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Research Article

ON PRE-$T_{\frac{3}{4}}$ SPACES

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ABSTRACT

Since every topological space is pre-$T_{\frac{1}{2}}$ space, we aim to define a space that is weaker than pre-T_1 space; namely pre-$T_{\frac{3}{4}}$ space, when we use the notions of regular open sets and preopen sets. We prove that a pre-$T_{\frac{3}{4}}$ space is weaker than $T_{\frac{3}{4}}$-space, regular space and pre-R_0 space, additionally, we investigate the topological properties of pre-$T_{\frac{3}{4}}$ space, as the hereditary property and their images by some paritcular functions; moreover we discuss the behavior of pre-$T_{\frac{3}{4}}$ axiom in some special spaces as; submaximal spaces, regular spaces, extremelly disconnected spaces and hyperconnected spaces.

Keywords: Topological space and generalizations, generalized continuity, separation axioms, subspaces, submaximal spaces, regular spaces, hyperconnected space, extremely disconnected space.

AMS Subject Classification (2000): 54A05, 54C08, 54D70, 54B05, 54D10, 54D65.

INTRODUCTION

In 1937 Stone [1], introduced regular open sets using the closure and interior operators, and he showed that regular open set is stronger than open set, then he applied the family of regular open sets in the semi regularization space and in a generalization for algebraic openings and closings in a complete lattice. More details can be found in [2-8]. Corson and Michael in 1964 [9] defined the concepts of preopen sets by the name "locally dense" sets, as a generlaizations of open sets, this class of sets plays a significant role in general topology. See more in [10-15]. In the literature many studies have been made on defining different separation axioms using some generalizations of open sets and closed sets, where the characterization of these spaces found to be useful in the study of computer science and digital topology. Levine, Maki, Dunham, Dontchev, Araki and Arenas, introduced several low separation axioms that lie between T_0 and T_1-spaces, in particular, they used the concepts of ʌ-sets, generalized ʌ-sets, λ-sets, generalized closed sets and regular open sets to defined the axioms of $T_{\frac{1}{4}}$, $T_{\frac{1}{3}}$, $T_{\frac{1}{2}}$ and $T_{\frac{3}{4}}$ spaces [16-23]. The concept of generalized closed set was due to Levine in 1970 [16], when he used this notation to define a space called $T_{\frac{1}{2}}$-space, and he showed that $T_{\frac{1}{2}}$ is strictly between the spaces T_1 and T_0, in addition, he produced new separation axiom which lies between $T_{\frac{1}{2}}$-space and T_1-space, called $T_{\frac{3}{4}}$-space. After that, the authors in [17] characterized $T_{\frac{1}{2}}$-spaces as those spaces where every subset is λ-closed. Dunham [18] showed that a space is $T_{\frac{1}{2}}$ if and only if each singletons is open or closed.

Pre-separation axioms consist the spaces; pre-T_0, pre-T_1, pre-T_2, pre-regular and pre-normal spaces and they were defined by many researchers as Kar and Bhattacharyye [24], by replacing the notion of open sets with preopen sets in the classical definitions. In 1995 [25], Dontchev defined pre-$T_{\frac{1}{2}}$ space as a space in which every singleton is either preopen or preclosed, and he used the property that any singleton is either preopen or nowhere dense set which due to Jankovic and Reilly [26], to prove that any topological space is pre-$T_{\frac{1}{2}}$ space. Since any topological space is pre-$T_{\frac{1}{2}}$, we aim to define a space that is weaker than pre-T_1 space and $T_{\frac{3}{4}}$-space; called pre-$T_{\frac{3}{4}}$ space, when we use the concepts of regular open sets and preopen sets. In this paper, we study the topological properties of this space, as the hereditary property and their images by some particular functions. Moreover, we characterized this space by the new notion of pgδ-closed set. Finally, we investigate the behavior of pre-$T_{\frac{3}{4}}$ space in some special spaces as; regular spaces, submaximal spaces, partition spaces, hyperconnected spaces and extremely disconnected spaces. We divided this article into six sections, as follows: in section one we present the introduction, then we review some preliminaries in section two, regarding regular open sets, preopen sets and some types of separation axioms that we need in our study. In section three, we define pre-$T_{\frac{3}{4}}$ space using the notion of regular open sets and preclosed sets, and then we study their relation with the classical separation axioms, and investigate the subspace behavior and their images by some particular maps. Section four contains a new concepts of pgδ-closed set as a generalization of closed set, and then we used this concept to study pre-$T_{\frac{3}{4}}$ space. In section five, we investigate the characterization of pre-$T_{\frac{3}{4}}$ spaces and establish their implication with some known spaces. Finally, we outline our results in the conclusion.

