem 1.5.1 *The following statements are valid:*

(i) if $p < a+1$, furthermore the positive equilibrium point $\overline{x}$ of Eq.(1.16) is locally asymptotically stable, and is called a sink.

(ii) If $p > a+1$, then the positive equilibrium point $\overline{x}$ of Eq.(1.16) is unstable, and is called a repeller.

(iii) If $p = a+1$, then the positive equilibrium point $\overline{x}$ of Eq.(1.16) is unstable, and is called a nonhyperbolic point.

Proof. (i) We set $p_1 = \frac{p}{\overline{x}}$, and $p_2 = -\frac{p}{\overline{x}}$. So by Theorem A (a)

$$|\ p_1\ | -1 + p_2 < 0 \Leftrightarrow \frac{p}{a+1} - \frac{p}{a+1} - 1 < 0 \Longleftrightarrow -1 < 0.$$

Also

$$1 + p_2 - 2 < 0 \Longleftrightarrow -1 + \frac{p}{a+1} < 0 \Leftrightarrow \frac{p}{a+1} < 1.$$

which is valid iff

$$p < a + 1.$$

So $\overline{x}$ is locally asymptotically stable when $p < a+1$.

(ii) By Theorem A (d) we have

$$|\ p_2\ | -1 = \frac{p}{a+1} - 1 > 0 \Longleftrightarrow \frac{p}{a+1} > 1,$$

and
$$| p_1 | - | 1 - p_2 |= \frac{p}{a+1} - 1 - \frac{p}{a+1} = -1.$$
Thus $\overline{x}$ is unstable (repeller point) when $p > a+1$.

(iii) By Theorem A (e) we have
$$p_2 = -1 \Leftrightarrow -\frac{p}{1+a} = -1 \Leftrightarrow -p = -(a+1) \Leftrightarrow p = a+1,$$
and
$$| p_1 | -2 \leq 0 \Leftrightarrow \frac{p}{a+1} - 2 \leq 0 \Leftrightarrow p \leq 2(a+1).$$
Thus $\overline{x}$ is unstable (repeller point) when $p > a+1$. ■

1.5.2 Boundedness of Solutions of Eq.(1.16)

In this subsection we discuss the suffiction conditions for bounded solution of Eq.(1.16).

Theorem 1.5.2 *If* $0 < p < 1$*, consequently every positive solution of Eq.(1.16) is bounded and persists.*

Proof. We obtain from Eq.(1.16) that
$$x_{n+1} > a, \quad n \geq 0.$$
Hence $\{x_n\}$ persists. It follows again of Eq.(1.16) that
$$x_{2n+1} \leq a + (\frac{x_{2n}}{a})^p, \quad n = 0, 1, \ldots .$$

Now we suppose the difference equation
$$y_{n+1} = a + (\frac{y_n}{a})^p, \quad n \geq 0. \tag{1.17}$$

Let $\{y_n\}$ be a solution of Eq.(1.17) with $y_0 = x_0$. Thus, cleary
$$x_{2n+1} \leq y_{n+1} \quad (resp \quad x_{2n+2} \leq y_{n+1}), \quad n = 0, 1, \ldots .$$
We will establish that the sequence $\{y_n\}$ is bounded. Let
$$f(x) = a + \frac{x^p}{a^p}.$$
Then
$$f^{'}(x) = \frac{1}{a^p} p x^{p-1} > 0, \text{ and } f^{''}(x) = \frac{1}{a^p} p(p-1) x^{p-2} < 0.$$

Therefore the function f is increasing and concave . Thus we obtain that there is a unique fixed point y^* of the equation $f(y) = y$. Likewise the function f satisfies

$$(f(y) - y)(y - y^*) < 0, \quad y \in (0, \infty).$$

By Theorem E y^* is a global attractor of all positive solutions of Eq.(1.17) and so $\{y_n\}$ is bounded. Then from Eq.(1.17) the sequence $\{x_n\}$ is so bounded. This finishes the proof of the theorem. ■

Example 1.5.3 *Figure (5) shows the bounded solutions of the equilibrium point* $\overline{x} = 24$ *of Eq.(1.16) whenever* $x_{-1} = 1.0323, x_0 = 2.441, a = 23,$ *and* $p = 0.000000002.$

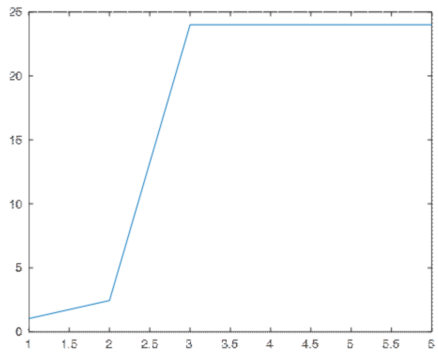


Figure (5)

Theorem 1.5.4 *Assume that* $p \geq 4$. *Then Eq.(1.16) has unbounded solutions.*

Proof. Note that for every solution $\{x_n\}_{n=-1}^{\infty}$of Eq.(1.16) the following inequality holds:

$$x_{n+1} > \frac{x_n^p}{x_{n-1}^p}, \text{ for } n \in N. \tag{1.18}$$

Let $y_n = \ln x_n$. It follows from (1.18) that

$$y_{n+1} - py_n + py_{n-1} > 0. \tag{1.19}$$

Note that the roots of the polynomial

$$p(\lambda) = \lambda^2 - p\lambda + p,$$

are given by

$$\lambda_1, \lambda_2 = \frac{p \pm \sqrt{p^2 - 4p}}{2},$$

Since $p \geq 4$ we have that $\lambda_1 > 1$. On the other hand we have

$$\lambda_2 = \frac{2p}{p + \sqrt{p^2 - 4p}}.$$

Hence if $p \geq 4$, both roots of $p(\lambda)$ are positive. Note that (1.19) can be rewritten in the form

$$y_{n+1} - \lambda_1 y_n - \lambda_2(y_n - \lambda_1 y_{n-1}) > 0.$$

Then we see that

$$\frac{x_{n+1}}{x_n^{\lambda_1}} > (\frac{x_n}{x_{n-1}^{\lambda_1}})^{\lambda_2}. \tag{1.20}$$

It follows that

$$\frac{x_n}{x_{n-1}^{\lambda_1}} > (\frac{x_{n-1}}{x_{n-2}^{\lambda_1}})^{\lambda_2} > ... > (\frac{x_1}{x_0^{\lambda_1}})^{\lambda_2} > (\frac{x_0}{x_{-1}^{\lambda_1}})^{\lambda_2}.$$

Select x_{-1} and x_0 so that

$$x_0 > 1, \quad x_0 = x_{-1}^{\lambda_1}.$$

Then it follows by (1.20) that

$$x_n > (\frac{x_0}{x_{-1}^{\lambda_1}})^{\lambda_2} x_{n-1}^{\lambda_1} = x_{n-1}^{\lambda_1} > ... > x_0^{\lambda_1^n},$$

and consequently $x_n > x_0^{\lambda_1^n}$, $n \in N$. Letting $n \to \infty$, then $x_n \to \infty$. From which the outcome takes after. ■

1.5.3 Global Stability of Eq.(1.16)

Here we study the characteristic task of global stability of Eq.(1.16).

Theorem 1.5.5 *Suppose that $a \geq 1$ and $0 < p < 1$. Then the unique positive equilibrium point of Eq.(1.16) is globally asymptotically stable.*

Proof. By Theorem 1.5.1 (i) $\bar{x}$ is locally asymptotically stable. Thus it is suffices prove that every positive solution of Eq.(1.16) tends to the unique positive equilibrium $\bar{x}$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(1.16). By Theorem 1.5.2 $\{x_n\}_{n=-1}^{\infty}$ is bounded. Thus we have

$$a \leq s = \liminf x_n, \quad \text{and} \quad S = \limsup x_n < \infty .$$

Then we get from (1.16)

$$S \leq a + \frac{S^p}{s^p}, \quad \text{and} \quad s \geq a + \frac{s^p}{S^p}. \tag{1.21}$$

We claim that $S = s$, otherwise $S > s$. We obtain from (1.21)

$$s^p S \leq s^p a + S^p, \quad \text{and} \quad s S^p \geq S^p a + s^p. \tag{1.22}$$

Thus we have

$$s^{1-p} < S^{1-p},$$

or equivalently

$$sS^p < Ss^p. \tag{1.23}$$

It follows from Eq.(1.22) and Eq.(1.23) that

$$S^p a + s^p \leq s^p a + S^p.$$

Hence

$$S^p(a-1) \leq s^p(a-1).$$

which is impossible for $a \geq 1$. Hence the result follows. ■

Example 1.5.6 *Figure (6) shows the global attractivity of the equilibrium point* $\overline{x} = 1.2000$ *of Eq.(1.16) whenever* $x_{-1} = 1.03$, $x_0 = 2.441$, $a = 1.1$, *and* $p = 0.9$.

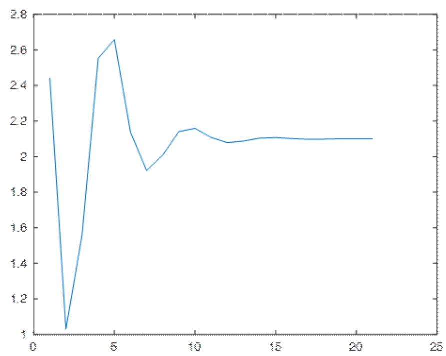


Figure (6)

1.5.4 Oscillatory Solutions of Eq.(1.16)

Here we present the characteristic task of oscillatory solution of Eq.(1.16).

Theorem 1.5.7 *Assume that* $0 < p \leq 1$, *then every positive solution of Eq.(1.16) oscillates about the equilibrium point* $\overline{x} = a + 1$ *with semicycles of length two or three and the extreme of every semicycle occurs at the first or the second term.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1.16). First, we present every positive semicycle except possibly the first term has two or three terms. Assume that $x_{N-1} < \overline{x}$, and $x_N \geq \overline{x}$, for some $N \in \mathbb{N}$. We obtain

$$x_{N+1} = a + \frac{x_N^p}{x_{N-1}^p} > a + 1 = \overline{x}.$$

If $x_{N+1} > x_N$, so we get

$$x_{N+2} = a + \frac{x_{N+1}^p}{x_N^p} > a + 1 = \overline{x}.$$

Since $p \in (0, 1]$, we include that

$$x_{N+2} = a + \frac{x_{N+1}^p}{x_N^p} \leq a + \frac{x_{N+1}^p}{\overline{x}^p} \leq a + \frac{x_{N+1}^p}{a+1} \leq x_{N+1}.$$

So $\overline{x} < x_{N+2} < x_{N+1.}$ Therefore

$$x_{N+3} = a + \frac{x_{N+2}^p}{x_{N+1}^p} < a + 1 = \overline{x}.$$

Then the proof is completed. ■

Theorem 1.5.8 *Eq.(1.16) has no periodic solutions of prime period two.*

Proof. As the sake of contradiction. Assume that $..., x, y, x, y, ...$ be a periodic solution of period two of Eq.(1.16). It press that

$$x = a + (\frac{y}{x})^p, \quad \text{and} \quad y = a + (\frac{x}{y})^p, \tag{1.24}$$

which suggest that

$$y = a + \frac{1}{x - a}. \tag{1.25}$$

Substituting from (1.24) into (1.25) and after some calculation we get

$$(x - a)^{p+1} x^p = (a(x - a) + 1)^p. \tag{1.26}$$

Taking the logarithm on both sides of (1.26), we acquire

$$f(x) = (p + 1)\ln(x - a) + p \ln x - p \ln[a(x - a) + 1] = 0. \tag{1.27}$$

It is obvious that $x = a + 1$ is an obvious solution of (1.27). Presently we examine that this is the unique solution of the equation (1.27). Now

$$\acute{f}(x) = \frac{(x - a)(ax + p(a(x - a) + 1)) + (p + 1)x}{x(x - a)(a(x - a) + 1)}.$$

Thus $\acute{f}(x) > 0$, for $x \in (a, \infty)$, which implies that the f is strictly increasing on the interval (a, ∞). Hence, the equilibrium point $\overline{x} = a + 1$ is the unique solution of (1.27). From Eq.(1.26) we obtain $y = a + 1$ and consequently. This means $(a + 1, a + 1)$ is the unique solution of System (1.24). Finishing the proof of the theorem. ■

1.6 Case 2. When a_n be a periodic sequence of period two

In this section we study the behavior of solution of Eq.(1.1) while a_n is a periodic sequence of period two with $\alpha, \beta \in (0, \infty)$ and $\alpha \neq \beta$. Consider $a_{2n} = \alpha$, and $a_{2n+1} = \beta$. Then we have

$$\begin{aligned} x_{2n+1} &= \alpha + \frac{x_{2n}^p}{x_{2n-1}^p}, \\ x_{2n+2} &= \beta + \frac{x_{2n+1}^p}{x_{2n}^p}. \end{aligned} \tag{1.28}$$

Now Eq.(1.1) can be rewritten in the following form:

$$\begin{aligned} u_{n+1} &= \alpha + \frac{u_n^p}{v_n^p}, \\ v_{n+1} &= \beta + \frac{v_n^p}{u_n^p}. \end{aligned} \tag{1.29}$$

1.6.1 Locally stability

Here we discuss the local stability of System (1.29). It is easy to see that $(\overline{u}, \overline{v}) = (\alpha + 1, \beta + 1)$ is the unique positive equilibrium point of System (1.29).

Theorem 1.6.1 *If* $p < \frac{(\beta+1)(\alpha+1)}{(\alpha+1)^p(\beta+1)^p}$, *then the positive equilibrium point* $(\overline{u}, \overline{v}) = (\alpha + 1, \beta + 1)$ *of System (1.29) is locally asymptotically stable.*

Proof. We consider the map T on $[0, \infty) \times [0, \infty)$ such that

$$T(u, v) = \begin{bmatrix} T_1(u, v) \\ T_2(u, v) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{u^p}{v^p} \\ \beta + \frac{v^p}{u^p} \end{bmatrix}.$$

Then we have

$$\frac{\partial T_1(u, v)}{\partial u} = -\frac{pu^{p-1}v^p}{(u^p)^2}, \quad \text{and} \quad \frac{\partial T_1(u, v)}{\partial v} = \frac{pv^{p-1}}{u^p},$$

and

$$\frac{\partial T_2(u, v)}{\partial u} = \frac{pu^{p-1}}{v^p}, \quad \text{and} \quad \frac{\partial T_2(u, v)}{\partial v} = -\frac{pv^{p-1}u^p}{(v^p)^2}.$$

Therefore the Jacobian matrix of T at $(\overline{u}, \overline{v}) = (\alpha + 1, \beta + 1)$ is

$$J(E_{\alpha,\beta}) = \begin{bmatrix} -\frac{pu^{p-1}v^p}{(u^p)^2} & \frac{pv^{p-1}}{u^p} \\ \frac{pu^{p-1}}{v^p} & -\frac{pv^{p-1}u^p}{(v^p)^2} \end{bmatrix},$$

and the characteristic equation associated with $(\overline{u}, \overline{v})$ is

$$p(\lambda) = \lambda^2 - \lambda p(\frac{(\beta+1)^{p-1}}{(\alpha+1)} + \frac{(\alpha+1)^{p-1}}{(\beta+1)}).$$

Then we obtain

$$\lambda_1 = 0, \quad \lambda_2 = p(\frac{(\beta+1)^{p-1}}{(\alpha+1)} + \frac{(\alpha+1)^{p-1}}{(\beta+1)}).$$

It follows by Theorem D that the equilibrium point $(\overline{u}, \overline{v}) = (\alpha+1, \beta+1)$ of System (1.29) is locally asymptotically stable if $p < \frac{(\beta+1)(\alpha+1)}{(\alpha+1)^p + (\beta+1)^p}$. Then the proof is completed. ■

Example 1.6.2 *Figure (7) shows the local stability of the equilibrium point*

$(\overline{u}, \overline{v}) = (21.6073, 0.0780)$ *of System (1.29) whenever* $u_0 = 2.43, v_0 = 0.4562$, $\alpha = 0.76$, $\beta = 0.03$, *and* $p = 0.54$.

