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# Stability of Solutions of the Double Obstacle Problem on Metric Spaces

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**Abstract.** We study the regularity properties of solutions to the double obstacle problem in a metric space. Our main results are a global reverse Hölder inequality, and stability of solutions. We assume the space supports a weak Poincaré inequality and a doubling measure. Furthermore we assume that the complement of the domain is uniformly thick in a capacitary sense.

**Keywords.** double obstacle problem, higher integrability, metric measure space, Poincaré inequality, stability.

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## 1. Introduction

One of the most important elliptic variational problems is to minimize the $p$-energy functional

$$\int_\Omega |Du|^p \, dx \tag{1.1}$$

with $1 < p < \infty$ in an open subset $\Omega$ of $\mathbf{R}^n$ among all functions $u\colon \Omega \to \mathbf{R}$ which belong to a suitable Sobolev space. This is equivalent to solving the $p$-harmonic equation.

In a general metric measure space that only has a doubling measure and supports a Poincaré inequality, it is not clear how to define the $p$-harmonic equation. However, the variational approach to $p$-harmonic functions is available. The reason for this is that the Sobolev spaces on general metric measure spaces can be defined without the notion of partial derivatives, and the absolute value

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of the gradient in (1.1) can be replaced by the notion of an upper gradient; see, e.g. [11], [23].

Obstacle problems naturally appear in the nonlinear potential theory; see, for example, [13] and [16]. By a solution to the double obstacle problem, we mean a function that minimizes the $p$-Dirichlet integral among all the functions restricted by a given function from below and above, see Section 3 for the exact definition. Previously, the obstacle problem in the metric setting has been studied, for example, in [1] and the double obstacle problem in [6], [8].

In this note, we study integrability and stability for solutions to the double obstacle problem in complete metric measure spaces that have a doubling measure and support a Poincaré inequality. The case of a single obstacle problem is included, since one obstacle function can be chosen to be identically infinity. We are especially interested in the stability of solutions to the double obstacle problem, when the exponent $p$ varies. In the euclidean setting, stability problems have been studied by Li and Martio – first for the single obstacle problem and then for the double obstacle problem; see [18], [19], [20]. The purpose of this note is to extend their results to metric spaces.

Proofs of stability results can be often divided in two parts. First, the solutions are shown to be better integrable than a priori assumed. This usually requires a higher integrability result such as the Gehring lemma. The techniques that are needed in the second part of the proof vary more. Many of the techniques needed in our note are similar to the techniques used in [17] by Maasalo and Zatorska-Goldstein, where the higher integrability and stability results are proved for quasiminimizers in metric spaces.

## 2. Notation and preliminaries

Throughout this paper, we assume that the measure $\mu$ is doubling, that is there exists $C_\mu > 0$ such that

$$\mu(B(x, 2r)) \le C_\mu \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

A nonnegative Borel function $g$ is said to be an *upper gradient* of an extended real-valued function $f$ on $X$ if for all rectifiable curves $\gamma : [0, l_\gamma] \to X$ parameterized by arc length $ds$, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If $g$ is a nonnegative measurable function on $X$ and if (2.1) holds for $p$-almost every curve, then $g$ is a *$p$-weak upper gradient* of $f$.

By saying that (2.1) holds for $p$-almost every curve we mean that it fails only for a curve family with zero $p$-modulus; see, for example, Definition 2.1 in [23]. If $f$ has a $p$-weak upper gradient in $L^p(X)$, then it has a *minimal $p$-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every $p$-weak upper gradient $g \in L^p(X)$ of $f$, $g_f \le g$ a.e.; see, for example, Corollary 3.7 in [24].

In [23], upper gradients have been used to define Sobolev type spaces on metric spaces. We will use the following equivalent definition.

**Definition 2.2.** Let $u \in L^p(X)$ with $1 \le p < \infty$. We define

$$\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p},$$

where $g_u$ is the minimal $p$-weak upper gradient of $u$. The *Newtonian space* on $X$ is the quotient space

$$N^{1,p}(X) = \left\{ u : \|u\|_{N^{1,p}(X)} < \infty \right\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p. 249 in [23]. We also have the following lemma about minimal $p$-weak upper gradients; see, for example, [1] or [2].