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PRELIMINARIES

In this section, we recall some details that we need in the sequel regarding: regular open sets, preopen sets, some low separation axioms and pre-separation axioms. Throughout this paper (X, τ) or simply X denotes topological space, and we shall denote the closure and the interior of a set A with respect to the topological space (X, τ) as $\overline{A}$ and A^o; respectively, and the complement of the set A in X, the difference of A and B, and the power set of X are denoted by A^c, A/B and $P(X)$; respectively.

Regular Open Sets and Preopen Sets

Definition 2.1.1. [8,11] A subset M of a space (X, τ) is called:

1. regular open (briefly r-open) if $W = \overline{W}^o$, while the set is called δ-open set if it is the union of r-open sets.
2. preopen if $W \subseteq \overline{W}^o$. The complement of preopen set is called preclosed.

The family of all regular open sets in X is denoted by $RO(X, \tau)$, while the family of all preopen sets and preclosed sets in X are denoted by $PO(X, \tau)$ and $PC(X, \tau)$, respectively.

Corollary 2.1.2. [8,11] In a space X, we have:

- every r-open set is δ-open set.
- every δ-open set is open.
- every open set is preopen.
- every denes set is preopen.

r-open set $\Longrightarrow$ δ-open set $\Longrightarrow$ open set $\Longrightarrow$ preopen set

Corollary 2.1.3. [11] A subset B of a space X is preclosed iff $\overline{B^o} \subseteq B$.

Corollary 2.1.4. [8,11] R-closed sets, δ-closed sets, closed sets and preclosed sets are weakly ordered as:

r-closed set $\Longrightarrow$ δ-closed set $\Longrightarrow$ closed set $\Longrightarrow$ preclosed set

Definition 2.1.5. [8,11] Let B be a subset of a topological space (X, τ), then:

1. the pre-closure of B defined as the intersection of all preclosed sets containing B, and denoted by $\overline{B}^p$.
2. the δ-closure of B defined as the intersection of all δ-closed sets containing B, and denoted by $\overline{B}^\delta$.

Theorem 2.1.6. [8,11] Let X be a space and $B \subseteq X$, then:

- $B \subseteq \overline{B}^p \subseteq \overline{B} \subseteq \overline{B}^\delta$.
- B is preclosed set if and only if $\overline{B}^p = B$.
- B is δ-closed set if and only if $\overline{B}^\delta = B$.
- $\overline{B}^p$ is preclosed set, while $\overline{B}^\delta$ is δ-closed set.

Theorem 2.1.7. [8,11] If (X, τ) is a topological space, and $W \subseteq Y \subseteq X$; where:

- Y is an open subspace of X, then W is r-open in Y iff W is r-open in X.
- Y is an preopen subspace of X, then W is preopen in Y iff W is preopen in X.

Definition 2.1.8. [4,11] A map $F: (X, \tau) \rightarrow (Y, \sigma)$ is called:

1. Strongly regular open map if the image of each r-open subset of X is r-open in Y.
2. preclosed map if the image of each preclosed subset of X is preclosed in Y.

Separation Axioms

Definition 2.2.1. [27,16] A topological space $(X, \tau)$is said to be:

1. T_1-space if whenever x, y are distinct points in X, there exist open sets U and V such that U containing x but not y, and V containing y but not x.
2. $T_{\frac{1}{2}}$-space if $\{x\}$ is open or closed for any point x in X.
3. $T_{\frac{3}{4}}$-space if $\{x\}$ is r-open or closed for any point x in X.

Theorem 2.2.2. [27] A topological space X is T_1-space if any singleton $\{x\}$ is closed in X.

Corollary 2.2.3. [16]

- Every T_1-space is $T_{\frac{3}{4}}$.
- Every $T_{\frac{3}{4}}$-space is $T_{\frac{1}{2}}$.

Definition 2.2.4. [24, 25] A topological space $(X, \tau)$is said to be:

1. pre-T_1 space if whenever x, y are distinct points in X there exist preopen sets A and B such that A containing x but not y, and B containing y but not x.
2. pre-$T_{\frac{1}{2}}$ space if $\{x\}$ is preopen or preclosed for any point x in X.
3. pre-T_0 space if whenever x, y are distinct points in X there exists a preopen set that containing one and not the other.