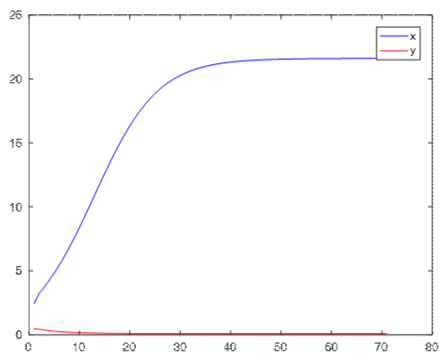


Figure (7)

1.6.2 Periodicity of Eq.(1.1)

In this subsection we investigate the excitons of periodic solutions of Eq.(1.1).

Theorem 1.6.3 *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then Eq.(1.1) has periodic solution of prime period two.*

Proof. To prove that Eq.(1.1) possess a periodic solution $\{x_n\}$ of prime period two, we must find positive numbers x_{-1}, x_0 such that

$$x_{-1} = \frac{\alpha x_{-1}^p + x_0^p}{x_{-1}^p}, \quad \textit{and} \quad x_0 = \frac{\beta x_0^p + x_{-1}^p}{x_0^p}. \tag{1.30}$$

Let $x_{-1} = x$, and $x_0 = y$, then we obtain from (1.30)

$$x = \alpha + \frac{y^p}{x^p}, \quad \text{and} \quad y = \beta + \frac{x^p}{y^p}. \tag{1.31}$$

Now we want to prove that (1.31) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (1.31) we have

$$y = (x - \alpha)^{\frac{1}{p}} x. \tag{1.32}$$

From (1.32) and the second relation of (1.31) we get

$$x(x-\alpha)^{\frac{1}{p}} = \beta + \frac{x^p}{x^p(x-\alpha)},$$

or

$$x(x-\alpha)^{\frac{p+1}{p}} - \beta(x-\alpha) - 1 = 0.$$

Now define the function

$$f(x) = x(x-\alpha)^{\frac{p+1}{p}} - \beta(x-\alpha) - 1, \quad x > \alpha. \tag{1.33}$$

Then

$$\lim_{x\to\alpha^+} f(x) = -1, \ \ and \ \lim_{x\to\infty} f(x) = \infty.$$

Hence Eq.(1.33) has at least one solution $x > \alpha$. Then if $y = (x-\alpha)^{\frac{1}{p}} x$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. ∎

1.7 Case 3. When a_n is a positive bounded sequence

In this section we assume that a_n is a positive bounded sequence

$$\lim_{n\to\infty} \inf a_n = a \geq 0, \quad \text{and} \quad \lim_{n\to\infty} \sup a_n = b < \infty. \tag{1.34}$$

1.7.1 Boundedness

The primary theorem indicate to the boundedness and the persistence of the positive solutions of Eq.(1.1).

Theorem 1.7.1 *Assume* $0 < p < 1$. *Therefore every positive solution of Eq.(1.1) is bounded and persists.*

Proof. The proof is similar to the proof of Theorem 1.5.2 and will be omitted. ∎

Lemma 1.7.2 *Assume that* $0 < p \leq 1$. *Let* $\lim_{n\to\infty} \inf a_n = a \geq 0$, *and* $\lim_{n\to\infty} \sup a_n = b < \infty$ *and* $\{x_n\}$ *be a positive solution of Eq.(1.1). Then*

$$\frac{ab-1}{b-1} \leq \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n \leq \frac{ab-1}{a-1}.$$

Proof. Assume

$$\lim_{n\to\infty} \inf x_n = \lambda, \quad \text{and} \quad \lim_{n\to\infty} \sup x_n = \mu. \tag{1.35}$$

Let $\epsilon > 0$ for $n \geq N_0(\epsilon)$ we get

$$\lambda - \epsilon \leq x_n \leq \mu + \epsilon, \quad \text{and} \quad a - \epsilon \leq a_n \leq b + \epsilon.$$

Therefore

$$x_{n+1} \geq a - \epsilon + (\frac{\lambda - \epsilon}{\eta + \epsilon})^p. \tag{1.36}$$

Taking the $\lim_{n\to\infty} \inf$ for Eq.(1.36). We obtain

$$\lambda \geq a - \epsilon + (\frac{\lambda - \epsilon}{\eta + \epsilon})^p.$$

Since $\epsilon > 0$ is arbitrary,

$$\lambda \geq a + (\frac{\lambda}{\eta})^p. \tag{1.37}$$

Similarly

$$\eta \leq b + (\frac{\eta}{\lambda})^p. \tag{1.38}$$

We get from equations (1.37) and (1.38) that

$$\lambda\eta^p \geq a\eta^p + \lambda^p, \quad \text{and} \quad \eta\lambda^p \leq b\lambda^p + \eta^p. \tag{1.39}$$

Since $0 < p < 1$ holds. Then we have

$$\lambda^{1-p} \leq \eta^{1-p},$$

or equivalently

$$\lambda\eta^p \leq \eta\lambda^p. \tag{1.40}$$

It follows from equations (1.39) and (1.40) that

$$a\eta^p + \lambda^p \leq b\lambda^p + \eta^p.$$

So

$$\eta^p(a-1) \leq \lambda^p(b-1),$$

and we have

$$(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}, \quad \text{and} \quad (\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}.$$

We obtain from Eq.(1.37) for all $n > N_0(\epsilon)$ that

$$\lambda \geq a + (\frac{\lambda}{\eta})^p \geq a + \frac{a-1}{b-1} = \frac{ab-1}{b-1}.$$

Similarly from Eq.(1.38) we get

$$\eta \leq \frac{ab-1}{a-1}.$$

Thus the proof is completed. ■

Now define the sequence $\{y_n\}$ to be

$$y_n = \frac{x_n}{\overline{x}_n},\ n = -1, 0, 1, ...,$$

where $\overline{x}_n$ be a fixed solution of Eq.(1.1). Then Eq.(1.1) will be rewritten as

$$y_{n+1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p(\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}. \tag{1.41}$$

Lemma 1.7.3 *Let $\{\overline{x}_n\}$ be a fixed positive solution of Eq.(1.41). Then the following statements are true.*

(i) Eq.(1.41) has a positive equilibrium solution $\overline{y} = 1$.

(ii) Let $\{y_n\}$ be a solution of Eq.(1.41). Then except possibly for the first semicycle, every solution of Eq.(1.41) has semicycle of length one.

Proof. (i) trivial.

(ii) Assume that for some n, $y_{n-1} \geq y_n$. Then $(\frac{y_n}{y_{n-1}}) < 1$ and

$$y_{n+1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p(\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} < \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} = 1. \tag{1.42}$$

Let $\{y_n\}$ be an finally oscillatory solution of Eq.(1.41) such as $y_{n-1} < 1$ and $y_n \geq 1$. From part (1.42) it follows that $y_{n+1} < 1$. Therefore the positive semicycle has exactly one term. The proof for negative semicycle is similar. ■

Lemma 1.7.4 *Let $\{y_n\}$ be a fixed positive solution of Eq.(1.41). Suppose that there exists an $m \in \{1, 2, ...\}$ such that*

$$y_{2m-1} < 1, \quad \textit{and} \quad y_{2m} \geq 1. \tag{1.43}$$

Then

$$y_{2n-1} < 1, \quad \textit{and} \quad y_{2n} \geq 1, \textit{ for } n = m, m+1, \ldots . \tag{1.44}$$

Moreover, if

$$y_{2m-1} \geq 1, \quad \textit{and} \quad y_{2m} < 1. \tag{1.45}$$

Then

$$y_{2n-1} \geq 1, \quad \textit{and} \quad y_n < 1, \textit{ for } n = m, m+1, \ldots . \tag{1.46}$$

Proof. Let $\{y_n\}$ be a solution of Eq.(1.41) such that Eq.(1.43) holds for an $m \in \{1, 2, ...\}$. we have

$$y_{2m-1} = \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p (\frac{y_n}{y_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} \geq \frac{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p}{a_n + (\frac{\overline{x}_n}{\overline{x}_{n-1}})^p} = 1.$$

Working inductively we can easily prove that Eq.(1.44) is satisfied. Similarly we can prove that if Eq.(1.45) holds for an $m \in \{1, 2, ...\}$, then Eq.(1.46) is satisfied. This completes the proof of the lemma. ■

1.7.2 Global attractor of the solutions

Here we investigate the global stability of Eq.(1.1).

Theorem 1.7.5 *Let $\{\overline{x}_n\}$ be a fixed solution of Eq.(1.1). Suppose that one of the following holds:*

(i) $0 < p \leq \frac{1}{2}$.

(ii) $\frac{1}{2} < p < 1$, $a > 1$, *and* $a(a-1) > b - 1$.*Then for every solution* $\{x_n\}$ *of Eq.(1.1) the relation* $\lim\limits_{n\to\infty} \frac{x_n}{\overline{x}_n} = 1$ *is true.*

Proof. (i) Let $\{y_n\}$ be a solution of Eq.(1.41). It is sufficient to prove that

$$\lim_{n\to\infty} y_n = 1$$

Suppose that there exists an $m \in \{1, 2, ...\}$ such that (1.43) or (1.45) . Without loss of generality we may assume that (1.43) holds for an $m \in \{1, 2, ...\}$ and $0 < p \leq \frac{1}{2}$ is satisfied.

Let

$$\mu = \liminf_{n\to\infty} y_n, \quad \text{and} \quad \zeta = \limsup_{n\to\infty} y_n. \tag{1.47}$$

also

$$\tau = \liminf_{n\to\infty} \overline{x}_n, \quad \text{and} \quad \omega = \limsup_{n\to\infty} \overline{x}_n, \tag{1.48}$$

and

$$\delta = \frac{\omega}{\tau}. \tag{1.49}$$

Define rhe function F by

$$F(x, y, z) = \frac{x + y^p z^p}{x + y^p} \tag{1.50}$$

for $x, y.z > 0$. Then we have

$$\frac{\partial F}{\partial x} = \frac{y^p(1 - z^p)}{(x + y^p)^2}, \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{pxy^{p-1}(z^p - 1)}{(x + y^p)^2}$$

Let $n \geq m$. Using Eq.(1.41) we have

$$\begin{aligned} y_{2n+1} &= F(a_{2n}, \tfrac{\overline{x}_{2n}}{\overline{x}_{2n-1}}, \tfrac{y_{2n}}{y_{2n-1}}), \\ y_{2n+2} &= F(a_{2n+1}, \tfrac{\overline{x}_{2n+1}}{\overline{x}_{2n}}, \tfrac{y_{2n+1}}{y_{2n}}). \end{aligned} \tag{1.51}$$

Since (1.43) holds by Lemma 3 we obtain the following:

$$\frac{y_{2n-1}}{y_{2n}} < 1, \quad \text{and} \quad \frac{y_{2n}}{y_{2n-1}} \geq 1, \quad \text{for } n \geq m.$$

using Eq.(1.34), (1.47)-(1.51) and monotonic properties of F we have

$$\theta \leq F(a, \delta, \frac{\zeta}{\mu}) = \frac{a + (\frac{\zeta}{\mu})^p \delta^p}{a + \delta^p}, \quad \text{and} \quad \mu \geq F(a, \delta, \frac{\mu}{\zeta}) = \frac{a + (\frac{\mu}{\zeta})^p \delta^p}{a + \delta^p},$$

or

$$\zeta \mu^p \leq \frac{a\mu^p + \zeta^p \delta^p}{a + \delta^p}, \quad \text{and} \quad \mu \zeta^p \geq \frac{a\zeta^p + \mu^p \delta^p}{a + \delta^p}.$$

Then

$$a\zeta^p \mu^{p-1} + \mu^{2p-1}\delta^p \leq \zeta^p \mu^p \leq a\mu^p \zeta^{p-1} + \zeta^{2p-1}\delta^p.$$

Hence

$$a\zeta^p \mu^{p-1} + \mu^{2p-1}\delta^p \leq a\mu^p \zeta^{p-1} + \zeta^{2p-1}\delta^p,$$

and so

$$\zeta^p(a\mu^{p-1} + \mu^{p-1}(\frac{\mu}{\zeta})^p \delta^p) \leq \mu^p(a\zeta^{p-1} + \zeta^{p-1}(\frac{\zeta}{\mu})^p \delta^p),$$

or

$$(\frac{\zeta}{\mu})^p(a(\frac{\mu}{\zeta})^{p-1} - \delta^p) \leq a - (\frac{\mu}{\zeta})^{p-1}\delta^p.$$

Thus

$$a\frac{\zeta}{\mu} - \delta^p(\frac{\zeta}{\mu})^p \leq a - (\frac{\zeta}{\mu})^{1-p}\delta^p.$$

Since $0 < p \leq \frac{1}{2}$, we obtain

$$a(\frac{\zeta}{\mu} - 1) \leq \delta^p((\frac{\zeta}{\mu})^p - (\frac{\zeta}{\mu})^{1-p}).$$

Therefore

$$a(\frac{\zeta}{\mu} - 1) \leq 0,$$

which implies that

$$\zeta \leq \mu.$$

Thus we get that $\zeta = \mu$. The proof is completed. (ii)Now suppose $\frac{1}{2} < p < 1$, $a > 1$, and $a(a-1) > b-1$. Note that $(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}$ and $(\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}$. Then it follows by that (1.34), (1.47-1.51) and $(\frac{\eta}{\lambda})^p \leq \frac{b-1}{a-1}$ and $(\frac{\lambda}{\eta})^p \geq \frac{a-1}{b-1}$ hold. Then we obtain

$$\theta \leq F(a, \frac{\eta}{\lambda}, \frac{\zeta}{\mu}) = \frac{a + (\frac{\eta}{\lambda})^p (\frac{\zeta}{\mu})^p}{a + (\frac{\eta}{\lambda})^p} \leq \frac{a + (\frac{b-1}{a-1})(\frac{\zeta}{\mu})^p}{a + (\frac{b-1}{a-1})},$$

and

$$\mu \geq F(a, \frac{\eta}{\lambda}, \frac{\mu}{\zeta}) = \frac{a + (\frac{\eta}{\lambda})^p (\frac{\mu}{\zeta})^p}{a + (\frac{\eta}{\lambda})^p} \geq \frac{a + (\frac{b-1}{a-1})(\frac{\mu}{\zeta})^p}{a + (\frac{b-1}{a-1})}.$$

Then

$$\mu^p \zeta \leq \frac{a\mu^p}{a + (\frac{b-1}{a-1})} + \frac{(\frac{b-1}{a-1})\zeta^p}{a + (\frac{b-1}{a-1})}, \quad \textit{and} \quad \mu\zeta^p \geq \frac{a\zeta^p}{a + (\frac{b-1}{a-1})} + \frac{(\frac{b-1}{a-1})\mu^p}{a + (\frac{b-1}{a-1})}. \tag{1.52}$$

Since $\mu \leq \zeta$ it follows that $\mu\zeta^p \leq \zeta\mu^p$. Therefore from (1.52) we get

$$\frac{a\zeta^p}{a + (\frac{b-1}{a-1})} + \frac{(\frac{b-1}{a-1})\mu^p}{a + (\frac{b-1}{a-1})} \leq \frac{a\mu^p}{a + (\frac{b-1}{a-1})} + \frac{(\frac{b-1}{a-1})\zeta^p}{a + (\frac{b-1}{a-1})}.$$

Then

$$(\frac{a}{a + (\frac{b-1}{a-1})} - \frac{(\frac{b-1}{a-1})}{a + (\frac{b-1}{a-1})})\zeta^p \leq (\frac{a}{a + (\frac{b-1}{a-1})} - \frac{(\frac{b-1}{a-1})}{a + (\frac{b-1}{a-1})})\mu^p. \tag{1.53}$$

Since $\frac{1}{2} < p < 1$, $a > 1$, and $a(a-1) > b-1$, we obtain from (1.53) that $\zeta \leq \mu$ and so $\zeta = \mu$. Then the proof is completed. ■

Example 1.7.6 *Figure (8) shows the global attractivity of the equilibrium point* $\overline{x} = 1$ *of Eq.(1.41) whenever* $\overline{x}_{-1} = 2.091$, $\overline{x}_0 = 23.0192$, $y_{-1} = 4.341$, $y_0 = 2.3134$, $a = 0.2145$, *and* $p = 0.441$.