**Lemma 2.3.** *If $u, v \in N^{1,p}(X)$, then*

$$g_u = g_v \quad \text{a.e. on } \{x \in X : u(x) = v(x)\}.$$

*Moreover, if $c \in \mathbf{R}$ is a constant, then $g_u = 0$ a.e. on $\{x \in X : u(x) = c\}$.*

For $\Omega \subset X$ open we define the space $N^{1,p}(\Omega)$ with respect to the restrictions of the metric $d$ and the measure $\mu$ to $\Omega$. It is well known that the restriction to $\Omega$ of a minimal $p$-weak upper gradient in $X$ remains minimal with respect to $\Omega$. By $N_0^{1,p}(\Omega)$ we denote the space of functions $u \in N^{1,p}(\Omega)$ whose zero extension is in $N^{1,p}(X)$.

The space of Lipschitz-functions is denoted by $\mathrm{Lip}(X)$. A function $u$ belongs to $\mathrm{Lip}_0(\Omega)$ if $u \in \mathrm{Lip}(\Omega)$ and its zero extension belongs to $\mathrm{Lip}(X)$.

**Definition 2.4.** The *capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on $E$.

We say that a property holds *$p$-quasieverywhere* ($p$-q.e.) in $X$, if it holds everywhere except on a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u$, $v \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$

q.e. Moreover, Corollary 3.3 in [23] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

Let $\Omega \subset X$ be open and bounded and let $E \Subset \Omega$, that is $\overline{E} \subset \Omega$. The *relative $p$-capacity* of $E$ with respect to $\Omega$ is defined by

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on $E$.

**Definition 2.5.** Let $1 \leq p < \infty$. We say that the space $X$ supports a weak $(1,p)$-*Poincaré inequality*, if there exist constants $C > 0$ and $\lambda > 1$ such that for all balls $B(x,r)$ in $X$, all locally integrable functions $u$ on $X$ and all upper gradients $g$ of $u$ we have

$$\left( \fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \right) \leq Cr \left( \fint_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p},$$

where

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

Note that by the Hölder inequality, a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for every $q \geq p$.

**Lemma 2.6.** *Let $X$ be a doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality. Then $X$ supports a weak $(t,p)$-Poincaré inequality, i.e., there exist constants $C'$ and $\lambda'$ such that*

$$\left( \fint_{B(x,r)} |u - u_{B(x,r)}|^t \, d\mu \right)^{1/t} \leq C' r \left( \fint_{B(x,\lambda' r)} g^p \, d\mu \right)^{1/p},$$

*for all balls $B$ in $X$ and all $t$ such that*

$$\begin{cases} 1 \leq t \leq Qp/(Q-p) & \text{if } q < Q, \\ 1 \leq t & \text{if } p \geq Q, \end{cases}$$

*where $Q = \log_2 C_\mu$.*

From now on we assume that $X$ supports a weak $(1,p)$-Poincaré inequality. In [15] it was shown that, in a complete doubling metric measure space, a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for some $1 < q < p$. Moreover, by increasing $q$ if necessary, we may assume that $p \in (q, q^*)$, where $q^* = qQ/(Q-q)$ and $Q = \log_2 C_\mu$. This and Lemma 2.6 imply that the space $X$ supports the $(p,q)$-Poincaré inequality, for some $1 < q < p$.

Throughout the rest of this paper we assume that the space $X$ satisfies the *local linear connectivity* property (LLC-property), that is, there exist constants $C \geq 1$ and $r_0 > 0$ such that for all balls $B$ in $X$ whose radius at most $r_0$, every two points in the annulus $2B \setminus \overline{B}$ can be connected by a curve lying in the annulus $2CB \setminus C^{-1}\overline{B}$.

In general, the $(1,p)$-Poincaré inequality does not imply LLC. However, this holds when $X$ is a complete and the measure satisfies in addition to the doubling condition that

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \leq C\left(\frac{r}{R}\right)^s$$

for all $0 < r < R$ and $x \in B(y,R)$ with some $s > p$. For a proof, see [12].

We say that the set $E \subset X$ is uniformly $p$-fat if there exist constants $C > 0$ and $r_0 > 0$ such that for all $x \in E$ and $0 < r < r_0$, we have

$$\operatorname{cap}_p(E \cap B(x,r); B(x,2r)) \geq C \operatorname{cap}_p(B(x,r); B(x,2r)).$$

Under our assumption, if $\Omega \subset X$ is open and bounded such that $C_p(X \setminus \Omega) > 0$ and $X \setminus \Omega$ is uniformly $p$-fat, then $X \setminus \Omega$ is also $p_0$-fat for some $p_0 < p$; see [4]. Note also that $p$-fatness always implies $p + \varepsilon$-fatness for every $\varepsilon \geq 0$.