Theorem 2.2.5. [24] A spaceX is pre-T_1 iff for any point $x \in X$, the singleton set $\{x\}$ is preclosed.

Corollary 2.2.6. [24] Every T_i-space is pre-T_i, where (i=0,$\frac{1}{2}$,1).

Definition 2.2.7. [24] A space X is said to be pre-R_0iff for each preopen set G and $x \in G$ implies $\overline{\{x\}}^p \subseteq G$.

Corollary 2.2.8. [24] In a space X, we have:

- Any pre-T_1space is pre-R_0.
- In pre-R_0 space, X ispre-T_1 iff X is pre-T_0.

PRE-$T_{\frac{3}{4}}$ SPACES

In this section, we define a new space that is weaker than both $T_{\frac{3}{4}}$-space and pre-T_1 space; called pre-$T_{\frac{3}{4}}$ space, then we study the topological properties of this new axiom as their implication with the classical separation axioms, additionally, we discuss their subspaces behavior and their images by some particular maps.

Definition 3.1. A topological space (X, τ) is called pre-$T_{\frac{3}{4}}$ space if every singleton is regular open or preclosed.

Theorem 3.2. Every pre-T_1space is pre-$T_{\frac{3}{4}}$.

Proof. From theorem (2.2.5), every singleton is preclosed set in pre-T_1 space, it follows that the space is pre-$T_{\frac{3}{4}}$.

Pre-T_1Space $\Longrightarrow$Pre-$T_{\frac{3}{4}}$ Space

Theorem 3.3. Every $T_{\frac{3}{4}}$-space is p-$T_{\frac{3}{4}}$.

Proof. From definition (2.2.1) and corollary (2.1.4) any singleton $\{x\}$ is r-open or preclosed in $T_{\frac{3}{4}}$-space, it follows that the space is pre-$T_{\frac{3}{4}}$.

$T_{\frac{3}{4}}$-Space $\Longrightarrow$ Pre-$T_{\frac{3}{4}}$ Space

Examples 3.4.

1. If $X = \{a,b,c\}$ with $\tau = \{\emptyset, X, \{a,b\}\}$, then $RO(X,\tau) = \{\emptyset, X\}$ and $PO(X,\tau) = P(X)/\{\{c\}\}$. Therefore, the space X is pre-$T_{\frac{3}{4}}$ but not $T_{\frac{3}{4}}$-space since the singleton $\{a\}$ is not r-open nor closed.
2. Let $X = \mathbb{R}$ with $\tau = \{A \subseteq \mathbb{R}: 0 \notin A\} \cup \{\emptyset\}$, then $PO(X,\tau) = RO(X,\tau) = \tau$, so any singelton $\{x\}$ where $x \neq 0$ is r-open, and $\{0\}$ is preclosed. ThusX is pre-$T_{\frac{3}{4}}$ space.
3. Let $X = \{a,b,c\}$ with $\tau = \{\emptyset, X, \{a\}\}$, the $RO(X,\tau) = \{X, \emptyset\}$ and $PO(X,\tau) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$, so X is not pre-$T_{\frac{3}{4}}$, since the singleton $\{a\}$ is not r-open nor preclosed.

Lemma 3.5. In a space X, any singleton $\{x\}$ is r-open iff $\{x\}$ is δ-open.

Proof. Suppose $\{x\}$ is δ-open, then $\{x\} = \cup_\alpha W_\alpha$; where W_α is r-open set, hence there is α such that $W_\alpha = \{x\}$; so $\{x\}$ is r-open. Conversely, direct since any r-open set is δ-open.

$\{x\}$ is r-open $\Longrightarrow$ $\{x\}$ is δ-open

Corollary 3.6. A space X is pre-$T_{\frac{3}{4}}$ iff any singelton $\{x\}$ is δ-open or preclosed.

Proof. Direct from the previous lemma.

Lemma 3.7. If (X,τ) is a topological space, then any singelton $\{x\}$ is open or preclosed.

Proof. Suppose $\{x\}$is not open in X, then $\{x\}^c$ is not closed, i.e. $\overline{\{x\}^c} \neq \{x\}^c$ hence $\overline{\{x\}^c} = X$, so $\overline{\{x\}^c}^{\circ} = \overline{X}^{\circ} = X$, therefore $\{x\}^c$ is preopen set, hence $\{x\}$ is preclosed.