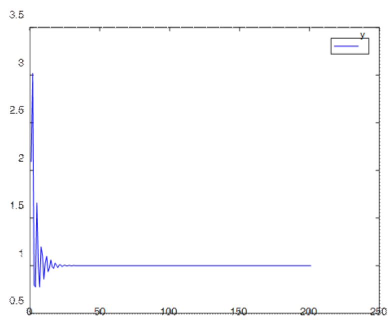


Figure (8)

1.7.3 Periodicity of Eq.(1.1)

In the following theorem we find the sufficient conditions for the existence of two-periodic solutions for Eq.(1.1).

Theorem 1.7.7 *Assume that $0 < p < 1$ and $\{a_n\}$ is a periodic sequence of period twos. Then Eq.(1.1) has a periodic solution of prime period two.*

Proof. For Eq.(1.1) posses a periodic solution $\{x_n\}$ of prime period two, we must find some positive numbers x_{-1}, x_0. Assume that $\{a_n\} = \{a_0, a_1, a_0, a_1, ...\}$, such that

$$x_{-1} = x_1 = a_0 + (\frac{x_0}{x_{-1}})^p, \text{ and } x_0 = x_2 = a_1 + (\frac{x_1}{x_0})^p, \tag{1.54}$$

We shall show that System (1.54) is consistent. We get from Eq.(1.54)

$$(x_{-1} - a_0)(x_0 - a_1) = 1. \tag{1.55}$$

It follows that

$$(x_{-1} - a_0)^{p+1} = \frac{(a_1(x_{-1} - a_0) + 1)^p}{x_{-1}^p}, \text{ and } (x_0 - a_1)^{p+1} = \frac{(a_0(x_0 - a_1) + 1)^p}{x_0^p}. \tag{1.56}$$

We define a function F by

$$F(x) = (x - a_0)^{p+1} - \frac{(a_1(x - a_0) + 1)^p}{x^p}, \quad x > a_0.$$

Then

$$F(a_0) = -\frac{1}{a_0} < 0, \quad \text{and } F(a_0 + 1) = 1 - \frac{(a_1 + 1)^p}{(a_0 + 1)^p} > 0.$$

Now let $a_1 < a_0$, then F has a zero, say x_{-1}, in the interval $(a_0, a_0 + 1)$, and in view of equations (1.55) and (1.56) we get that Eq.(1.1) has a two-periodic solution. Assume now that $a_1 > a_0$. We define a function G such that

$$G(x) = (x - a_1)^{p+1} - \frac{(a_0(x - a_1) + 1)^p}{x^p}, \quad x > a_1.$$

Then

$$G(a_1) = -\frac{1}{a_1} < 0, \quad F(a_1 + 1) = 1 - \frac{(a_0 + 1)^p}{(a_1 + 1)^p} > 0.$$

Thus, G has a zero, say x_0, in the interval $(a_1, a_1 + 1)$, and in view of equations (1.55) and (1.56) we get that Eq.(1.1) has a two-periodic solution. ∎

Theorem 1.7.8 *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$.*Then every solution of Eq.(1.1) converges to a period two solution of Eq.(1.1).*

Proof. We know by Theorem 1.7.1 that every positive solution of Eq.(1.1) is bounded, therefore there are some positive constants l, L, s and S such that

$$l = \lim_{n\to\infty} \inf x_{2n+1}, \quad \text{and} \quad L = \lim_{n\to\infty} \sup x_{2n+1},$$

$$s = \lim_{n\to\infty} \inf x_{2n}, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_{2n}.$$

Now we get from Eq.(1.1) that

$$\begin{aligned} x_{2n+1} &= a_{2n} + \tfrac{x_{2n}^p}{x_{2n-1}^p}, \\ x_{2n+2} &= a_{2n+1} + \tfrac{x_{2n+1}^p}{x_{2n}^p}. \end{aligned} \tag{1.57}$$

Therefore, it is easy to see from System (1.57) that

$$l \geq a_0 + \frac{s^p}{L^p}, \quad \text{and} \quad L \leq a_0 + \frac{S^p}{l^p},$$

and

$$s \geq a_1 + \frac{l^p}{S^p}, \quad \text{and} \quad S \leq a_1 + \frac{L^p}{s^p}.$$

Then we obtain

$$L^p l \geq a_0 L^p + s^p, \quad \text{and} \quad L l^p \leq a_0 l^p + S^p,$$

and

$$S^p s \geq a_1 S^p + l^p\ , \quad \text{and} \quad S s^p \leq a_1 s^p + L^p.$$

So, we get

$$a_0 L^p + s^p \leq L l^p \leq L^p l \leq a_0 l^p + S^p,$$

and

$$a_1 S^p + l^p \leq S^p s \leq S s^p \leq a_1 s^p + L^p.$$

Thus, we have

$$a_0(L^p - l^p) \leq S^p - s^p, \quad \text{and} \quad a_1(S^p - s^p) \leq L^p - l^p. \tag{1.58}$$

Thus it is clear from (1.58) that $s = S$ and $l = L$. Now assume that $\lim_{n\to\infty} x_{2n+1} = S$ and $\lim_{n\to\infty} x_{2n} = L$. We want to proof that $S \neq L$. From System(1.57) we get

$$S = \alpha + \frac{L^P}{S^P}, \quad \text{and} \quad L = \beta + \frac{S^P}{L^P}.$$

As thae sake of contradiction assume that $L = S$, then

$$L = \alpha + 1, \quad \text{and} \quad S = \beta + 1$$

thus $\alpha = \beta$ which is a contradiction. So $\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}$. The proof is so completed. ■

Chapter 2

On Some Second Order Difference Equations

2.1 Introduction

In this chapter we study the local stability, boundedness, global attractivity, oscillatory, and the periodicity for the solutions of the rational difference equations

$$x_{n+1} = ax_n + \frac{bx_n}{1 + x_{n-1}^p}, \quad n \geq 0,$$

and

$$x_{n+1} = ax_{n-1} + \frac{b}{1 + x_n^p}, \quad n \geq 0,$$

where the parameters $a, b,$and $p \in (0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers.

2.2 On the Equation $x_{n+1} = ax_n + \frac{bx_n}{1+x_{n-1}^p}$,

Our aim in this section is to investigate the locally , boundedness, and the global attractively for the positive solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{1 + x_{n-1}^p}, \quad n \geq 0, \tag{2.1}$$

with $a, b \in [0, \infty), p \in (0, \infty)$, and the initial conditions x_{-1}, x_0 are arbitrary positive numbers.

2.2.1 Local Stability of the Equilibrium Points

Here we investigate the equilibrium points of Eq.(2.1).

The equilibrium points of Eq.(2.1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b\overline{x}}{1+\overline{x}^p}$$

If $b + a > 1$, and $a < 1$, then Eq.(2.1) has the equilibrium points $\overline{x} = 0$ and $\overline{x} = \sqrt[p]{\frac{b}{1-a} - 1}$. Now let $f\ : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = au + \frac{bu}{1+v^p}.$$

Therefore,

$$\frac{\delta f(u,v)}{\delta u} = a + \frac{b}{1+v^p} \qquad \text{and} \qquad \frac{\delta f(u,v)}{\delta u} = -\frac{bpuv^{p-1}}{(1+v^p)^2}.$$

Then we see that

$$\frac{\delta f(\overline{x},\overline{x})}{\delta u} = a + \frac{b}{1+\overline{x}^p} = p_1, \qquad \text{and} \qquad \frac{\delta f(\overline{x},\overline{x})}{\delta v} = -\frac{bp\overline{x}^p}{(1+\overline{x}^p)^2} = p_2.$$

Then the linearized equation of Eq.(2.1) about $\overline{x}$ is

$$y_{n+1} - (a + \frac{b}{1+\overline{x}^p})y_n + (\frac{bp\overline{x}^p}{(1+\overline{x}^p)^2})y_{n-1} = 0.$$

Theorem 2.2.1 *The following statements are true:*

(i) The equilibrium point $\overline{x} = 0$ of Eq.(2.1) is locally asymptotically stable if $a + b < 1$.

(ii) The equilibrium point $\overline{x} = 0$ of Eq.(2.1) is unstable if $a + b > 1$.

(iii) The equilibrium point $\overline{x} = 0$ of Eq.(2.1) is stable if $a + b = 1$.

Proof. Since the linearized equation of Eq.(2.1) about the equilibrium point $\overline{x} = 0$ can be written in the following form

$$y_{n+1} = (a+b)y_n, \quad n \geq 0,$$

so, the characteristic equation of Eq.(2.1) about $\overline{x} = 0$, is

$$\lambda^2 - (a+b)\lambda = 0.$$

Then, the proof of (i),(ii) follows immediately from Theorem A.

(iii) Let $\epsilon > 0$, and consider $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(2.1) such that

$$\mid x_{-1} \mid + \mid x_0 \mid < \delta.$$

It suffices to show that

$$\mid x_1 \mid < \epsilon.$$

Now,

$$0 < |x_1| = \left| ax_0 + \frac{bx_0}{1 + x_{-1}^p} \right| < |(a+b)x_0| = |x_0| < \delta.$$

Chose $\delta = \epsilon$ then $|x_1| < \epsilon$ whenever $a + b < 1$ holds. Then, the result follows by induction. ■

2.2.2 Boundedness of the Solutions

Here we discuss the boundedness nature of the solutions of Eq.(2.1)

Lemma 2.2.2 *Assume that $a + b \leq 1$, then every positive solution of Eq.(2.1) is bounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(2.1). It follows from Eq.(2.1) that

$$x_{n+1} = ax_n + \frac{bx_n}{1 + x_{n-1}^p} \leq (a+b)x_n.$$

Then in view of the proof of Theorem 4.2.1 , we have

$$x_1 \leq x_0.$$

Similary it is easy to see that

$$... \leq x_n \leq ... \leq x_2 \leq x_1 \leq x_0.$$

So every solution of Eq.(2.1) is bounded from above. ■

Lemma 2.2.3 *If $a > 1$, then the Eq.(2.1) has unbounded solutions.*

Proof. Consider $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(2.1), then it follows that

$$x_{n+1} > ax_n,$$

so

$$x_n > ax_{n-1} > ... > a^n x_{0.}$$

Then

$$\lim_{n\to\infty} x_n = \infty.$$

Thus, the proof is completed. ■

2.2.3 Global Attractor of the Equilibrium Points of Eq.(2.1)

This subsection is devoted to investigate the global attractivity character of solution of Eq.(2.1).

Theorem 2.2.4 *Assume that* $a+b \leq 1,$ *then the zero equilibrium point of Eq.(2.1) is globally asymptotically stable.*

Proof. We know by Theorem 2.2.1 that $\overline{x} = 0$ is locally asymptotically stable equilibrium point of Eq.(2.1) if $a + b \leq 1$, and so it suffices to show that $\overline{x} = 0$ is global attractor of Eq.(2.1) as follows

$$0 \leq x_{n+1} = ax_n + \frac{bx_n}{1 + x_{n-1}^p} \leq x_n,$$

Then the sequence $\{x_n\}$ is decreasing ,and bounded from below by zero and since there is a unique equilibrium point $\overline{x} = 0$ in this case, then $\lim\limits_{n\to\infty} x_n = 0$. Then the proof is completed. ■

Theorem 2.2.5 *If* $a < 1$ *and* $a + b > 1$,*Then the positive equilibrium point* $\overline{x}$ *is a global attractor of Eq.(2.1).*

Proof. We can easily see that the function

$$g(u, v) = au + \frac{bu}{1 + v^p}$$

is increasing in u and decreasing in v.Suppose that (m, M) is a solution of the system

$$M = g(M, m), \quad \text{and} \quad m = g(m, M).$$

We can see from Eq.(2.1), that

$$M = aM + \frac{bM}{1+m^p}, \quad \text{and} \quad m = am + \frac{bm}{1+M^p},$$

or

$$1 - a = \frac{b}{1+m^p}, \quad \text{and} \quad 1 - a = \frac{b}{1+M^P}.$$

We obtain from this

$$m = M.$$

By Theorem B, we can see that $\overline{x}$ is a global attractor of Eq.(2.1). Then the proof is complete. ■

Theorem 2.2.6 *Let $\{x_n\}_{n=-1}^{\infty}$ be a nontrivial solution of Eq.(2.1). Then the following statements are true:*

(i) Every semicycle, except perhaps for the first one, has at least two terms.

(ii) The extreme in each semicycle occur at either the first term or the second. Furthermore after the first, the remaining terms in a positive semicycle are strictly decreasing and in a negative semicycle are strictly increasing.

Proof. We present the proofs for positive semicycles only. The proofs for negative semicycles are similar and will be omitted.

(i) Assume that for some $N \geq 0$,

$$x_{N-1} < \overline{x} \quad \text{and} \quad x_N \geq \overline{x}.$$

Then

$$x_{N+1} = ax_N + \frac{bx_N}{1+x_{N-1}} > a\overline{x} + \frac{b\overline{x}}{1+\overline{x}^p} = \overline{x}.$$

(ii) Assume that for some $N \geq 0$, the first two terms in a positive semicycle are x_N and x_{N+1}. Then

$$x_N \geq \overline{x}, \quad x_{N+1} > \overline{x}$$

and

$$\frac{x_{N+2}}{x_{N+1}} = \frac{1}{x_{N+1}}[ax_{N+1} + \frac{bx_{N+1}}{1+x_N^P}] = a + \frac{b}{1+x_N^P} < a + \frac{b}{1+\overline{x}^p} = 1.$$

The proof is completed. ■

Example 2.2.7 *Figure (9) shows the global attractivity of the equilibrium point* $\overline{x} = 0$ *of Eq.(2.1) whenever* $x_{-1} = 0.87$, $x_0 = 0.9539$, $a = 0.2$, $b = 0.5$, *and* $p = 0.442$.

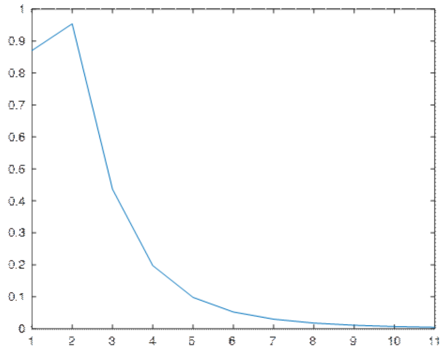


Figure (9)

Example 2.2.8 *Figure (10) shows that Eq.(2.1) has unbounded solutions with the values* $x_{-1} = 1.537$, $x_0 = 3.019$, $a = 5$, $b = 0.9$, *and* $p = 2$.

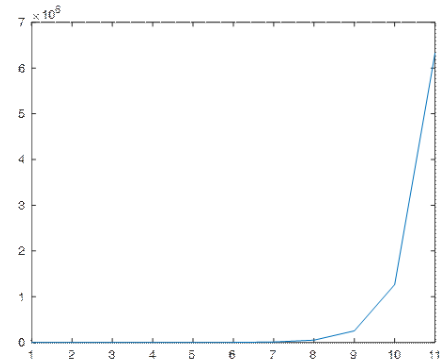


Figure (10)

Example 2.2.9 *Figure (11) shows the global attractivity of the equilibrium point* $\overline{x} = 4.1231$ *of Eq.(2.1) whenever* $x_{-1} = 6.012$, $x_0 = 2.34$, $a = 0.5$, $b = 9$, *and* $p = 2$.

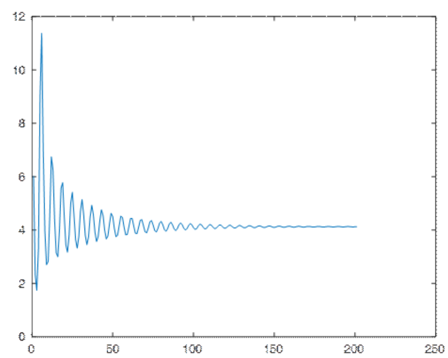


Figure (11)

2.3 On the Equation $x_{n+1} = ax_{n-1} + \frac{b}{1+x_n^p}$,

In this section we deal the locally, global attractivity, and the boundedness for the solutions of the rational difference equation

$$x_{n+1} = ax_{n-1} + \frac{b}{1+x_n^p}, \quad n \geq 0, \tag{2.2}$$

where the parameters $a, b,$ and p are nonnegative real numbers and initial conditions x_{-1}, and x_0 are nonnegative real numbers.