The following lemma will be needed later. For a proof, see for example [16].

**Lemma 2.7.** *Let $u, v \in N^{1,p}(X)$ and $\eta \in \operatorname{Lip}(X)$ be such that $0 \leq \eta \leq 1$. Set $w = u + \eta(v-u) = (1-\eta)u + \eta v$. Then*

$$g_w \leq (1-\eta)g_u + \eta g_v + |v-u| g_\eta$$

*almost everywhere in $X$, where $g_w$, $g_u$, $g_v$ and $g_\eta$ are the $p$-weak upper gradients of $w$, $u$, $v$ and $\eta$ respectively.*

We shall use the following two results. For a proof, see [5] and [22].

**Lemma 2.8.** *Let $X$ be doubling metric measure space supporting a weak $(1,q)$-Poincaré inequality and let $E \subset B = B(x_0,r)$ with $0 < r < \operatorname{diam} X/8$. Then there exists a $C > 0$ such that*

$$\frac{\mu(E)}{Cr^q} \leq \operatorname{cap}_q(E, 2B) \leq \frac{C\mu(B)}{r^q}$$

*and*

$$\frac{C_q(E)}{C(1+r^q)} \leq \operatorname{cap}_q(E, 2B) \leq 2^{q-1}\left(1 + \frac{1}{r^q}\right)C_q(E).$$

**Proposition 2.9.** *Let $X$ be a doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality and let $u \in N^{1,p}(X)$. Then there exists a constant*

*$C > 0$ such that for all balls $B$ in $X$ and $S = \{x \in \frac{1}{2}B : u(x) = 0\}$ the following inequality holds*

$$\left(\fint_B |u|^t \, d\mu\right)^{1/t} \le \left(\frac{C}{\operatorname{cap}_p(S,B)} \fint_{\lambda' B} g_u^p\right)^{1/p},$$

*where $t$ and $\lambda'$ are as in Lemma 2.6.*

We will use the following Poincaré type inequality. For a proof, see for example [17], Lemma 2.1.

**Lemma 2.10.** *Let $X$ be a doubling metric measure space supporting a weak $(1,p)$- Poincaré inequality. Let $\Omega$ be a bounded open subset of $X$ such that $C(X \setminus \Omega) > 0$. There exists a constant $C > 0$ such that for all $u \in N_0^{1,p}(\Omega)$, we have*

$$\int_\Omega |u|^p \, d\mu \le C \int_\Omega g_u^p \, d\mu.$$

*The constant $C$ depends on the diameter of $\Omega$.*

The proofs of the next two lemmas can be found, for example, in [14] and [22].

**Lemma 2.11.** *Let $X$ be a proper, doubling, LCC metric space supporting a weak $(1,q)$-Poincaré inequality for some $1 < q < p$. Let also $\Omega \subset X$ is open and bounded such that $X \setminus \Omega$ is uniformly $p$-fat. Assume that $\{u_i\}_{i=1}^\infty$ is bounded sequence in $N_0^{1,p}(\Omega)$ such that $u_i \to u$ q.e. in $\Omega$. Then $u \in N_0^{1,p}(\Omega)$.*

**Lemma 2.12.** *Let $X$ be as in Lemma 2.11. Then*

$$N_0^{1,p}(\Omega) = N^{1,p}(\Omega) \cap \bigcap_{s<p} N_0^{1,s}(\Omega).$$

We achieve the growth of integrability for upper gradients of solutions of the obstacle problem using the Gehring Lemma 2.13 below. For a proof see for example [21] or [25].