Theorem 3.8. If (X,τ) is a pre-$T_{\frac{3}{4}}$ space and $Y \subseteq X$ is an open subspace of X, then Y is pre-$T_{\frac{3}{4}}$.

Proof. Let $y \in Y$, since X is pre-$T_{\frac{3}{4}}$, then $\{y\}$ is r-open or preclosed set in X. If $\{y\}$ is r-open inX, then from theorem (2.1.7) we obtain $\{y\}$ is r-open in Y. If $\{y\}$ is preclosed in X, then from corollary (2.1.3) we have $\overline{\{y\}^{o}} \subseteq \{y\}$: **Case one** when $\{y\}$ is not open in X, then $\{y\}$ is not open in Y (since Y is open subspace), then from lemma (3.7) we obtain $\{y\}$ is preclosed in Y. **Case two** when $\{y\}$ is open in X, and from $\overline{\{y\}^{o}} \subseteq \{y\}$ we obtain $\overline{\{y\}} \subseteq \{y\}$, hence $\overline{\{y\}} = \{y\}$, i.e. $\{y\}$ is open and closed, so$\{y\}$ is clopen in X, then $\{y\}$ is also clopen in Y (since $\{y\} = \{y\} \cap Y$), hence $\{y\}$ is preclosed in Y. Thus complete the prove.

Example 3.9. Subspace of pre-$T_{\frac{3}{4}}$ space need not be pre-$T_{\frac{3}{4}}$ in general, for example if $X = \{a,b,c\}$ with $\tau = \{\emptyset, X, \{a,b\}\}$, then the space X is pre-$T_{\frac{3}{4}}$ while the subspace $Y = \{b,c\}$ where $\tau_Y = \{\emptyset, Y, \{b\}\}$ is not pre-$T_{\frac{3}{4}}$, since the singleton $\{b\}$ is not r-open nor preclosed in Y.

Theorem 3.10. The image of a pre-$T_{\frac{3}{4}}$ space by preclosed and strongly regular open map is pre-$T_{\frac{3}{4}}$ space.

Proof. Let $F: (X,\tau) \to (Y,\sigma)$be an onto map from a pre-$T_{\frac{3}{4}}$ space X, and suppose $y \in Y$, then there exists a point $x \in X$ such that $F(x) = y$, since X is pre-$T_{\frac{3}{4}}$ we have $\{x\}$ is r-open or preclosed in X, since the map F is preclosed and strongly regular open, we have $F(x) = \{y\}$ is r-open or preclosed in Y.

Example 3.11. Strongly regular open image of pre-$T_{\frac{3}{4}}$ space need not be pre-$T_{\frac{3}{4}}$ space, for example: Let $X = Y = \mathbb{R}$ and let τ_1 be the trivial topology on $\mathbb{R}$, and $\tau_2 = \{\emptyset, \mathbb{R}, \{1\}\}$. Then $PO(\mathbb{R},\tau_1) = P(\mathbb{R})$ and $RO(\mathbb{R},\tau_1) = \{\emptyset, \mathbb{R}\}$, while $PO(\mathbb{R},\tau_2) = \{A \subseteq \mathbb{R}: 1 \in A\}$ and $RO(\mathbb{R},\tau_2) = \{\emptyset, \mathbb{R}\}$. Then the identity map from $(\mathbb{R},\tau_1)$ onto the space $(\mathbb{R},\tau_2)$ is strongly regular open map, however the space $(\mathbb{R},\tau_1)$ is pre-$T_{\frac{3}{4}}$, since, while $(\mathbb{R},\tau_2)$ is not pre-$T_{\frac{3}{4}}$ space, since $\{1\}$ is not r-open nor preclosed.

PRE-GENERALIZED δ-CLOSED SETS AND $T_{\frac{3}{4}}$-SPACES

In this section, we introduce a new generalization of δ-closed sets using the notions of preopen sets; namely pre-generalized δ-closed sets. We investigate their properties and characterized the space of pre-$T_{\frac{3}{4}}$ using this class of sets.

Definition 4.1. A subset B ofa topological space $(X,\tau)$is called pre-generalized δ-closed (pgδ-closed) if $\overline{B}^{\delta} \subseteq N$ whenever $B \subseteq N$ and N is preopen set.