2.3.1 Local Stability of the Equilibrium Points

This subsection deals with study the local stability character of the positive equilibrium point of Eq.(2.2).

The equilibrium points of Eq.(2.2) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b}{1+\overline{x}^p}.$$

If $a < 1$, then the uniqne positive equilibrium point of Eq.(2.2) is given by

$$\overline{x} + \overline{x}^{p+1} = \frac{b}{1-a}.$$

Let $f : (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(u, v) = av + \frac{b}{1+u^p}$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = -\frac{bpv^{p-1}}{(1+v^p)^2}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = a.$$

Set

$$p_1 = -\frac{p}{b}\overline{x}^{p+1}(1-a)^2, \quad \text{and} \quad p_2 = a$$

Then the linearized equation of Eq.(2.2) about $\overline{x}$ is

$$y_{n+1} + \frac{p}{b}\overline{x}^{p+1}(1-a)^2 y_n - ay_{n-1} = 0.$$

Theorem 2.3.1 *Assume that $a < 1$ and $p > 1$. Then the following statments are true*

(i) If b
$$b > \frac{p}{(p-1)^{\frac{1}{p}}}(\frac{1-a}{p-1}), \tag{2.3}$$
then the equilibrium point of Eq.(2.2) is locally asymptotically stable.

(ii) If b
$$b < \frac{p}{(p-1)^{\frac{1}{p}}}(\frac{1-a}{p-1}), \tag{2.4}$$
then the equilibrium point of Eq.(2.2) is unstable, in fact is a saddle point.

(iii) If b
$$b = \frac{p}{(p-1)^{\frac{1}{p}}}(\frac{1-a}{p-1}), \tag{2.5}$$
then the equilibrium point of Eq.(2.2) is a nonhyperbolic point.

Proof. (i) By Theorem A we get
$$|p_1| + |p_2| = a + \frac{bp\overline{x}^{p-1}}{(1+\overline{x}^p)^2} < 1 \Leftrightarrow a + \frac{p}{b}\overline{x}^{p+1}(1-a) < 1 \Leftrightarrow \overline{x} < (\frac{b}{p(1-a)})^{\frac{1}{p+1}}.$$

Let $g(x) = x(1-a) + x^{p+1}(1-a) - b$.A simple calculation, using condition (2.3), shows that
$$g((\frac{b}{p(1-a)})^{\frac{1}{p+1}}) = (\frac{b}{p(1-a)})^{\frac{1}{p+1}}(1-a) + \frac{b}{p(1-a)}(1-a) - b < 0$$
$$\Leftrightarrow (\frac{b}{p})^{\frac{1}{p+1}}(1-a)^{\frac{p}{p+1}} + \frac{b}{p} < b \Leftrightarrow b \geq \frac{p}{(p-1)^{\frac{1}{p}}}(\frac{1-a}{p-1}).$$

Then, since $\lim\limits_{x\to\infty} g(x) = \infty$, $\overline{x}^{p+1} < \frac{b}{p(1-a)}$.

(ii) The condition $p_1^2 + 4p_2 > 0$ of Theorem A is always satisfied and so $\overline{x}$ is unstable if $\overline{x}^{p+1} < \frac{b}{p(1-a)}$. By condition (2.4), we have
$$g((\frac{b}{p(1-a)})^{\frac{1}{p+1}}) = (\frac{b}{p(1-a)})^{\frac{1}{p+1}}(1-a) + \frac{b}{p(1-a)}(1-a) - b > 0.$$

Then since $g(0) < 0, \overline{x}^{p+1} < \frac{b}{p(1-a)}$.

(iii) The condition$\mid p_1 \mid = \mid 1 - p_2 \mid$ is equivalent to $\overline{x}^{p+1} = \frac{b}{p(1-a)}$. Similarly by condition (2.5), we have $g((\frac{b}{p(1-a)})^{\frac{1}{p+1}}) = 0$. Then $\overline{x}^{p+1} = \frac{b}{p(1-a)}$. Then the proof is completed. ■

2.3.2 Boundedness Charactor of Eq.(2.2)

In this subsection, we study the characteristic task of boundedness of solutions Eq.(2.2).

Theorem 2.3.2 *If $a < 1$, then Eq.(2.2) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(2.2) it follows from Eq.(2.2) that

$$x_{n+1} = ax_{n-1} + \frac{b}{1 + x_n^p} < ax_{n-1} + b.$$

Therefore

$$\lim_{n\to\infty} \sup x_n \leq \frac{b}{1-a} = M.$$

Thus $\{x_n\}_{n=-1}^{\infty}$ is bounded from above. Again it follows from Eq.(2.2) that

$$x_{n+1} = ax_{n-1} + \frac{b}{1 + x_n^p} > \frac{b}{1 + x_n^p} \geq \frac{b}{1 + (\frac{b}{1-a})^p} = m.$$

Then $\{x_n\}$ is bounded from blow too. Then the result is followed. ■

Theorem 2.3.3 *If $a > 1$, then Eq.(2.2) has unbounded solutions.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(2.2). We obtain from Eq.(2.2) that

$$x_{n+1} = ax_{n-1} + \frac{b}{1 + x_{n-1}^p} > ax_{n-1},$$

that is

$$x_n > ax_{n-1} > a^2 x_{n-1} > ... > a^n x_0.$$

It follows that $\lim_{n\to\infty} x_n = \infty$. Then the proof is completed. ■

2.3.3 Global attractor

This subsection is devoted to investigate the global attractivity character of solutions of Eq.(2.2).

Theorem 2.3.4 *Let $0 < p < 1$ and $a < 1$.Then every positive solution of Eq.(2.2) converges to $\overline{x}$.*

Proof. It was shown by Theorem 2.3.2 that every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2.2) is bounded. Thus it follows

$$\lim_{n\to\infty} \inf x_n = l, \quad \text{and} \quad \lim\sup x_n = L$$

As the sake of contraduction assume that $l < L$. We see from Eq.(2.2) that

$$al + \frac{b}{1+L^p} \le l < L \le aL + \frac{b}{1+l^p}$$

which implies that

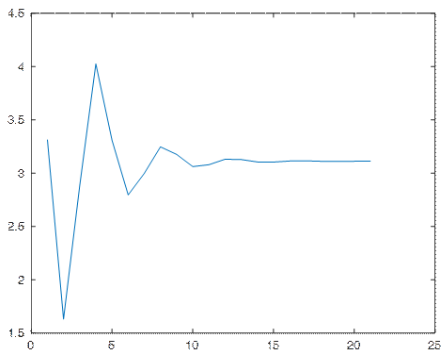
$$\frac{b}{1-a} - l \le lL^p < Ll^p \le \frac{b}{1-a} - L$$

i.e.,

$$(L-l) < 0$$

which gives a contradiction. Hence the result follows. ■

Example 2.3.5 *Figure (12) shows the global attractivity of the equilibrium point $\overline{x}$ = 3.1118 of Eq.(2.2) whenever x_{-1} =3.3124 , x_0 = 1.63, $a = 0.1$, $b = 9$, and $p =$ 0.7.*



2.3.4 Oscillatory Solutions of Eq.(2.2)

In this subsection, we study the characteristic task of oscillatory solutions of Eq.(2.2).

Theorem 2.3.6 *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2.2) which consists of at least two semicycles. Then $\{x_n\}_{n=-1}^{\infty}$ is oscillatory. Moreover, with the possible exception of the first semi-cycle has length 1.*

Proof. It suffices to consider the following two cases.

Case 1. Suppose $x_{N-1} < \overline{x} < x_N$. Then

$$x_{N+1} = ax_{N-1} + \frac{b}{1 + x_N^p} < a\overline{x} + \frac{b}{1 + \overline{x}^p} = \overline{x} \ ,$$

and

$$x_{N+2} = ax_N + \frac{b}{1 + x_{N+1}^p} > a\overline{x} + \frac{b}{1 + \overline{x}^p} = \overline{x}.$$

Case 2. Suppose $x_N < \overline{x} < x_{N-1}$. Then

$$\begin{aligned} x_{N+1} &= ax_{N-1} + \frac{b}{1 + x_N^p} > a\overline{x} + \frac{b}{1 + \overline{x}^p} = \overline{x} \\ \text{and} \ \ x_{N+2} &= ax_N + \frac{b}{1 + x_{N+1}^p} < a\overline{x} + \frac{b}{1 + \overline{x}^p} = \overline{x}. \end{aligned}$$

The proof is complete. ■

Chapter 3

On Some Higher Order Difference Equations

3.1 Introduction

In this chapter we investigate the global attractivity, and the boundedness for the solutions of the rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{A x_{n-k}^p + B x_{n-m}^q}, \quad n \geq 0, \tag{3.1}$$

where the parameters $\alpha, \beta, \gamma, A, B, p,$ and $q \in (0, \infty)$ and the initial conditions $x_{-l}, x_{-l+1}, ..., x_{-1}, x_0$ where $l = \max\{k, m\}$ are positive real numbers.

The work of this section divided into two parts; Part **I** concerned with the special cases of Eq.(3.1) and Part Π deals with the general Eq.(3.1).

Part I

Here, we study the following cases of Eq.(3.1).

1. Whenever $A = \gamma = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \quad n \geq 0. \tag{3.2}$$

2. Whenever $A = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{B x_{n-m}^q},$$

or

$$x_{n+1} = C + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \ n \geq 0, \tag{3.3}$$

where $C = \alpha + \frac{\gamma}{B}$.

3. Whenever $\beta = B = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p}, \quad n \geq 0. \tag{3.4}$$

4. Whenever $B = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{A x_{n-k}^p},$$

or

$$x_{n+1} = D + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p}, \ n \geq 0, \tag{3.5}$$

where $D = \alpha + \frac{\beta}{A}$.

5. Whenever $\beta = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p + B x_{n-m}^q}, \quad n \geq 0. \tag{3.6}$$

6. Whenever $\gamma = 0$ then Eq.(3.1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{A x_{n-k}^p + B x_{n-m}^q}, \quad n \geq 0. \tag{3.7}$$

In this part we study the special cases of Eq.(3.1).

3.2 Case 1. Study of Eq.(3.2)

In this section, we study the local stability, the boundedness, global attractivity, oscillatery, and periodicity for the solutions of the equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \ \ n \geq 0.$$

3.2.1 Local Stability of the Equilibrium Point of Eq.(3.2)

It is easy to see that Eq.(3.2) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta \overline{x}^p}{B\overline{x}^q}.$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = \alpha + \frac{\beta u^p}{Bv^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = E\frac{pu^{p-1}}{v^q}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -E\frac{qv^{q-1}u^p}{(v^q)^2},$$

where $E = \frac{\beta}{B}$. Set

$$p_1 = Ep\overline{x}^{p-q-1}, \quad \text{and} \quad p_2 = -Eq\overline{x}^{p-q-1}.$$

Then the linearized equation of Eq.(3.2) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x},\overline{x})$, and $p_1 = -f_v(\overline{x},\overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2\lambda^{k-m} + p_1 = 0.$$

Theorem 3.2.1 *If $\overline{x} < \frac{1}{\sqrt[p-q-1]{E(p+q)}}$, then the positive equilibrium point $\overline{x}$ of Eq.(3.2) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = Ep\overline{x}^{p-q-1}$, and $p_2 = -Eq\overline{x}^{p-q-1}$. Then

$$|p_1| + |p_2| < 1 \Leftrightarrow Ep\overline{x}^{p-q-1} + Eq\overline{x}^{p-q-1} < 1.$$

which is valid iff

$$\overline{x}^{p+q-1} < \frac{1}{E(p+q)}.$$

So by Theorem A $\overline{x}$ is locally asymptotically stable when $\overline{x} < \frac{1}{\sqrt[p-q-1]{E(p+q)}}$. ∎

3.2.2 Boundedness of Eq.(3.2)

Here, we investigate the bounded character of Eq.(3.2).

Theorem 3.2.2 *If $0 < p < 1$, then the Eq.(3.2) is bounded and persists.*

Proof. Assume that $\{x_n\}$ be a solution of Eq.(3.2). We obtain from Eq.(3.2) that

$$x_{n+1} > \alpha, \quad \text{for } n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(3.2) that

$$x_{n+1} \leq \alpha + Lx_{n-k}^p,$$

where $L = \frac{\beta}{B\alpha^q}$. Now we consider the difference equation

$$y_{n+1} = \alpha + Ly_n^p, \quad \text{for } n \geq 0. \tag{3.8}$$

Let $\{y_n\}$ be a solution of Eq.(3.8) with $y_0 = x_0$. Then obviously

$$x_{n+1} \leq y_{n+1}, \quad \text{for } n = 0, 1, \ldots .$$

We shall prove that the sequence $\{y_n\}$ is bounded. Let

$$f(x) = \alpha + Lx^p.$$

Then

$$f^{'}(x) = Lpx^{p-1} > 0, \quad \text{and} \quad f^{''}(x) = Lp(p-1)x^{p-2} < 0.$$

Therefore the function f is increasing and concave . Thus we obtain that there is a unique fixed point y^* of the equation $f(y) = y$. Also the function f satisfies

$$(f(y) - y)(y - y^*) < 0, \quad y \in (0, \infty).$$

It follows by Theorem C that y^* is a global attractor of all positive solutions of Eq.(3.8) and so $\{y_n\}$ is bounded. Therefore from Eq.(3.2) the sequence $\{x_n\}$ is also bounded. This completes the proof of the theorem. ■

3.2.3 Global attractor

Here we study the global asymptotic stability of the positive solutions of Eq.(3.2).

Theorem 3.2.3 *Assume that $0 < p < 1 < q$, $\alpha > E(p + q - 1)^{\frac{1}{q-p+1}}$. Then every positive solution of Eq.(3.2) converges to the unique positive equilibrium point $\overline{x}$ of Eq.(3.2).*

Proof. Note that when $0 < p < 1 < q$, it was shown in Theorem 3.2.2 that every positive solution of Eq.(3.2) is bounded. Then we have the following

$$s = \lim_{n\to\infty} \inf x_n, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_n.$$

It is clear that $s \leq S$. We want to proof that $s \geq S$. Now it is easy to see from Eq.(3.2) that

$$s \geq \alpha + E\frac{s^p}{S^q}, \quad \text{and} \quad S \leq \alpha + E\frac{S^p}{s^q}.$$

Thus we have

$$sS^q \geq \alpha S^q + Es^p, \quad \text{and} \quad s^q S \leq as^q + ES^p.$$

Thus

$$\alpha s^{q-1}S^q + Es^p s^{q-1} \leq \alpha s^q S^{q-1} + ES^p S^{q-1}.$$

Then we get

$$\alpha S^{q-1}s^{q-1}(S-s) \leq E(S^{p+q-1} - s^{p+q-1}).$$

So

$$\alpha S^{q-1}s^{q-1} \leq E\frac{S^{p+q-1} - s^{p+q-1}}{S-s}. \tag{3.9}$$

If we consider the function x^{p+q-1}, then there exists a $c \in (s, S)$ such that

$$\frac{S^{p+q-1} - s^{p+q-1}}{S-s} = (p+q-1)c^{p+q-2} \leq (p+q-1)S^{p+q-2}. \tag{3.10}$$

Theen from (3.9) and (3.10) we get

$$\alpha S^{q-1}s^{q-1} \leq E(p+q-1)S^{p+q-2}.$$

or

$$\alpha S^{1-p}s^{q-1} \leq E(p+q-1).$$

Since $S \geq \alpha$ and $s \leq \alpha$. Then we obtain

$$\alpha\alpha^{1-p}\alpha^{q-1} = \alpha^{q-p+1} \leq E(p+q-1).$$

which contradicts to $0 < p < 1 < q$. Which implies that $s = S$. Thus the proof is complete. ■

Example 3.2.4 *Figure (13) shows the global attractivity of the equilibrium point* $\overline{x} = 1.1837$ *of Eq.(3.2) whenever* $x_{-1} = 5.6487$, $x_0 = 1.0231$, $p = 0.5$, $q = 0.9$, $\alpha = 0.7$, $\beta = 0.19$, *and* $B = 0.52$.