**Lemma 2.13** (Gehring lemma)**.** *Let $1 < s_0 < s_1$ be fixed and let $s \in [s_0, s_1]$. Let $g \in L^s_{\mathrm{loc}}(X)$ and $f \in L^{s_1}_{\mathrm{loc}}(X)$ be non-negative functions. Assume that there exists a constant $D > 1$ such that for every ball $B \subset \sigma B \subset X$ the inequality*

$$\fint_B g^s \, d\mu \le D\left[\left(\fint_{\sigma B} g \, d\mu\right)^s + \fint_{\sigma B} f^s \, d\mu\right]$$

*holds for some $\sigma > 1$. Then there exists an $\varepsilon_0 > 0$ such that $g \in L^{\tilde{s}}_{\mathrm{loc}}(X)$ for $\tilde{s} \in [s, s+\varepsilon_0)$ and moreover*

$$\left(\fint_B g^{\tilde{s}} \, d\mu\right)^{1/\tilde{s}} \le C\left[\left(\fint_{\sigma B} g^s \, d\mu\right)^{1/s} + \left(\fint_{\sigma B} f^{\tilde{s}} \, d\mu\right)^{1/\tilde{s}}\right],$$

*where* $C = C(s_0, s_1, \sigma, C_\mu, D)$.

## 3. Higher integrability of upper gradients for solutions of the double obstacle problem

Recall that we assume that $X$ is a complete metric measure space supporting a $(1,p)$-Poincaré inequality and that $X$ satisfies the LLC property. Moreover, the measure $\mu$ is doubling.

Throughout the rest of this paper we make the additional assumptions that $\Omega \subset X$ is a nonempty bounded open set such that $C_p(X \setminus \Omega) > 0$. Also the letter $C$ represents various constants and can change even within the same line of a calculation. Recall also that $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

We define the double obstacle problem as follows. We first fix the obstacle functions $\psi_1, \psi_2 \in N^{1,p}(\Omega)$ and the boundary values $f \in N^{1,p}(\Omega)$. Then let

$$\mathcal{K}^p_{\psi_1,\psi_2,f} = \{v \in N^{1,p}(\Omega) : v - f \in N^{1,p}_0(\Omega) \text{ and } \psi_1 \le v \le \psi_2 \text{ q.e. in } \Omega\}$$

be the space of admissible functions. We say that $u$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem if

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu$$

for all $v \in \mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$. A unique solution (up to sets of capacity zero) of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem exists if $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega) \neq \varnothing$. See [6], [7], [8], [9] for more details about the double obstacle problem in metric spaces.

**Theorem 3.1.** *Let* $\Omega$ *be open and bounded subset of* $X$ *such that* $C_p(X \setminus \Omega) > 0$ *and* $X \setminus \Omega$ *is* $p$*-fat. Let* $f, \psi_1, \psi_2 \in N^{1,s}(\Omega)$ *for some* $s > p$ *and such that* $\psi_1 \le f \le \psi_2$ *q.e. in* $\Omega$.

*If* $u \in N^{1,p}(\Omega)$ *is a solution of the* $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$*-problem, then there exists* $\delta_0 = \delta_0(p) \le s - p$ *such that* $g_u \in L^{p+\delta}(\Omega)$ *for all* $0 < \delta \le \delta_0$ *and*

$$\left(\int_\Omega g_u^{p+\delta} \, d\mu\right)^{1/(p+\delta)} \le C \left[\left(\int_\Omega g_u^p \, d\mu\right)^{1/p} + \left(\int_\Omega (g_{\psi_1} + g_{\psi_2} + g_f)^{p+\delta} \, d\mu\right)^{1/(p+\delta)}\right].$$

**Proof.** First, remember that by Lemma 2.6 and the discussion after it, $(1,p)$-Poincaré implies a $(p,p_0)$-Poincaré with some $p_0 < p$. By the self-improving property of $p$-fatness, we may also choose $p_0$ so that $X \setminus \Omega$ is $p_0$-fat.

Choose a ball $B_0 \subset X$ such that $\Omega \Subset B_0 \Subset 2B_0$. Fix $r > 0$ and let $B = B(x_0, r)$ be a ball such that $4\lambda B \subset 2B_0$. We have two cases: either $2\lambda B \subset \Omega$ or $2\lambda B \setminus \Omega \neq \varnothing$.