Corollary 4.2. Any δ-closed set is pgδ-closed, but not conversely.

Proof. Suppose B is δ-closed set and $B \subseteq N$, where N is preopen, then $\overline{B}^{\delta} \subseteq B \subseteq N$, hence B is δg-preclosed.

δ-closed set $\Longrightarrow$ pgδ-closed set

Example 4.3. Let $X = \{a,b,c\}$ with $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$, then $A = \{c\}$ is pgδ-closed but it is not δ-closed.

Examples 4.4. No general relation between closed set and pgδ-closed set, for example:

1. If $X = \mathbb{R}$ with $\tau = \{\mathbb{R}, \mathbb{Q}, \emptyset\}$, then $RO(X,\tau) = \{\mathbb{R}, \emptyset\}$ while $PO(X,\tau) = \{A \subseteq \mathbb{R}: A \cap \mathbb{Q} \neq \emptyset\} \cup \{\emptyset\}$, so the set $\mathbb{K}$ is closed but not pgδ-clsoed, since $\mathbb{K} \subseteq \mathbb{K} \cup \{2\}$ and $\mathbb{K} \cup \{2\}$ is preopen set but $\overline{\mathbb{K}}^{\delta} = \mathbb{R} \nsubseteq \mathbb{K} \cup \{2\}$.
2. In the usual topology on $\mathbb{R}$ the interval $[0,1)$ is pgδ-clsoed set but not closed, since $\overline{[0,1)}^{\delta} = [0,1]$, and the smallest preopen set that contains $[0,1)$ is the closed interval $[0,1]$.

Theorem 4.5. A space X is pre-$T_{\frac{3}{4}}$ iff any pgδ-closed set is δ-closed.

Proof. Suppose A is pgδ-closed set but not δ-closed, i.e. $\overline{A}^{\delta} \neq A$, hence there exists a point $x \in \overline{A}^{\delta}$ but $x \notin A$, then any δ-open set contains x intersects A, but $\{x\} \cap A = \emptyset$, hence $\{x\}$ is not δ-open, so it is not r-open, and since X is pre-$T_{\frac{3}{4}}$, we have $\{x\}$ is preclosed. Then $A \subseteq \{x\}^c$, where $\{x\}^c$ is preopen, so $\overline{A}^{\delta} \subseteq \{x\}^c$, i.e. $x \notin \overline{A}^{\delta}$ which contradict the assumption. Conversely, suppose $\{x\}$ is not preclosed set, then $\{x\}^c$ is not preopen, so $\{x\}^c$ is pgδ-closed set since the only preopen set contains $\{x\}^c$ is X, hence $\{x\}^c$ is δ-closed, i.e. $\{x\}$ is δ-open, therefore X is pre-$T_{\frac{3}{4}}$. constant.

PRE-$T_{\frac{3}{4}}$ SPACES AND SOME RELATED SPACES

In the present section, we illustrate the implication between pre-$T_{\frac{3}{4}}$ space with some known spaces as; submaximal space, regular space, partition space, hyperconnected space and extremely disconnected space.

In Submaximal Spaces

Definition 5.1.1. [11] A space X is called submaximal if any dense set in X is open.

Theorem 5.1.2. [11] A space X is submaximal iff any preopen is open.

Corollary 5.1.3. In submaximal space X, these conditions are equivalent:

- X is pre-$T_{\frac{3}{4}}$.
- X is $T_{\frac{3}{4}}$.

Proof. Direct since any preopen set is open.

$$\text{Pre-}T_{\frac{3}{4}}\text{ Space} \xLeftrightarrow{\text{Submaximal}} T_{\frac{3}{4}}\text{-Space}$$

Corollary 5.1.4. In submaximal space X, we have:

1. X is pre-T_0 iff X is T_0.
2. X is pre-T_1 iff X is T_1.
3. X is pre-T_2 iff X is T_2.
4. X is pre-$T_{\frac{1}{2}}$ iff X is $T_{\frac{1}{2}}$.

Corollary 4.1.5. Any submaximal space is $T_{\frac{1}{2}}$.

Proof. Since any space is pre-$T_{\frac{1}{2}}$ and in submaximal space, the axioms pre-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are coincid, thus complete the prove.

$$\text{Submaximal Space} \Longrightarrow T_{\frac{1}{2}}\text{-Space}$$

In Regular Spaces

Theorem 5.2.1. [6] In regular space X, any open set can be expressed as a union of r-open sets, equivalently; a subsetA of a regular space X is open set if and only if A is δ-open

Corollary 5.2.2. In regular space X, the singelton $\{x\}$ is r-open iff $\{x\}$ is open.