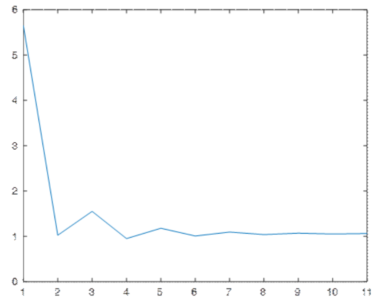


Figure (13)

3.2.4 Oscillatery of the solutions for Eq.(3.2)

In the next theorem, we study the oscillatery character of Eq.(3.2).

Theorem 3.2.5 *Assume that* k *is odd and* m *is even and* $m < k$, *then Eq.(3.2) has oscillatory solutions.*

Proof. Case (1) let $\{x_n\}$ be a solution of Eq.(3.2)with

$$x_{-k}, x_{-k+1}, ..., x_{-1} \geq \overline{x}, \quad \text{and} \quad x_{-m+1}, x_{-m+1}, ..., x_0 < \overline{x}.$$

We get from Eq.(3.2) that

$$x_1 = \alpha + \frac{\beta x^p_{-k}}{B x^q_{-m}} \geq \alpha + \frac{\beta \overline{x}^p}{B \overline{x}^q} = \overline{x},$$

and

$$x_2 = \alpha + \frac{\beta x^p_{-m+1}}{B x^q_{-k+1}} < \alpha + \frac{\beta \overline{x}^p}{B \overline{x}^q} = \overline{x}$$

Then, the result follows by induction.

Case (2) let

$$x_{-m}, x_{-m+1}, ..., x_0 \geq \overline{x}, \quad \text{and} \quad x_{-k+1}, x_{-k+1}, ..., x_{-1} < \overline{x}.$$

is similary the case (1). Then it will be omitted. ∎

Example 3.2.6 *Figure (14) shows the oscillatory solutions of Eq.(3.2) whenever* $x_{-1} = 1.6487$, $x_0 = 2.0231$, $\alpha = 0.23$, $p = 0.2$, $q = 2$, $\beta = 0.9$, *and* $B = 0.5$.

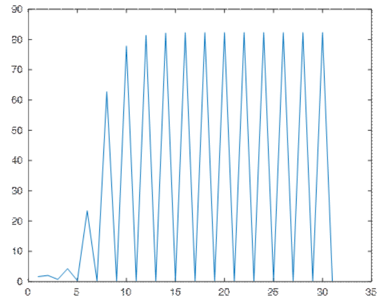


Figure (14)

3.2.5 Periodicity of the solutions

The next theorem deals with the existence of periodic solutions to Eq.(3.2).

Theorem 3.2.7 *Let* k *is odd and* m *is even. If* $0 < p < 1 < q$, *then Eq.(3.2) has periodic solutions of period two.*

Proof. Let $\{x_n\}$ be a solution of Eq.(3.2) with the initial values x_{-1}, and x_0 such that

$$x_{-1} = \frac{\alpha B x_0^q + \beta x_{-1}^p}{B x_0^q}, \quad \text{and} \quad x_0 = \frac{\alpha B x_{-1}^q + \beta x_0^p}{B x_{-1}^q}. \tag{3.11}$$

Let $x_- = x$, and $x_0 = y$, then we obtain from (3.11)

$$x = \alpha + \frac{\beta x^p}{B y^q}, \quad \text{and} \quad y = \alpha + \frac{\beta y^p}{B x^q}. \tag{3.12}$$

Now we want to prove that (3.12) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (3.12) we have

$$y = \frac{\beta^{\frac{1}{q}} x^{\frac{p}{q}}}{B^{\frac{1}{q}} (x - \alpha)^{\frac{1}{q}}}. \tag{3.13}$$

From (3.13) and the second relation of (3.12) we get

$$\frac{\beta^{\frac{1}{q}} x^{\frac{p}{q}}}{B^{\frac{1}{q}} (x - \alpha)^{\frac{1}{q}}} - \frac{\beta^{\frac{p+q}{q}} x^{\frac{p^2-q^2}{q}}}{B^{\frac{p+q}{q}} (x - \alpha)^{\frac{p}{q}}} - \alpha = 0.$$

Now define the function

$$f(x)=\frac{1}{(x-\alpha)^{\frac{1}{q}}}\left(\left(\frac{\beta}{B}\right)^{\frac{1}{q}}x^{\frac{p}{q}}-\left(\frac{\beta}{B}\right)^{\frac{p+q}{q}}x^{\frac{p^2-q^2}{q}}(x-\alpha)^{\frac{1-p}{q}}\right)-\alpha,\quad x>\alpha. \tag{3.14}$$

Then

$$\lim_{x\to\alpha^+}f(x)=\infty,\quad \lim_{x\to\infty}f(x)=-\alpha.$$

Hence Eq.(3.14) has at least one solution $x>\alpha$. Then if $\overline{y}=\frac{\beta^{\frac{1}{q}}\overline{x}^{\frac{p}{q}}}{B^{\frac{1}{q}}(\overline{x}-\alpha)^{\frac{1}{q}}}$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. Thus the proof is complete. ■

Example 3.2.8 *Figure (15) shows the periodicity solutions of Eq.(3.2) whenever* $x_{-1}=1.737$, $x_0=2.423$, $\alpha=0.7$, $p=0.2$, $q=4$, $\beta=0.5$, *and* $B=0.32$.

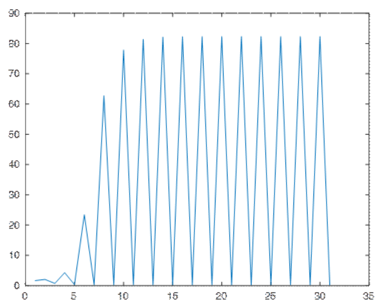


Figure (15)

3.3 Case 2. Study of Eq.(3.3)

This equation is similar of Eq.(3.2) and its investigation is similar to Eq.(3.2) and so will be omitted.

3.4 Case 3. Study of Eq.(3.4)

The proofs of the theorems in this section are similar to the proofs of the theorems in section 2 and will be left to the reader.

Theorem 3.4.1 *If* $\overline{x}<\frac{1}{\sqrt[q-p-1]{F(p+q)}}$, *then the positive equilibrium point* $\overline{x}$ *of Eq.(3.4) is locally asymptotically stable, and is called a sink.*

Theorem 3.4.2 *If $0 < q < 1$, then the Eq.(3.4) is bounded and persists.*

Theorem 3.4.3 *Assume that $0 < q < 1 < p$, $\alpha > F(q+p-1)^{\frac{1}{p-q+1}}$. Then every positive solution of Eq.(3.4) converges to the unique positive equilibrium point $\overline{x}$ of Eq.(3.4).*

Theorem 3.4.4 *Assume that m is odd and k is even and $k < m$, then Eq.(3.4) has oscillatory solutions.*

Theorem 3.4.5 *Let m is odd and k is even. If $0 < q < 1 < p$, then Eq.(3.4) has periodic solutions of period two.*

3.5 Case 4. Study of Eq.(3.5)

This equation is similar of Eq.(3.4) and its investigation is similar to Eq.(3.4) and so will be omitted.

3.6 Case 5. Study of Eq.(3.6)

Eq.(3.6) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta \overline{x}^p}{A\overline{x}^p + B\overline{x}^q}.$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = \alpha + \frac{\beta u^p}{Au^p + aBv^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = \frac{A\beta p v^q u^{p-1}}{(Au^p + Bv^q)^2}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -\frac{\beta B q v^{q-1} u^p}{(Au^p + Bv^q)^2},$$

Set

$$p_1 = \frac{A\beta p \overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}, \quad \text{and} \quad p_2 = -\frac{B\beta q \overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}.$$

Then the linearized equation of Eq.(3.6) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x},\overline{x})$, and $p_1 = -f_v(\overline{x},\overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2 \lambda^{k-m} + p_1 = 0.$$

3.6.1 Local Stability of the Equilibrium Points of Eq.(3.6)

Here we establish the local stability of the equilibrium points of Eq.(3.6).

Theorem 3.6.1 *If* $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p+B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}$, *then the positive equilibrium point* $\overline{x}$ *of Eq.(3.6) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = \frac{A\beta p\overline{x}^{q+p-1}}{(A\overline{x}^p+B\overline{x}^q)^2}$, and $p_2 = -\frac{B\beta q\overline{x}^{q+p-1}}{(A\overline{x}^p+B\overline{x}^q)^2}$. Therefore

$$|p_1| + |p_2| < 1 \Leftrightarrow \frac{A\beta p\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2} + \frac{B\beta q\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < 1.$$

which is valid iff

$$\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}.$$

So by Theorem A $\overline{x}$ is locally asymptotically stable when $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p+B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}$.
■

3.6.2 Boundedness of Eq.(3.6)

Here, we investigate the bounded character of Eq.(3.6).

Theorem 3.6.2 *If* $0 < p < 1$, *then the Eq.(3.6) is bounded and persists.*

Proof. Assume that $\{x_n\}$ be a solution of Eq.(3.6). We obtain from Eq.(3.6) that

$$x_{n+1} > \alpha, \quad \text{for } n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(3.6) that

$$x_{n+1} \leq \alpha + \frac{\beta x_{n-k}^p}{A\alpha^p + B\alpha^q} \leq \alpha + \frac{\beta x_{n-k}^p}{B\alpha^q}, \quad for \quad n \geq 0.$$

The rest of the proof is similar to the proof of the Theorem 3.2.2 and will be omitted. ■

3.6.3 Global Stability of Eq.(3.6)

In this section we investigate the global asymptotic stability of Eq.(3.6).

Theorem 3.6.3 *The positive equilibrium point $\overline{x}$ is a global attractor of Eq.(3.6). If*

$$(AM^P + Bm^q)(Am^p + BM^q) \neq \beta B(\sum_{i=1}^{\infty} \alpha^{i-1} M^{p+q-i} + \sum_{i=1}^{\infty} \alpha^{p+q-i} M^{i-1}), \qquad (3.15)$$

where M is given by $M = \alpha + \frac{\beta M^p}{A\alpha^p + B\alpha^q}$.

Proof. We can see that the function

$$f(u, v) = \alpha + \frac{\beta u^p}{Au^p + Bv^q},$$

is increasing in u and decreasing in v. Since Eq.(3.6) is bounded by Theorem 3.5.2. Suppose that (m, M) is a solution of the system

$$M = f(M, m), \quad \text{and} \quad m = f(m, M).$$

We obtain from Eq.(3.6) that

$$M = \alpha + \frac{\beta M^p}{AM^P + Bm^q}, \quad \text{and} \quad m = \alpha + \frac{\beta m^p}{Am^P + BM^q}.$$

Thus

$$(M - m)(AM^P + Bm^q)(Am^p + BM^q) - B\beta(M^{p+q} - m^{p+q}) = 0.$$

Then we obtain

$$(M - m)[(AM^P + Bm^q)(Am^p + BM^q) - B\beta(\sum_{i=1}^{\infty} \alpha^{i-1} M^{p+q-i} + \sum_{i=1}^{\infty} \alpha^{p+q-i} M^{i-1})] = 0.$$

Seine the condition (3.15) holds, then we get

$$M = m$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(3.6), and then the proof is complete. ∎

Example 3.6.4 *Figure (16) shows the global attractivity of the equilibrium point of Eq.(3.6) whenever* $x_{-1} = 5.4235$, $x_0 = 8.987$, $p = 0.2$, $q = 0.3$, $\alpha = 0.6$, $\beta = 0.4$,

$A = 0.4521$, ***and*** $B = 1.563$.

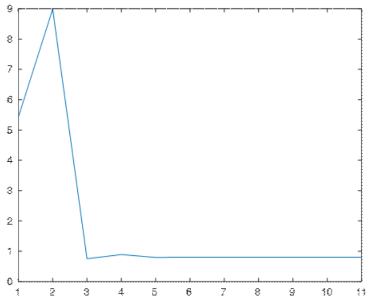


Figure (16)

3.7 Case 6: Study of Eq.(3.7)

This equation is the same of Eq.(3.6) and its investigation is similar to Eq.(3.6) and so will be omitted.

Part II

Now we will investigate the behavior of the solutions of Eq.(3.1).

3.7.0.1 Local Stability of Equilibrium Points

In this section we study the local stability character of the positive equilibrium points of Eq.(3.1). Eq.(3.1) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta\overline{x}^p + \gamma\overline{x}^q}{A\overline{x}^p + B\overline{x}^q}.$$

Let $f : (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(u, v) = \alpha + \frac{\beta u^p + \gamma v^q}{Au^p + Bv^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = \frac{pv^q u^{p-1}(B\beta - A\gamma)}{(Au^p + Bv^q)^2}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -\frac{qu^p v^{q-1}(B\beta - A\gamma)}{(Au^p + Bv^q)^2}.$$

Set

$$p_1 = \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}, \quad \text{and} \quad p_2 = -\frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}.$$

Then the linearized equation of Eq.(3.1) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x}, \overline{x})$, and $p_1 = -f_v(\overline{x}, \overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2\lambda^{k-m} + p_1 = 0.$$

Theorem 3.7.1 *If $\frac{A}{B} < \frac{\beta}{\gamma}$ and $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}$, then the positive equilibrium point $\overline{x}$ of Eq.(3.1) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}$, and $p_2 = -\frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}$. So by Theorem A

$$|p_1| + |p_2| < 1 \Leftrightarrow \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2} + \frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2} < 1.$$

which is valid iff

$$\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}.$$

So $\overline{x}$ is locally asymptotically stable when $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}$. ■

3.7.0.2 Boundedness of Eq.(3.1)

Here, we study the bounded character of Eq.(3.1).

Theorem 3.7.2 *Every solution of Eq.(3.1) is bounded and persists.*

Proof. Let $\{x_n\}$ be a positive solution of Eq.(3.1). We obtain from Eq.(3.1) that

$$x_{n+1} > \alpha, \quad \text{for} \quad n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(3.1) that

$$\begin{aligned} x_{n+1} &= \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{A x_{n-k}^p + B x_{n-m}^q} \\ &\leq \alpha + \frac{\max\{\beta, \gamma\}(x_{n-k} + x_{n-m})}{\min\{A, B\}(x_{n-k} + x_{n-m})} = \alpha + \frac{\max\{\beta, \gamma\}}{\min\{A, B\}} = M. \end{aligned}$$

Thus we get

$$0 < \alpha \leq x_n < \alpha + \frac{\max\{\beta,\gamma\}}{\min\{A,B\}} = M < \infty, \quad \text{for all } n \geq 1.$$

Therefore every solution of Eq.(3.1) is bounded and persists. Hence the result holds. ■

3.7.0.3 Global Stability of Eq.(3.1)

In this section we investigate the global asymptotic stability of Eq.(3.1).

Theorem 3.7.3 *If* $(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) \neq (AM^P + Bm^q)(Am^p + BM^q)$*, and* $\frac{A}{B} < \frac{\beta}{\gamma}$*, then the positive equilibrium point* $\overline{x}$ *is a global attractor of Eq.(3.1).*

Proof. We can see that the function

$$f(u,v) = \alpha + \frac{\beta u^p + \gamma v^q}{Au^p + Bv^q},$$

is increasing in u and decreasing in v. Suppose that (m, M) is a solution of the system

$$M = f(M,m), \quad \text{and} \quad m = f(m,M).$$

We obtain from Eq.(3.1) that

$$M = \alpha + \frac{\beta M^p + \gamma m^q}{AM^P + Bm^q}, \quad \text{and} \quad m = \alpha + \frac{\beta m^p + \gamma M^q}{Am^P + BM^q}.$$

Thus

$$(M-m)[(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) - (AM^P + Bm^q)(Am^p + BM^q)] = 0.$$

Since $B\beta > A\gamma$,

$(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) \neq (AM^P + Bm^q)(Am^p + BM^q)$ hold. Then we obtain

$$m = M.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(3.1), and then the proof is complete. ■

Example 3.7.4 *Figure (17) shows the global attractivity of the equilibrium point of Eq.(3.1) whenever* $x_{-1} = 2.4235$, $x_0 = 1.987$, $p = 0.7$, $q = 0.9$, $\alpha = 0.6$, $\beta = 0.4$, $\gamma = 0.2$, $A = 0.4521$, ***and*** $B = 0.52$.