Let us start with the first case. Let $\eta \in \mathrm{Lip}_0(2B)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B$ and $g_\eta \le C/r$. Write

$$u_{2B} = \fint_{2B} u\,d\mu = \frac{1}{\mu(2B)} \int_{2B} u\,d\mu,$$

and let $v = (1-\eta)(u - u_{2B}) + \eta w$, where

$$w = (\psi_1 - u_{2B})^+ - (\psi_2 - u_{2B})^- = \begin{cases} (\psi_1 - u_{2B})^+ & \text{if } \psi_2 \ge u_{2B}, \\ \psi_2 - u_{2B} & \text{if } \psi_2 < u_{2B}. \end{cases}$$

Then we have

$$\psi_1 - u_{2B} \le v \le \psi_2 - u_{2B}$$

and

$$v - (u - u_{2B}) = -\eta(u - u_{2B}) + \eta w \in N_0^{1,p}(2B).$$

Therefore $v \in \mathcal{K}^p_{\psi_1 - u_{2B}, \psi_2 - u_{2B}, u - u_{2B}}(2B)$. By Lemma 2.7 we have that

$$g_v \le (1-\eta) g_u + \eta g_w + |w - (u - u_{2B})| g_\eta$$

a.e. in $2B$. Hence using that $u - u_{2B}$ is a solution of the $\mathcal{K}^p_{\psi_1 - u_{2B}, \psi_2 - u_{2B}, u - u_{2B}}(2B)$-problem we get that

$$\begin{aligned} \int_B g_u^p\,d\mu &\le \int_{2B} g_u^p\,d\mu = \int_{2B} g_{u-u_{2B}}^p\,d\mu \le \int_{2B} g_v^p\,d\mu \\ &\le C\left[\int_{2B} (1-\eta)^p g_u^p\,d\mu + \int_{2B} \eta^p g_w^p\,d\mu + \int_{2B} |w - (u-u_{2B})|^p g_\eta^p\,d\mu\right] \\ &\le C \int_{2B \setminus B} g_u^p\,d\mu + C\int_{2B} g_w^p\,d\mu + \frac{C}{r^p} \int_{2B} \big(|w|^p + |u - u_{2B}|^p\big)\,d\mu. \end{aligned}$$

Adding $C \int_B g_u^p\,d\mu$ to both sides and dividing by $C+1$ we get, with $\theta = C/(C+1)$,

$$\int_B g_u^p\,d\mu \le \theta \int_{2B} g_u^p\,d\mu + \frac{\theta}{r^p} \int_{2B} \big(|w|^p + |u - u_{2B}|^p\big)\,d\mu + \theta \int_{2B} g_w^p\,d\mu.$$

Lemma 3.1 in [10] now implies that

$$\int_B g_u^p\,d\mu \le \frac{C}{r^p} \int_{2B} \big(|w|^p + |u - u_{2B}|^p\big)\,d\mu + C\int_{2B} g_w^p\,d\mu. \tag{3.2}$$

Note that we have $g_w \le g_{\psi_1} + g_{\psi_2}$ and that

$$|w| = \begin{cases} (\psi_1 - u_{2B})^+ & \text{if } \psi_2 \ge u_{2B}, \\ u_{2B} - \psi_2 & \text{if } \psi_2 < u_{2B}, \end{cases}$$

from which we see that $|w| \le |u - u_{2B}|$. It follows from (3.2) that

$$\int_B g_u^p\,d\mu \le \frac{C}{r^p} \int_{2B} |u - u_{2B}|^p\,d\mu + C \int_{2B} (g_{\psi_1} + g_{\psi_2})^p\,d\mu.$$

The doubling condition implies that

$$\fint_B g_u^p \, d\mu \le \frac{C}{r^p} \fint_{2B} |u - u_{2B}|^p \, d\mu + C \fint_{2B} (g_{\psi_1} + g_{\psi_2})^p \, d\mu.$$

Next, we apply the $(p, p_0)$-Poincaré inequality to the first integral on the right-hand side to get

$$\left( \frac{C}{r^p} \fint_{2B} |u - u_{2B}|^p \, d\mu \right)^{1/p} \le C \left( \fint_{2\lambda B} g_u^{p_0} \, d\mu \right)^{1/p_0}.$$

Hence

$$\fint_B g_u^p \, d\mu \le C \left( \fint_{2\lambda B} g_u^{p_0} \, d\mu \right)^{p/p_0} + C \fint_{2\lambda B} (g_{\psi_1} + g_{\psi_2})^p \, d\mu. \tag{3.3}$$