Proof. From corollary (2.1.2) any r-open set is open. Conversely, any open set in regular space is δ-open, and from lemma (3.5) any δ-open singelton is r-open set, that complete the prove.

Theorem 5.2.3. Any regular space X is pre-$T_{\frac{3}{4}}$.

Proof. From lemma (3.7) any singelton$\{x\}$ is open or preclosed, and inreguler space any singelton open set is r-open, so we have $\{x\}$ is r-open or preclosed, hence X is pre-$T_{\frac{3}{4}}$.

$$\text{Regular Space} \Longrightarrow \text{Pre-}T_{\frac{3}{4}}\text{ Space}$$

Example 5.2.4. Pre-$T_{\frac{3}{4}}$ space need not be regular, for expample: Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, b\}\}$, then $RO(X, \tau) = \{\emptyset, X, \{a, b\}, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ and $PC(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Therefore the space X is p-$T_{\frac{3}{4}}$ but not regular.

Corollary 5.2.5 In regular-submaximal space X, these stataments are equivalent:

1. X is pre-$T_{\frac{3}{4}}$.
2. X is $T_{\frac{3}{4}}$.
3. X is $T_{\frac{1}{2}}$.

Proof.

1. $\Longrightarrow$2) Direct since in submaximal space any preclosed set is closed.
2. $\Longrightarrow$3) Direct.
3. $\Longrightarrow$1) Direct since in regular space any singelton open set is r-open.

$$\text{Pre-}T_{\frac{3}{4}}\text{ Space} \xLeftrightarrow{\text{Regular submaximal}} T_{\frac{3}{4}}\text{-Space} \xLeftrightarrow{\text{Regular submaximal}} T_{\frac{1}{2}}\text{-Space}$$

Theorem 5.2.6. Any regular submaximal space X is $T_{\frac{3}{4}}$.

Proof. Since any regular space X is pre-$T_{\frac{3}{4}}$, and any pre-$T_{\frac{3}{4}}$ submaximal space is $T_{\frac{3}{4}}$, thus complete the prove.

$$\text{Regular submaximal Space} \Longrightarrow T_{\frac{3}{4}}\text{-Space}$$

In Partition Spaces

Definition 5.3.1. [14] A space (X, τ) is called partition if any open set is closed.

Remark 5.3.2. In partition space (X, τ), we have $RO(X, \tau) = \tau$ and $PO(X, \tau) = P(X)$.

Proof. Suppose V is a non-empty open subset of X, then $\overline{V}^{o} = V^{o} = V$, so $RO(X, \tau) = \tau$. Moreover, $N \subseteq \overline{N}$ for any subset N of X, since in partition space any closed set is open, we get $\overline{N}^{o} = \overline{N}$, hence $N \subseteq \overline{N}^{o}$, i.e. $PO(X, \tau) = P(X)$.

Corollary 5.3.3. Any partition space (X, τ) is p-$T_{\frac{3}{4}}$ (and pre-T_1).

Proof. Any singelton in partition space is preclosed, since $PO(X, \tau) = P(X)$.

$$\text{Partition Space} \Longrightarrow \text{Pre-}T_{\frac{3}{4}}\text{ Space}$$

Corollary 5.3.4. Any $T_{\frac{1}{2}}$ partition space is $T_{\frac{3}{4}}$.

Proof. Since any singelton in $T_{\frac{1}{2}}$-space is open or closed, and in partition space any open is r-open, then X is $T_{\frac{3}{4}}$.

$$T_{\frac{1}{2}}\text{-Space} \xLeftrightarrow{\text{Partition}} T_{\frac{3}{4}}\text{-Space}$$

In Hyperconnected Spaces

Definition 5.4.1. [14] A space X is called hyperconnected if any open set is dense.

Lemma 5.4.2. A space X is called hyperconnected space (X, τ), we have $RO(X, \tau) = \{\emptyset, X\}$.

Proof. Suppose V is a non-empty open set, then V is dense in X, so $\overline{V}^{o} = X^{o} = X$, hence V is not r-open set.

Theorem 5.4.3. Any pre-$T_{\frac{3}{4}}$ hyperconnected space X is pre-T_1.