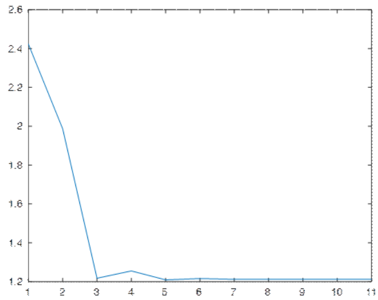


Figure (17)

Chapter 4

On Some Systems of Difference Equations

4.1 Introduction

In this chapter, we investigate the dynamic behavior of the positive solutions of the following system of difference equations.

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^{p} w_{n-1}^{p_1}}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^{q} u_{n-1}^{q_1}}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^{r} v_{n-1}^{r_1}}, n \in \mathbb{N}_0 \quad (4.1)$$

where the initial conditions u_{-i}, v_{-i}, w_i $(i = 0, 1, 2, 3)$ are non-negative real numbers and the parameters $a, b, c, d, e, f, g, h, I, p, q, r$ are positive real numbers.

The hypothesis of discrete dynamic of systems of grew enormously amid the most recent thirty years of the twentieth century. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology.

[7] investigated the periodicity of the positive solutions of the system

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

[32] el al. studied the system of two nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

We will study the following cases:

Case 1. If $p_1 = q_1 = r_1 = 0$.

Case 2. If $p_1 = q_1 = r_1 = 1$.

4.2 Case 1. System (4.1) when $p_1 = q_1 = r_1 = 0$.

We will investigate the stability of the two equilibrium points of System (4.1) when $p_1 = q_1 = r_1 = 0$. Then from System (4.1) we get

$$u_{n+1} = \frac{au_{n-1}}{b + cv_{n-3}^{p}}, v_{n+1} = \frac{dv_{n-1}}{e + fw_{n-3}^{q}}, w_{n+1} = \frac{gw_{n-1}}{h + Iu_{n-3}^{r}}, \quad n \in \mathbb{N}_0. \tag{4.2}$$

By the change of variables $u_n = (\frac{h}{I})^{\frac{1}{r}} x_n, v_n = (\frac{b}{c})^{\frac{1}{p}} y_n, w_n = (\frac{e}{f})^{\frac{1}{q}} z_n$. System (4.2) can be rewritten as

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + y_{n-3}^{p}}, \quad y_{n+1} = \frac{\beta y_{n-1}}{1 + z_{n-3}^{r}}, \quad z_{n+1} = \frac{\gamma y_{n-1}}{1 + x_{n-3}^{q}}, \quad n \in \mathbb{N}_0 \tag{4.3}$$

where $\alpha = \frac{a}{b}, \beta = \frac{g}{h}, \gamma = \frac{d}{e}$.

In this section, we investigate the stability of the two equilibrium points of System (4.3). When $\alpha, \beta, \gamma \in (0, 1)$, it is easy to see that $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ is the unique equilibrium point of System (4.3). When $\alpha, \beta, \gamma \in (1, \infty)$, the unique positive equilibrium point of System (4.3) is given by $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma - 1)^{\frac{1}{q}}, (\alpha - 1)^{\frac{1}{p}}, (\beta - 1)^{\frac{1}{r}})$.

4.2.1 Local stability of the Equilibrium Points

In this subsection we find conditions so that the zero equilibrium $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$ of System (4.3) is stable and the positive equilibrium $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ of System (4.3) is unstable.

Theorem 4.2.1 *The following statements hold:*

(i) If $\alpha, \beta, \gamma \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (4.3) is locally asymptotically stable.

(ii) If $\alpha \in (1, \infty)$ or $\beta \in (1, \infty)$ or $\gamma \in (1, \infty)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (4.3) is unstable.

(iii) If $\alpha, \beta, \gamma \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma - 1)^{\frac{1}{q}}, (\alpha - 1)^{\frac{1}{p}}, (\beta - 1)^{\frac{1}{r}})$ of System (4.3) is unstable.

Proof. We will rewrite System (4.3) in the form

$$X_{n+1} = F(X_N), \tag{4.4}$$

where $X_n = (x_n, ..., x_{n-3}, y_n, ..., y_{n-3}, z_n, ..., z_{n-3})^T$ and the map F is given by

$$F\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ s_0 \\ s_1 \\ s_2 \\ s_3 \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha t_1}{1+s_3^p} \\ t_0 \\ t_1 \\ t_2 \\ \frac{\beta s_1}{1+k_3^r} \\ s_0 \\ s_1 \\ s_2 \\ \frac{\gamma k_1}{1+t_3^q} \\ k_0 \\ k_1 \\ k_2 \end{pmatrix}.$$

The linearized System of (4.4) about the equilibrium point $\overline{X} = (0, ..., 0)^T$ is given by

$$X_{n+1} = J_F(\overline{X}_0)X_n,$$

where

$$J_F(\overline{X}_0) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the characteristic equation of $J_F(\overline{X}_0)$ is given by

$$\lambda^6(\lambda^2 - \alpha)(\lambda^2 - \beta)(\lambda^2 - \gamma) = 0. \tag{4.5}$$

Then we have the following:

(i) If $\alpha, \beta, \gamma \in (0, 1)$, all the roots of the Eq.(4.5) lie inside the open unit disk $|\lambda| < 1$. So, the unique equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (4.3) is locally asymptotically stable.

(ii) It is clearly that if $\alpha \in (1, \infty)$ *or* $\beta \in (1, \infty)$ *or* $\gamma \in (1, \infty)$, then some roots of Eq.(4.5) have absolute value greater that one. Thus, the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (4.3) is unstable.

(iii) The linearized system of (4.4) about the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ is given by $X_{n+1} = J_F(\overline{X}_{\alpha,\beta,\gamma})X_n$, where

$$x_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \end{pmatrix}, \quad J_F(\overline{x}_0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$A = -\frac{p(\alpha -)^{\frac{p-1}{p}}(\beta-1)^{\frac{1}{r}}}{\alpha}, \; B = -\frac{r(\gamma -)^{\frac{1}{q}}(\beta-1)^{\frac{r-1}{r}}}{\beta}, \; and \;\; C = -\frac{q(\alpha -)^{\frac{1}{p}}(\gamma-1)^{\frac{q-1}{q}}}{\gamma}.$$

The characteristic equation of $J_F(\overline{X}_{\alpha,\beta,\gamma})$ is given by

$$p(\lambda) = \lambda^{12} - 3\lambda^{10} + 3\lambda^8 - \lambda^6 - rpq\frac{(\alpha-1)(\beta-1)(\gamma-1)}{\alpha\beta\gamma}.$$

Now

$$p(1) = -rpq\frac{(\alpha-1)(\beta-1)(\gamma-1)}{\alpha\beta\gamma} < 0 \quad \text{and} \quad \lim_{\lambda\to\infty} p(\lambda) = \infty.$$

Then $p(\lambda)$ has at least one root in the interval $(1,\infty)$. So, by Theorem D if $\alpha,\beta,\gamma \in (1,\infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma-1)^{\frac{1}{q}}, (\alpha-1)^{\frac{1}{p}}, (\beta-1)^{\frac{1}{r}})$ of System (4.3) is unstable. This completes the proof. ∎

4.2.2 Global Stability of System (4.3)

In the following theorem, we study the convergency of the solutions of System (4.3) to its zero equilibrium point.

Theorem 4.2.2 *If $\alpha,\beta,\gamma \in (0,1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of System (4.3) is globally asymptotically stable.*

Proof. We proved in Theorem 4.2.1 that if $\alpha, \beta, \gamma \in (0,1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of System (4.3) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\to\infty} (x_n, y_n, z_n) = (0,0,0).$$

We have from System (4.3) that

$$\begin{aligned} 0 &\leq x_{n+1} = \frac{\alpha x_{n-1}}{1+y_{n-3}^p} \leq \alpha x_{n-1},\ 0 \leq y_{n+1} = \frac{\beta y_{n-1}}{1+z_{n-3}^r} \leq \beta y_{n-1}, \\ 0 &\leq z_{n+1} = \frac{\gamma z_{n-1}}{1+x_{n-3}^q} \leq \gamma z_{n-1}, \quad for\ n \in \mathbb{N}_0. \end{aligned}$$

Then it follows by induction that

$$0 \leq x_{2n-i} \leq \alpha^n x_{-i},\ 0 \leq y_{2n-i} \leq \beta^n y_{-i},\ 0 \leq z_{2n-i} \leq \gamma^n z_{-i}, \tag{4.6}$$

where x_{-i}, y_{-i}, z_{-i} $(i = 0,1)$ are the initial conditions. Consequently, by taking limits of inequalities in (4.6) when $\alpha, \beta, \gamma \in (0,1)$, we get $\lim_{n\to\infty} (x_n, y_n, z_n) = (0,0,0)$. This completes the proof. ■

Example 4.2.3 *Figure (18) shows the global attractivity of the zero equilibrium point $\overline{x}$ of System (4.3) for the values $\alpha = 0.9$, $\beta = 0.2$, $\gamma = .5$, $p = 2$, $q = 0.3$, and $r = 5$ whenever $x_{-3} = 1.04$, $x_{-2} = 2.6$, $x_{-1} = 1.02$, $x_0 = 3.04$, $y_{-3} = 1.3$, $y_{-2} = 3.9$, $y_{-1} = 0.4$, $y_0 = 1.2$, $z_{-3} = 1.5$, $z_{-2} = 2.3$, $z_{-1} = 0.9$, and $z_0 = 0.006$.*

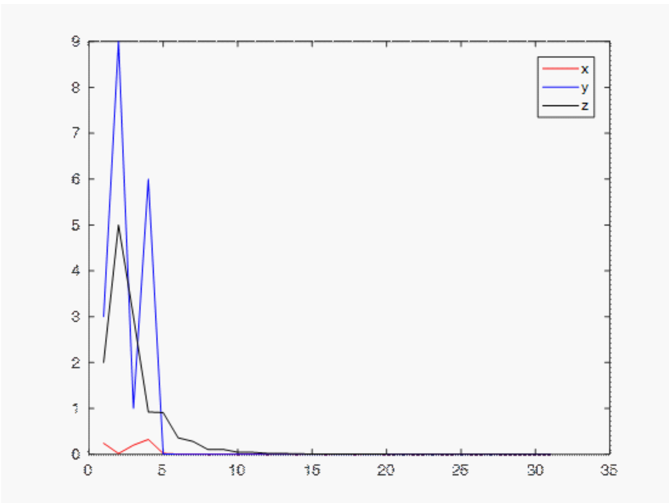


Figure (18)

4.2.3 Study of 2-Periodic solutions

Here we show that there is a prime two periodic solution.

Theorem 4.2.4 *If $\alpha = \beta = \gamma = 1$, then every solution of System (4.3) tends a period two solution.*

Proof. We get from System (4.3)

$$\begin{aligned} x_{2n+1} - x_{2n-1} &= -\frac{x_{2n-1} z_{n-3}^p}{1 + z_{n-3}^p} \leq 0,\ y_{2n+1} - y_{2n-1} = -\frac{y_{2n-1} x_{n-3}^q}{1 + x_{n-3}^q} \leq 0, \\ z_{2n+1} - z_{2n-1} &= -\frac{z_{2n-1} y_{n-3}^r}{1 + y_{n-3}^r} \leq 0. \end{aligned}$$

and

$$\begin{aligned} x_{2n+2} - x_{2n} &= -\frac{x_{2n} z_{2n-2}^p}{1 + z_{2n-2}^p} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{y_{2n} x_{2n-2}^q}{1 + x_{2n-2}^q} \leq 0, \\ z_{2n+2} - z_{2n} &= -\frac{z_{2n} y_{2n-2}^r}{1 + y_{2n-2}^r} \leq 0. \end{aligned}$$

Thus, we get

$$x_{2n+1} \leq x_{2n-1},\ y_{2n+1} \leq y_{2n-1},\ z_{2n+1} \leq z_{2n-1},\ x_{2n+2} \leq x_{2n},\ y_{2n+2} \leq y_{2n},$$

and

$$z_{2n+2} \leq z_{2n}.$$

The sequences $\{(x_{2n-1}, y_{2n-1}, z_{2n-1})\}_{n=-3}^{\infty}$ and $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ are non-increasing. Hence, while the odd-index terms tend to one periodic point, the even-index terms tend to another periodic point. This completes the proof. ∎

Theorem 4.2.5 *Assume that $\alpha = \beta = \gamma = 1$, then every solution $\{(x_n, y_n, z_n)\}_{n=-3}^{\infty}$ of System (4.3) converges to a period two solution. Moreover the sequence $\{x_n\}$ converges to a period solution of the form*

$$..., \varphi, \psi, \varphi, \psi, ...,$$

also the sequence $\{y_n\}$ converges to a period two solution

$$..., \gamma, \delta, \gamma, \delta, ...,$$

and the sequence $\{z_n\}$ converges to a period two solution

$$..., \lambda, \mu, \lambda, \mu, ...,$$

and the solution has the form

$$\{(0, 0, 0), (\psi, \delta, \mu), (0, 0, 0), ...\}.$$

Proof. We have from System (4.3)

$$
\begin{aligned}
x_{n+1}-x_{n-1} &= -\frac{x_{n-1}y_{n-3}^{p}}{1+y_{n-3}^{p}} \leq 0,\ y_{n+1}-y_{n-1} = -\frac{y_{n-1}z_{n-3}^{q}}{1+z_{n-3}^{q}} \leq 0,\\
z_{n+1}-z_{n-1} &= -\frac{z_{n-1}x_{n-3}^{r}}{1+x_{n-3}^{r}} \leq 0,
\end{aligned}
$$

which imply that $\{x_n\}$ converges to a period two solution

$$..., \varphi, \psi, \varphi, \psi, ...,$$

also $\{y_n\}$ converges to a period two solution

$$..., \gamma, \delta, \gamma, \delta, ...,$$

and $\{z_n\}$ converges to a period two solution

$$..., \lambda, \mu, \lambda, \mu, ...$$

If we assume that

$$\lim_{n\to\infty} x_{2n} = \varphi,\ \lim_{n\to\infty} x_{2n+1} = \psi,\ \lim_{n\to\infty} y_{2n} = \gamma,\ \lim_{n\to\infty} y_{2n+1} = \delta,\ \lim_{n\to\infty} z_{2n} = \lambda,$$

and

$$\lim_{n\to\infty} z_{2n+1} = \mu,$$

then we have

$$\varphi = \frac{\varphi}{1+\gamma^{p}},\ \psi = \frac{\psi}{1+\gamma^{p}},\ \gamma = \frac{\gamma}{1+\lambda^{r}},\ \delta = \frac{\delta}{1+\lambda^{r}},\ \lambda = \frac{\lambda}{1+\varphi^{q}},\ \mu = \frac{\mu}{1+\varphi^{q}}$$

which implies that $\gamma = \lambda = \varphi = 0$. Then the proof is completed. ■

Example 4.2.6 *Figure (19) shows that the solutions of System (4.3) tend to a period two solution of System (4.3) for the values* $\alpha = \beta = \gamma = 1$, $p = 3$, $q = 3$, *and* $r = 3$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 4$, $y_{-3} = 0.3$, $y_{-2} = 0.9$, $y_{-1} = 4$, $y_0 = 2$, $z_{-3} = 0.5$, $z_{-2} = 2.3$, $z_{-1} = 0.9$, *and* $z_0 = 6$.