Next assume that $2\lambda B \setminus \Omega \neq \varnothing$. Let $\eta \in \mathrm{Lip}_0(2B)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B$ and $g_\eta \le C/r$. Let also

$$v = u - \eta(u - f) = (1 - \eta)u + \eta f.$$

It follows that $\psi_1 \le v \le \psi_2$ in $2B \cap \Omega$ and that

$$v - u = -\eta(u - f) \in N_0^{1,p}(2B \cap \Omega).$$

Hence $v \in \mathcal{K}^p_{\psi_1,\psi_2,u}(2B \cap \Omega)$. As $u$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,u}(2B \cap \Omega)$-problem we get, using that $g_v \le (1 - \eta) g_u + \eta g_f + |u - f| g_\eta$,

$$\begin{aligned}
\int_{2B\cap\Omega} g_u^p \, d\mu &\le \int_{2B\cap\Omega} g_v^p \, d\mu \\
&\le C \int_{2B\cap\Omega} (1 - \eta)^p g_u^p \, d\mu + C \int_{2B\cap\Omega} \eta^p g_f^p \, d\mu + C \int_{2B\cap\Omega} |u - f|^p g_\eta^p \, d\mu \\
&\le C \int_{(2B\setminus B)\cap\Omega} g_u^p \, d\mu + C \int_{2B\cap\Omega} g_f^p \, d\mu + \frac{C}{r^p} \int_{2B\cap\Omega} |u - f|^p \, d\mu.
\end{aligned}$$

Adding $C \int_{B\cap\Omega} g_u^p \, d\mu$ to both sides and dividing by $C + 1$, we get

$$\int_{B\cap\Omega} g_u^p \, d\mu \le \theta \int_{2B\cap\Omega} g_u^p \, d\mu + \theta \int_{2B\cap\Omega} g_f^p \, d\mu + \frac{\theta}{r^p} \int_{2B\cap\Omega} |u - f|^p \, d\mu,$$

where $\theta = C/(C + 1) < 1$. As previously, Lemma 3.1 in [10] then implies that

$$\int_{B\cap\Omega} g_u^p \, d\mu \le \frac{C}{r^p} \int_{2B\cap\Omega} |u - f|^p \, d\mu + C \int_{2B\cap\Omega} g_f^p \, d\mu. \tag{3.4}$$

Now, we use Proposition 2.9, with $p = p_0$ and $t = p$, together with the doubling condition to conclude that

$$\begin{aligned}\left(\frac{C}{r^p} \fint_{4B} |u-f|^p \, d\mu\right)^{1/p} &\leq \frac{C}{r} \left(\frac{1}{\mathrm{cap}_{p_0}(S, 4B)} \int_{4\lambda B} g_{u-f}^{p_0} \, d\mu\right)^{1/p_0} \\ &\leq C \left(\frac{\mu(4B)}{\mathrm{cap}_{p_0}(S, 4B) r^{p_0}} \fint_{4\lambda B} g_{u-f}^{p_0} \, d\mu\right)^{1/p_0} \\ &\leq C \left(\frac{\mu(2B) r^{-p_0}}{\mathrm{cap}_{p_0}(S, 4B)} \fint_{4\lambda B} g_{u-f}^{p_0} \, d\mu\right)^{1/p_0},\end{aligned}$$

where $S = \{x \in 2B : u(x) = f(x)\}$. As $u = f$ $p$-q.e. (and thus $p_0$-q.e.) in $X \setminus \Omega$ we have $2B \setminus \Omega \subset S$. This with the fact that $X \setminus \Omega$ is uniformly $p_0$-fat imply that

$$\mathrm{cap}_{p_0}(S, 4B) \geq \mathrm{cap}_{p_0}(2B \setminus \Omega, 4B) \geq C \, \mathrm{cap}_{p_0}(2B, 4B) \geq C\mu(2B) r^{-p_0}.$$

Hence, as $g_{u-f} = 0$ a.e. in $X \setminus \Omega$, we get

$$\begin{aligned}\left(\frac{C}{r^p} \fint_{4B} |u-f|^p \, d\mu\right)^{1/p} &\leq C \left(\fint_{4\lambda B} g_{u-f}^{p_0} \, d\mu\right)^{1/p_0} \\ &= C \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_{u-f}^{p_0} \, d\mu\right)^{1/p_0} \\ &\leq C \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_u^{p_0} \, d\mu\right)^{1/p_0} + C \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_f^{p_0} \, d\mu\right)^{1/p_0}.\end{aligned} \tag{3.5}$$