Proof. Suppose X is pre-$T_{\frac{3}{4}}$, then any singelton $\{x\}$ is r-open or preclosed, but from the previous lemma $\{x\}$ is not r-open, hence $\{x\}$ is preclosed, thus X is pre-T_1.

Corollary 5.4.3. In hyperconnected space X, we have: X is pre-$T_{\frac{3}{4}}$ iff X is pre-T_1.

$$\text{Pre-}T_{\frac{3}{4}} \text{ Space} \xLeftrightarrow{\text{Hyperconnected}} \text{Pre-}T_1 \text{ Space}$$

In Extremely Disconnected Space

Definition 5.5.1. [14] A topological space (X, τ) is called extremelly discometed (namely e.d) if the closure of any open set is open, i.e. if V is open in X, then $\overline{V}$ is also open.

Corollary 5.5.2. [14] A space (X, τ) is extremelly discometed, then:

1. any r-open set is clopen.
2. any r-closed set is clopen .
3. $RO(X, \tau) = RC(X, \tau) = \tau \cap F$, where F is the collection of all closed subsets in X.

Theorem 5.5.3. Any pre-$T_{\frac{3}{4}}$ extremelly discometed space is pre-R_0.

Proof. Suppose N is preopen set in X and x$\in$ N, since X is pre-$T_{\frac{3}{4}}$ then $\{x\}$ is r-open or preclosed. **Case 1:** If $x \in N$ and $\{x\}$ is r-open, then since X is extremelly discometed the singelton $\{x\}$ is clopen, so $\{x\}$ is preclosed, i.e. $\overline{\{x\}}^p = \{x\} \subseteq N$, hence X is pre-$R_0$. **Case 2:** If $x \in N$ and $\{x\}$ is preclosed, then $\overline{\{x\}}^p = \{x\} \subseteq N$, hence X is pre-$R_0$.

$$\text{Pre-}T_{\frac{3}{4}} \text{ Space} \xRightarrow{\text{Extremelly disconneted}} \text{Pre-}R_0 \text{ Space}$$

Theorem 5.5.3. Any pre-R_0 space is pre-$T_{\frac{3}{4}}$.

Proof. If X is a space, then X is pre-$T_{\frac{1}{2}}$, so it is pre-T_0 space, and from corollary (2.2.8) any pre-T_0 pre-R_0 is pre-T_1.

Corollary 5.5.4. Any pre-R_0 space is pre-T_1.

$$\text{Pre-}R_0 \text{ Space} \Longrightarrow \text{Pre-}T_1 \text{ Space} \Longrightarrow \text{Pre-}T_{\frac{3}{4}} \text{ Space}$$

Remark 5.5.5. In extremelly disconnected space X, we have: X is pre-$T_{\frac{3}{4}}$ iff X is pre-R_0.

$$\text{Pre-}T_{\frac{3}{4}} \text{ Space} \xLeftrightarrow{\text{Extremelly disconnected}} \text{Pre-}R_0 \text{ Space}$$

CONCLUSION

Using the concept of regular open sets and preclosed sets we introduce a new space namely pre-$T_{\frac{3}{4}}$ space, this space is weaker than both pre-T_1 space and $T_{\frac{3}{4}}$-space. In this paper, we study the topological properties of this space, as the hereditary property and their images by some particular functions, moreover, we characterized this space by the new notion of pgδ-closed sets. Finally, we investigate the behavior of pre-$T_{\frac{3}{4}}$ space in some special spaces as; regular spaces, submaximal spaces, partition spaces, hyperconnected spaces and extremely disconnected spaces.

Outline some of our results:

- A space X is pre-$T_{\frac{3}{4}}$ iff any pgδ-closed set is δ-closed set.
- Pre-$T_{\frac{3}{4}}$ space is weaker than: $T_{\frac{3}{4}}$-space, partition space, regular space, pre-T_1 space and pre-R_0 space.
- In hyperconnected space; the spaces pre-$T_{\frac{3}{4}}$ space and pre-T_1 space are coincide.
- In extremely disconnected space; the spaces pre-$T_{\frac{3}{4}}$ space and pre-R_0 space are coincide.
- In submaximal space; the spaces pre-$T_{\frac{3}{4}}$ space and $T_{\frac{3}{4}}$-space are coincide.
- Any regular submaximal space is $T_{\frac{3}{4}}$-space.

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