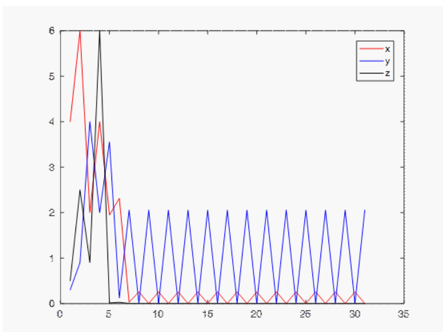


Figure (19)

4.2.4 Oscillatory Charactor

Here we dell with the oscillation of the positive solutions of System (4.3) about the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\gamma-1)^{\frac{1}{q}}, (\alpha-1)^{\frac{1}{p}}, (\beta-1)^{\frac{1}{r}})$.

Theorem 4.2.7 *Let $\alpha, \beta, \gamma \in (0, \infty)$ and $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ be a positive solution of System (4.3). Then, $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ oscillates about the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$. Moreover, with the possible exception of the first semicycle, every semicycle has length one.*

Proof. Assume that

(i) $x_{-1}, x_{-3} \geq \overline{x}_2,\ x_0, x_{-2} < \overline{x}_2$ *or* $x_{-1}, x_{-2} < \overline{x}_2,\ x_{-3}, x_0 \geq \overline{x}_2,\ y_{-1}, y_{-3} \geq \overline{y}_2,\ y_0, y_{-2} < \overline{y}_2,\ z_0, z_{-2} \geq \overline{z}_2,\ z_{-1}, z_{-3} < \overline{z}_2$

holds. Then we get

$$x_1 = \frac{\alpha x_{-1}}{1+y_{-3}^p} < \overline{x}_2, x_2 = \frac{\alpha x_0}{1+y_{-2}^p} \geq \overline{x}_2, x_3 = \frac{\alpha x_1}{1+y_{-1}^p} < \overline{x}_2, x_4 = \frac{\alpha x_2}{1+y_0^p} \geq \overline{x}_2$$

$$y_1 = \frac{\beta y_{-1}}{1+z_{-3}^r} \geq \overline{y}_2, y_2 = \frac{\beta y_0}{1+z_{-2}^r} < \overline{y}_2, y_3 = \frac{\beta y_1}{1+z_{-1}^r} \geq \overline{y}_2, y_4 = \frac{\beta y_2}{1+z_0^r} < \overline{y}_2$$

$$z_1 = \frac{\gamma z_{-1}}{1+x_{-3}^q} < \overline{z}_2, z_2 = \frac{\gamma z_0}{1+x_{-2}^q} \geq \overline{z}_2, z_3 = \frac{\gamma z_1}{1+x_{-1}^q} < \overline{z}_2, z_4 = \frac{\gamma z_2}{1+x_0^q} \geq \overline{z}_2$$

Then, the result follows by induction. (ii) $x_{-1}, x_{-3} < \overline{x}_2,\ x_0, x_{-2} \geq \overline{x}_2$ *or* $x_{-1}, x_{-2} \geq \overline{x}_2,\ x_{-3}, x_0 < \overline{x}_2,\ y_{-1}, y_{-3} < \overline{y}_2,\ y_0, y_{-2} \geq \overline{y}_2,\ z_0, z_{-2} < \overline{z}_2,\ z_{-1}, z_{-3} \geq \overline{z}_2$. The proof of this case is similarly to case (i) will be omitted. ■

4.2.5 Unboundedness of the Solutions of System (4.3)

In the following theorem, we show the existence of unbounded solutions for System (4.3)

Theorem 4.2.8 *If $\alpha, \beta, \gamma \in (1, \infty)$, then System (4.3) possesses an unbounded solution.*

Proof. Assume that $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ be a solution of System (4.3) with $x_{2n-3} < \overline{x}_2,\ x_{2n-2} \geq \overline{x}_2,\ y_{2n-3} \geq \overline{y}_2,\ y_{2n-2} < \overline{y}_2,\ z_{2n-3} < \overline{z}_2,$ *and* $z_{2n-2} \geq \overline{z}_2$ for $n \in \mathbb{N}_0$.Then, we have

$$x_{2n+2} = \frac{\alpha x_{2n}}{1+y_{2n-2}^p} \geq x_{2n}, y_{2n+1} = \frac{\beta y_{2n-1}}{1+z_{2n-3}^r} \geq y_{2n-1}, z_{2n+1} = \frac{\gamma z_{2n-1}}{1+x_{2n-3}^q} \geq z_{2n-1},$$

$$x_{2n+1} = \frac{\alpha x_{2n-1}}{1 + y_{2n-3}^{p}} < x_{2n-1}, y_{2n+2} = \frac{\beta y_{2n}}{1 + z_{2n-2}^{r}} < y_{2n}, z_{2n+1} = \frac{\gamma z_{2n}}{1 + x_{2n-2}^{q}} < z_{2n}.$$

from which it follows that $\lim_{n\to\infty}(x_{2n}, y_{2n-1}, z_{2n-1}) = (\infty, \infty, \infty)$ and $\lim_{n\to\infty}(x_{2n-1}, y_{2n}, z_{2n}) = (0, 0, 0)$.

This completes the proof. ■

Example 4.2.9 *Figure (20) shows that System (4.3) has unbounded solutions with the values* $\alpha = 1.02, \beta = 1.09, \gamma = 1.05$*, and* $p = q = r = 3$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$, $y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = 0.4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = 0.23$, *and* $z_0 = 3$.

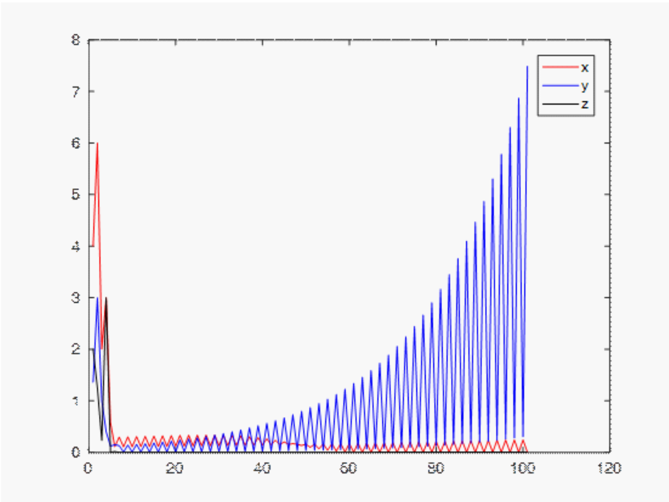


Figure (20)

4.3 Case 2. System (4.1) when $p_1 = q_1 = r_1 = 1$.

Now we will investigate the stability of the two equilibrium points of System (4.1) when $p_1 = q_1 = r_1 = 1$. Then from System (4.1) we get

$$u_{n+1} = \frac{a u_{n-1}}{b + c v_{n-3}^{p} w_{n-1}}, v_{n+1} = \frac{d v_{n-1}}{e + f w_{n-3}^{q} u_{n-1}}, w_{n+1} = \frac{g w_{n-1}}{h + I u_{n-3}^{r} v_{n-1}}, \quad n \in \mathbb{N}_0 \quad (4.7)$$

By the change of variables $u_n = (\frac{h}{I})^{\frac{1}{r}} x_n, v_n = (\frac{b}{c})^{\frac{1}{p}} y_n, w_n = (\frac{e}{f})^{\frac{1}{q}} z_n$. System (4.7) can be rewritten as

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + s y_{n-3}^{p} z_{n-1}}, \quad y_{n+1} = \frac{\beta y_{n-1}}{1 + t z_{n-3}^{r} x_{n-1}}, \quad z_{n+1} = \frac{\gamma z_{n-1}}{1 + x_{n-3}^{q} y_{n-1}} \qquad (4.8)$$

where $\alpha = \frac{a}{b}, \beta = \frac{d}{e}, \gamma = \frac{g}{h}$, and $s = (\frac{e}{f})^{\frac{1}{q}}, t = (\frac{h}{I})^{\frac{1}{r}}, k = (\frac{b}{c})^{\frac{1}{p}}$.

4.3.1 Stability of System (4.8)

In this subsection, we investigate the stability of the two equilibrium points of System (4.8). When $\alpha, \beta, \gamma \in (0, 1)$, it is easy to see that $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ is the unique equilibrium point of System (4.8). When $\alpha, \beta, \gamma \in (1, \infty)$, the unique positive equilibrium point of System (4.8) is $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = ((\frac{\gamma-1}{k})^{\frac{1}{r+1}}, (\frac{\alpha-1}{s})^{\frac{1}{p+1}}, (\frac{\beta-1}{t})^{\frac{1}{q+1}})$.

Theorem 4.3.1 *The following statements hold:*

(i) If $\alpha, \beta, \gamma \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of System (4.8) is locally asymptotically stable.

(ii) If $\alpha \in (1, \infty)$ or $\beta \in (1, \infty)$ or $\gamma \in (1, \infty)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$of System (4.8) is unstable.

(iii) If $\alpha, \beta, \gamma \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ of System (4.8) is unstable.

Proof. We rewrite System (4.8) in the form

$$X_{n+1} = F(X_n)$$

where $X_n = (x_n, ..., x_{n-3}, y_n, ..., y_{n-3}, z_n, ..., z_{n-3})^T$ and the map F is given by

$$F\begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ m_0 \\ m_1 \\ m_2 \\ m_3 \\ l_0 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha n_1}{1+sm_3^p l_1} \\ n_0 \\ n_1 \\ n_2 \\ \frac{\beta m_1}{1+tl_3^q n_1} \\ m_0 \\ m_1 \\ m_2 \\ \frac{\gamma l_1}{1+kn_3^r m_1} \\ l_0 \\ l_1 \\ l_2 \end{pmatrix}.$$

The linearized system of (4.4) about the equilibrium point $\overline{X} = (0, ..., 0)^T$ is given by

$$X_{n+1} = J_F(\overline{X}_0)X_n,$$

where

$$J_F(\overline{X}_0) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus the characteristic equation of $J_F(\overline{X}_0)$ is given by

$$\lambda^6(\lambda^2-\alpha)(\lambda^2-\beta)(\lambda^2-\gamma)=0. \tag{4.9}$$

We have the following: (i) If $\alpha,\beta,\gamma\in(0,1)$,all roots of the characteristic equation (4.9) lie inside the open unit disk $\mid\lambda\mid<1$. So, the unique equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ of System (4.8) is locally asymptotically stable.

(ii) If $\alpha\in(1,\infty)$ *or* $\beta\in(1,\infty)$ *or* $\gamma\in(1,\infty)$, then some roots of Eq.(4.9) have absolute values greater than one. Thus, the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)=(0,0,0)$ is unstable.

(iii) The linearized system of (4.4) about the positive equilibrium point $(\overline{x}_2,\overline{y}_2,\overline{z}_2)$ is given by

$$X_{n+1}=J_F(\overline{X}_{\alpha,\beta,\gamma})X_n.$$

where

$$X_n=\begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ z_n \\ z_{n-1} \\ z_{n-2} \\ z_{n-3} \end{pmatrix},\ J_F(\overline{X}_{\alpha,\beta,\gamma}) = \begin{pmatrix} 0 & A & 0 & 0 & 0 & 0 & 0 & B & 0 & C & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & F \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & H & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$A = \frac{\alpha t^{\frac{1}{q+1}}}{t^{\frac{1}{q+1}} + (s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}}}, \quad B = -\frac{p\alpha s^{\frac{2}{p+1}} t^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}(\beta-1)^{\frac{1}{q+1}}(\alpha-1)^{\frac{p-1}{p+1}}}{k^{\frac{1}{r+1}}(t^{\frac{1}{q+1}} + (s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}})^2},$$

$$C = -\frac{\alpha t^{\frac{2}{q+1}}(s(\alpha-1)^p)^{\frac{1}{p+1}}(\gamma-1)^{\frac{1}{r+1}}}{k^{\frac{1}{r+1}}(t^{\frac{1}{q+1}} + (s(\alpha-1)^p)^{\frac{1}{p+1}}(\beta-1)^{\frac{1}{q+1}})^2},$$

$$D = -\frac{\beta k^{\frac{2}{r+1}}(t(\beta-1)^q)^{\frac{1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}}(k^{\frac{1}{r+1}} + (t(\beta-1)^q)^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}})^2}, \quad E = \frac{\beta k^{\frac{1}{r+1}}}{k^{\frac{1}{r+1}} + (t(\beta-1)^q)^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}},$$

$$F = -\frac{\beta q t^{\frac{2}{q+1}} k^{\frac{1}{r+1}\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}}(\beta-1)^{\frac{q-1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}}(k^{\frac{1}{r+1}} + (t(\beta-1)^q)^{\frac{1}{q+1}}(\gamma-1)^{\frac{1}{r+1}})^2},$$

$$\begin{aligned} G &= -\frac{\gamma r k^{\frac{2}{r+1}} s^{\frac{1}{p+1}}(\gamma-1)^{\frac{r-1}{r+1}}(\beta-1)^{\frac{1}{q+1}}(\alpha-1)^{\frac{1}{p+1}}}{t^{\frac{1}{q+1}}(s^{\frac{1}{p+1}} + (k(\gamma-1)^r)^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}})^2}, \\ H &= -\frac{\gamma s^{\frac{2}{p+1}}(k(\gamma-1)^r)^{\frac{1}{r+1}}(\beta-1)^{\frac{1}{q+1}}}{t^{\frac{1}{q+1}}(s^{\frac{1}{p+1}} + (k(\gamma-1)^r)^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}})^2}, \end{aligned}$$

and

$$I = \frac{\gamma s^{\frac{1}{p+1}}}{s^{\frac{1}{p+1}} + (k(\gamma-1)^r)^{\frac{1}{r+1}}(\alpha-1)^{\frac{1}{p+1}}}.$$

The characteristic equation of $J_F(\overline{X}_{\alpha,\beta,\gamma})$ is given by

$$\begin{aligned} p(\lambda) &= \lambda^{12} - (A+E+I)\lambda^{10} + (EI+AE+AI)\lambda^8 \\ &\quad -(CG+FH+BD+CHD+AEI)\lambda^6 + (BDI+AFH+CGE)\lambda^4 - BFG. \end{aligned}$$

Therefor

$$p(0) = -BFG < 0 \quad \text{and} \quad \lim_{\lambda\to\infty} p(\lambda) = \infty.$$

Then $p(\lambda)$ has at least one root in the interval $(1,\infty)$. So by Theorem D we say that if $\alpha,\beta,\gamma \in (0,\infty)$, then the positive equilibrium point $(\overline{x}_2,\overline{y}_2,\overline{z}_2)$ of System (4.8) is unstable. This completes the proof. ■

4.3.2 Global Stability of System (4.8)

Here we investigate the global attractor of System (4.8) to its zero equilibrium point.

Theorem 4.3.2 *If $\alpha,\beta,\gamma \in (0,1)$, then the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1) = (0,0,0)$ of System (4.8) is globally asymptotically stable.*

Proof. We proved in Theorem 4.3.1 that if $\alpha, \beta, \gamma \in (0,1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of System (4.8) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\to\infty} (x_n, y_n, z_n) = (0,0,0).$$

We see from System (4.8) that, for $n \in N_0$

$$\begin{aligned} 0 &\leq x_{n+1} = \frac{\alpha x_{n-1}}{1 + s y_{n-3}^{p} z_{n-1}} \leq \alpha x_{n-1},\ 0 \leq y_{n+1} = \frac{\beta y_{n-1}}{1 + t z_{n-3}^{q} x_{n-1}} \leq \beta y_{n-1}, \\ 0 &\leq z_{n+1} = \frac{\gamma z_{n-1}}{1 + k x_{n-3}^{r} y_{n-1}} \leq \gamma z_{n-1}. \end{aligned}$$

Then it follows by induction that

$$0 \leq x_{2n-i} \leq \alpha^n x_{-i}, 0 \leq y_{2n-i} \leq \beta^n y_{-i}, 0 \leq z_{2n-i} \leq \gamma^n z_{-i}. \tag{4.10}$$

where $x_{-i}, y_{-i}, z_{-i} (i = 0, 1)$ are the initial conditions. Consequently, by taking limits of inequalities in (4.10), we get $\lim_{n\to\infty} (x_n, y_n, z_n) = (0,0,0)$. ■

Example 4.3.3 *Figure (21) shows the global attractivity of the zero equilibrium point $\overline{x}$ of System (4.8) for the values* $\alpha = 0.011$, $\beta = 0.827$, $\gamma = 0.021$, $p = 0.003$, $q = 0.01283$, $r = 0.343$, $s = 1$, $t = 3$, *and* $k = 2$ *whenever* $x_{-3} = 1.04$, $x_{-2} = 2.6$, $x_{-1} = 1.02$, $x_0 = 3.04$, $y_{-3} = 1.3$, $y_{-2} = 3.9$, $y_{-1} = 0.4$, $y_0 = 1.2$, $z_{-3} = 1.5$, $z_{-2} = 2.3$, $z_{-1} = 0.9$, *and* $z_0 = 0.006$.