It follows from the Hölder inequality that

$$\begin{aligned}\left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_f^{p_0} \, d\mu\right)^{1/p_0} &= \left(\fint_{4\lambda B} g_f^{p_0} \chi_{4\lambda B \cap \Omega} \, d\mu\right)^{1/p_0} \\ &\leq \left(\fint_{4\lambda B} g_f^{p} \chi_{4\lambda B \cap \Omega} \, d\mu\right)^{1/p} \\ &= \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_f^{p} \, d\mu\right)^{1/p}.\end{aligned} \tag{3.6}$$

The inequalities (3.4), (3.5) and (3.6) together with the doubling condition imply that

$$\begin{aligned}\left(\frac{1}{\mu(B)} \int_{B \cap \Omega} g_u^{p} \, d\mu\right)^{1/p} &\leq C \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_u^{p_0} \, d\mu\right)^{1/p_0} \\ &\quad + C \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B \cap \Omega} g_f^{p} \, d\mu\right)^{1/p},\end{aligned} \tag{3.7}$$

where the constant $C$ depends only on $p$, $\Omega$ and the space $X$.

Now let

$$g(x) = \begin{cases} g_u^p(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases} \qquad f(x) = \begin{cases} (g_{\psi_1} + g_{\psi_2} + g_f)^{p_0}(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

and $s = p/p_0$. Then from (3.3) and (3.7) we get

$$\fint_B g^s \, d\mu \le C \left( \fint_{4\lambda B} g \, d\mu \right)^s + C \fint_{4\lambda B} f^s \, d\mu,$$

with $s > 1$ and for all $B$ such that $4\lambda B \subset 2B_0$. The Gehring Lemma 2.13 now implies that

$$\left( \fint_B g^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \le C \left[ \left( \fint_{4\lambda B} g^s \, d\mu \right)^{1/s} + \left( \fint_{4\lambda B} f^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \right]. \tag{3.8}$$

Since the diameter of $\Omega$ is finite we may choose a finite number of balls $B(x_j, r_j)$, $j = 1, 2, \dots, N$, such that

$$B(x_j, 2\lambda r_j) \subset B_0 \quad \text{and} \quad \Omega \subset \bigcup_{j=1}^N B(x_j, r_j)$$

where $\lambda$ is the dilation constant in the Poincaré inequality. Now we multiply (3.8), with $B$ replaced by $B(x_j, r_j)$, by $\mu(4\lambda B(x_j, r_j))^{1/\tilde{s}}$ and sum over $B(x_j, r_j)$ to get the desired inequality. ■

**Theorem 3.9.** *Let $1 \le p_i < \infty$, $i = 1, 2, \dots$ and $p = \lim_{i\to\infty} p_i$. Let $\psi_1, \psi_2, f \in N^{1,s}(\Omega)$ for some $s > p$ and assume that $\psi_1 \le f \le \psi_2$ q.e. in $\Omega$. For $i = 1, 2, \dots$ let $u_i$ be a solution of the $\mathcal{K}^{p_i}_{\psi_1,\psi_2,f}(\Omega)$-problem. Then there exists an $\varepsilon_0 > 0$ and $u \in N^{1,p+\varepsilon_0}(\Omega)$ and a $(p + \varepsilon_0)$-weak upper gradient $g$ of $u$ such that $u_i, g_{u_i} \in L^{p+\varepsilon_0}(\Omega)$ and there is a subsequence such that*

$$u_{i_k} \to u \quad \text{in } L^{p+\varepsilon_0}(\Omega),$$
$$g_{u_{i_k}} \to g \quad \text{weakly in } L^{p+\varepsilon_0}(\Omega),$$

*Moreover, $u$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem.*

**Proof.** We know from Theorem 3.1 that for every $p_i$ there exists $\delta_i = \delta_i(p_i)$ such that $p_i$-weak upper gradient $g_{u_i}$ belongs to the space $L^{p_i+\delta_i}(\Omega)$ and

$$\left( \int_\Omega g_{u_i}^{p_i+\delta_i} \, d\mu \right)^{1/(p_i+\delta_i)} \le C \left( \int_\Omega g_{u_i}^{p_i} \, d\mu \right)^{1/p_i} + C \left( \int_\Omega (g_{\psi_1} + g_{\psi_2} + g_f)^{p_i+\delta_i} \, d\mu \right)^{1/(p_i+\delta_i)}. \tag{3.10}$$