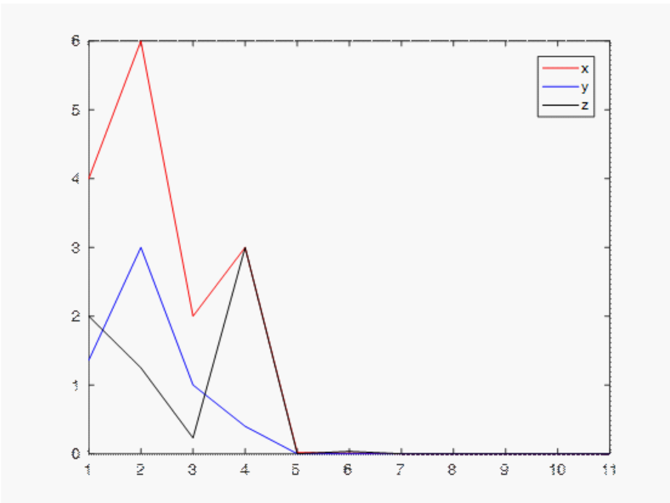


Figure (21)

4.3.3 Study of 2-Periodic Solutions of System (4.8

In the following theorem, we investigate the convergence of the period solutions period two of System (4.8).

Theorem 4.3.4 *If $\alpha = \beta = \gamma = 1$, then every solution of System (4.8) tends to a period two solution.*

Proof. We get from System (4.8) that

$$\begin{aligned} x_{2n+1} - x_{2n-1} &= -\frac{sx_{2n-1}y_{2n-3}^{p}z_{2n-1}}{1+sy_{2n-3}^{p}z_{2n-1}} \leq 0,\ y_{2n+1} - y_{2n-1} = -\frac{ty_{2n-1}z_{2n-3}^{q}x_{2n-1}}{1+tz_{2n-3}^{q}x_{2n-1}} \leq 0, \\ z_{2n+1} - z_{2n-1} &= -\frac{ky_{2n-1}x_{2n-3}^{r}z_{2n-1}}{1+kx_{2n-3}^{r}y_{2n-1}} \leq 0 \end{aligned}$$

and

$$\begin{aligned} x_{2n+2} - x_{2n} &= -\frac{sx_{2n}y_{2n-2}^{p}z_{2n}}{1+sy_{2n-2}^{p}z_{2n}} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{ty_{2n}z_{2n-2}^{q}x_{2n}}{1+tz_{2n-2}^{q}x_{2n}} \leq 0, \\ z_{2n+2} - z_{2n} &= -\frac{ky_{2n}x_{2n-2}^{r}z_{2n}}{1+kx_{2n-2}^{r}y_{2n}} \leq 0, \end{aligned}$$

also

$$\begin{aligned} x_{2n+2} - x_{2n} &= -\frac{sx_{2n}y_{2n-2}^{p}z_{2n}}{1+sy_{2n-2}^{p}z_{2n}} \leq 0,\ y_{2n+2} - y_{2n} = -\frac{ty_{2n}z_{2n-2}^{q}x_{2n}}{1+tz_{2n-2}^{q}x_{2n}} \leq 0, \\ z_{2n+2} - z_{2n} &= -\frac{ky_{2n}x_{2n-2}^{r}z_{2n}}{1+kx_{2n-2}^{r}y_{2n}} \leq 0. \end{aligned}$$

Thus we get

$$x_{2n+1} \leq x_{2n-1},\ y_{2n+1} \leq y_{2n-1},\ z_{2n+1} \leq z_{2n-1},\ x_{2n+2} \leq x_{2n},\ y_{2n+2} \leq y_{2n},$$

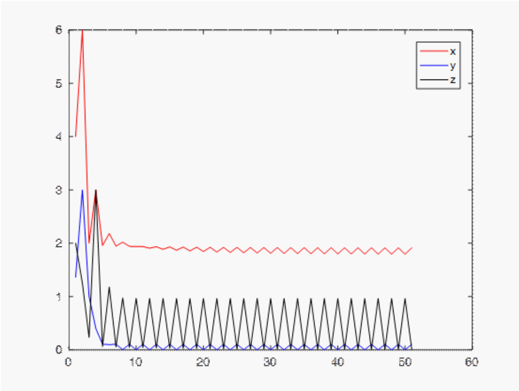
and

$$z_{2n+2} \leq z_{2n}.$$

That is , the sequences $\{(x_{2n-1}, y_{2n-1}, z_{2n-1})\}_{n=-3}^{\infty}$ and $\{(x_{2n}, y_{2n}, z_{2n})\}_{n=-3}^{\infty}$ are non-increasing. Hence, while the odd-index terms tend to one periodic point, the even-index terms tend to another periodic point. This completes the proof. ■

Example 4.3.5 *Figure (22) shows that the solutions of (4.8) tend to a period two solution of System (4.8) for the values $\alpha = \beta = \gamma = 1$, $p = 0.3$, $q = 0.8$, $r = 3$ and $s = 0.09$, $r = 0.54$, and $k = 0.922$ whenever $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$,*

$y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = 0.4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = 0.23$, *and* $z_0 = 3$.



$Figure$ (22)

Example 4.3.6 *Figure (23) shows that System (4.8) has an unbounded solution with* $\alpha = 1.02$, $\beta = 1.09$, $\gamma = 1.05$, $p = 3$, $q = 3$, $r = 3$**,** $s = 0.09$, $r = 1.54$, *and* $k = 0.922$ *whenever* $x_{-3} = 4$, $x_{-2} = 6$, $x_{-1} = 2$, $x_0 = 3$, $y_{-3} = 1.36$, $y_{-2} = 3$, $y_{-1} = 1$, $y_0 = 0.4$, $z_{-3} = 2$, $z_{-2} = 1.25$, $z_{-1} = 0.23$, *and* $z_0 = 3$.

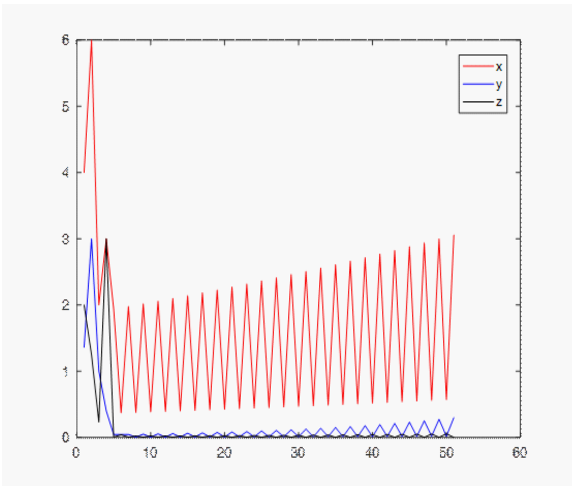


Figure (23)

Bibliography

[1] R. Agarwal, Difference Equation and Inequalities. Theory, Methods and Applications, Marcel Dekker Inc, New York, 1992.

[2] M. Aloqeili, Dynamics of a rational difference Equation, Appl. Math. Comp., 176(2) (2006), 768-774.

[3] A. M. Amleh, J.Hoag and G. Ladas, A difference Equation with eventually periodic solutions, Comp. Math. Appl., 36 (1998), 401-404.

[4] A. M. Amleh, V. Kirk and G. Ladas, On th dynamics of $x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_{n-2}}$, Math. Sci. Res. Hot-Line, 5 (2001), 1-15.

[5] A.M. Amleh, E.A. Grove, and G. Ladas, On the recursive sequence $x_{n+1} = a + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl.,vol. 233, pp. 790-798 (1999).

[6] L. S. Allen, M. K. Hannigan, and M. J. Strauss, Mathematical analysis of a model for a plant-herbivore system, Bulletin of Mathematical Biology, Vol. 55, 847-864 (1993).

[7] C. Cinar," on the positive of the difference equation system $x_{n+1} = \frac{1}{y_n}$, $y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$. Appl Math Comput. 158, 303-305 (2004). doi: 10.1016/j.amc.2003.08.073.

[8] D. Clark, M. R. S. Kulenovi c and J. F. Selgrade, Global asymptotic behavior of a two-dimensional difference equation modelling competition, Nonlinear Analysis, 52(2003), 1765-1776.

[9] R. Devault, E. A. Grove, G. Ladas, R. Levins, and C. Puccia, Oscillation and Stability in Models of a Perennial Grass, The Proceedings of Dynamic Systems and Applications-1, 87-93, Dynamic Publishers, Inc., U.S.A., 1994.

[10] R. Devault, V.L. Kocic & D. Stutson,"Global behavior of solution of the non-linear difference equation," Journal of Difference Equations and Applications, vol. 11, No. 8, July 2005, 707-719.

[11] R. Devault[a,*], C. Kent[b] and W. Kosmala[c] " On the recursive sequence $x_{n+1} = a + \frac{x_{n-k}}{x_n}$" ,J. Diff. Equa. Appl, Vol. 9(8), pp. 721-730 (2003).

[12] S. Elaydi, An Introduction to Difference Equations, Second Edition, Springer-Verlag, New York, (1999).

[13] E.M.Elabbasy, H.N.Agiza, A.A.Elsadany and H. El-Metwally, "The Dynamics of Triopoly Game with Heterogeneous Players", International Journal of Nonlinear Science Vol. 3 ,pp. 83-90 (2007).

[14] E.M. Elabbasy, H. El-Metwally and E.M. Elsayed, Qualitative Behavior of a Higher Order Difference Equation, Soochow Journal of Mathematical. Vol. 33, No. 4, (2007) 861-873.

[15] E.M. Elabbasy, H. El-Metwally and E.M. Elsayed, Some Properties and Expressions of Solutions for a Class of Nonlinear Difference Equation, Utilitas Mathematica, Vol. 87 (2012) 93-110.

[16] E. M. Elabbasy, M. Barsoum & H. S. Alshawee,"On the dynamics of a higher-order rational difference equation," Asial J. Math. And Applic, vol 2016, Article ID ama0291, 13 pages. ISSN 2307-7743.

[17] H. El-Metwally, Global Behavior of an Economic Model, Chaos Solitons & Fractals, Vol. 33, pp. 994-1005 (2007).

[18] H. El-Metwally and A. A. Elsadany, “Qualitative study of a bounded rationality duopoly game with exponential demand function", Utilitas Mathematica. vol. 88, pp. 337–354 (2012).

[19] H. El-Metwally, E. A. Grove and G. Ladas, A global convergence result with applications to periodic solutions, J. Math. Anal. Appl. vol. 245, pp. 161-170 (2000).

[20] H. El-Metwally, "Qualitative Proprieties of some Higher Order Difference Equations", Computers and Mathematics with Applications. Vol.58, No.4, (2009) 686-692.

[21] H. El-Metwally, On the Dynamics of a Higher Order Difference equation, Discrete Dynamics in Nature and Society Volume 2012 (2012), Article ID 263053, 8 pages doi:10.1155/2012/263053.

[22] H. El-Metwally, Solutions Form of some Rational Systems of Difference Equations, Discrete Dynamics in Nature and Society, Volume 2013, (2013), Article ID 903593, 11 pages, doi:10.1155/2013/903593.

[23] H. El-Metwally, M. M. El-Afifi, On the Behavior of some Extension Forms of some Population Models, Chaos Solitons & Fractals, 36 (2008) 104-114.

[24] H. El-Metwally, On the Structure and the Qualitative Behavior of an Economic Model, Advances in Difference Equations, 2013, 2013:169 doi:10.1186/1687-1847-2013-169.

[25] H. El-Metwally, I. Yalcinkaya and C. Cinar, Global stability of an economic model, Utilitas Mathematical, 95(2014), 235-244.

[26] Emmanouil Drymonis, Yevgeniy Kostrov & Zachary Kudlak, " On rational difference equations with nonnegative periodic coefficients," International Journal of Difference Equations, vol. 7, no. 1, pp. 19-34(2012).

[27] C. H. Gibbons, M. R. S. Kulenovi´c, G. Ladas and H. D. Voulov, On the Trichotomy Character of $x_{n+1} = \frac{\alpha+\beta x_{n-1}+\gamma x_{n-1}}{A+x_n}$, J. Diff. Equa. App., vol. 7 (2001).

[28] E. A. Grove, V. L. Kocic, G. Ladas and R. Levins, Periodicity in a simple genotype selection model, Differential Equations and Dynamical Systems, 1(1993), 35-50.

[29] Grove, E.A. and Ladas, G. Periodicities in Nonlinear Difference Equations. Chapman and Hall/CRC, Boca Raton, (2005).

[30] M. Gumus and Y. Soykan, Global character of a six-dimensional nonlinear system of difference equations, Discrete Dynamics in Nature and Society, Article ID 6842521, (2016).

[31] Y. K. Kuang and J. M. Cushing, Global stability in a Nonlinear Difference-delay Equation Model of Flour Beetle Population Growth, J. Differ. Equations Appl., 2(1)(1996), 31-37.

[32] A. S. Kurbanli, C. Cinar and I. Yalcinkaya, On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = x_{n-1}/(y_n x_{n-1}+1)$, $y_{n+1} = y_{n-1}/(x_n y_{n-1} + 1)$, Mathematical and Computer Modelling, 53(5-6)(2011), 1261-1267.

[33] M. R. S. Kulenović and G. Ladas. Dynamics of second order rational rational difference equations. Chapman & Hall/CRC. Boca Raton, FL, 2002. With open problems and conjectures.

[34] V. L. Kocić and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Mathematics and Its Applications, vol. 256, Kluwer Academic, Dorderecht, 1993.

[35] G. Ladas, A. Tovbis and G. Tzanetopoulos, On May's host Parasitoid Model, J. Differ. Equations Appl., 2(1996), 195-204.

[36] G. Papaschinopoulos*, C.J. Schinas, G. Stefanidou, "On the nonautonomous difference equation $x_{n+1} = a + \frac{x^p_{n-1}}{x^p_n}$", J. App. Math. and Comp. vol. 217, pp. 5573-5580 (2011).

[37] G. Papaschinopoulos, G. Ellina and K. B. Papadopoulos, Asymptotic behavior of the positive solutions of an exponential type system of difference equations, Applied Mathematics and Computation, 245(2014), 181-190.

[38] S. Stević, "On the recursive sequence $x_{n+1} = \frac{x_{n-1}}{g(x_n)}$", Taiwanese Journal of Mathematics, Vol. 6, No. 3, pp. 405-414, September 2002.

[39] S. Stević, "On the recursive sequence $x_{n+1} = A + \frac{x_n^p}{x_{n-1}^p}$", Discrete Dynamics in Nature Society, Article ID 34517, 9 pages. Doi: 10.1155/2007/34517.

[40] Özkan Öcalan & Mehmet Gümüs,"Global analysis of a non-autonomous difference equation with bounded coefficient," Electronic Journal of Mathematical Analysis and Applications, vol. 4(1) Jan. 2016, pp. 184-191.

[41] D. T. Tollu, Y. Yazlik and N. Taskara, On the solutions of two special types of Riccati difference equation via Fibonacci numbers, Advances in Difference Equations, Article ID 174, (2013).

[42] D. T. Tollu, Y. Yazlik and N. Taskara, On fourteen solvable systems of difference equations, Applied Mathematics and Computation, 233(2014), 310-319.

[43] I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, Discrete Dynamics in Nature and Society, vol. 2008, Article ID 860152, 12 pages.

[44] L. Yang and J. Yang, Dynamics of a system of two nonlinear difference equations, International Journal of Contemporary Mathematical Sciences, 6(2011), 209-214.

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