Using that $u_i$ is a solution of the $\mathcal{K}^{p_i}_{\psi_1,\psi_2,f}(\Omega)$-problem, and $f \in \mathcal{K}^{p_i}_{\psi_1,\psi_2,f}(\Omega)$, together with the Hölder inequality, we get

$$\int_\Omega g_{u_i}^{p_i}\,d\mu \le \int_\Omega g_f^{p_i}\,d\mu \le (\mu(\Omega))^{\delta_i/(p_i+\delta_i)}\biggl(\int_\Omega g_f^{p_i+\delta_i}\,d\mu\biggr)^{p_i/(p_i+\delta_i)}.$$

Hence

$$\biggl(\int_\Omega g_{u_i}^{p_i}\,d\mu\biggr)^{1/p_i} \le C_i\biggl(\int_\Omega g_f^{p_i+\delta_i}\,d\mu\biggr)^{1/(p_i+\delta_i)}.$$

This and (3.10) imply that

$$\biggl(\int_\Omega g_{u_i}^{p_i+\delta_i}\,d\mu\biggr)^{1/(p_i+\delta_i)} \le C_i\biggl(\int_\Omega (g_{\psi_1}+g_{\psi_2}+g_f)^{p_i+\delta_i}\,d\mu\biggr)^{1/(p_i+\delta_i)}. \tag{3.11}$$

Next, as $p_i \to p$ and $p \in (q, q^*)$ we may assume that $p_i \in (q, q^*)$. It then follows, as in [22], that

$$\delta_i \ge \delta_0 = \delta_0(p) \quad \text{and} \quad C_i \le C = C(p).$$

Let $\varepsilon_0 = \delta_0/2$. For $i$ large enough, we have

$$p + \varepsilon_0 \le p_i + \delta_0 \le p_i + \delta_i.$$

We can also choose $\delta_0$ and $\delta_i$ so that

$$p_i + \delta_i \le s.$$

By applying this and the Hölder inequality to (3.11), we get

$$\begin{aligned}\biggl(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,d\mu\biggr)^{1/(p+\varepsilon_0)} &\le C\biggl(\int_\Omega g_{u_i}^{p_i+\delta_i}\,d\mu\biggr)^{1/(p_i+\delta_i)}\\ &\le C_i\biggl(\int_\Omega (g_{\psi_1}+g_{\psi_2}+g_f)^{p_i+\delta_i}\,d\mu\biggr)^{1/(p_i+\delta_i)}\\ &\le C\biggl(\int_\Omega (g_{\psi_1}+g_{\psi_2}+g_f)^{s}\,d\mu\biggr)^{1/s}.\end{aligned}$$

It follows that

$$\begin{aligned}\biggl(\int_\Omega g_{u_i-f}^{p+\varepsilon_0}\,d\mu\biggr)^{1/(p+\varepsilon_0)} &\le \biggl(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,d\mu\biggr)^{1/(p+\varepsilon_0)} + \biggl(\int_\Omega g_f^{p+\varepsilon_0}\,d\mu\biggr)^{1/(p+\varepsilon_0)}\\ &\le C\biggl(\int_\Omega (g_{\psi_1}+g_{\psi_2}+g_f)^{s}\,d\mu\biggr)^{1/s} < \infty.\end{aligned} \tag{3.12}$$

This shows that the sequence $\{g_{u_i-f}\}_{i=1}^{\infty}$ is bounded in $L^{p+\varepsilon_0}(\Omega)$. Since $u_i - f \in N_0^{1,p}(\Omega)$ and the weak $(1, p+\varepsilon_0)$-Poincaré is satisfied for sufficiently large $i$ and all $\varepsilon_0 > 0$, Lemma 2.10 then implies that

$$\left(\int_\Omega |u_i - f|^{p+\varepsilon_0}\,d\mu\right)^{1/(p+\varepsilon_0)} \le C\left(\int_\Omega g_{u_i-f}^{p+\varepsilon_0}\,d\mu\right)^{1/(p+\varepsilon_0)} < \infty.$$

This and (3.12) imply that the sequence $\{u_i - f\}_{i=1}^{\infty}$ is bounded in $N^{1,p+\varepsilon_0}(\Omega)$.

Fix a ball $B_0$ such that $\Omega \subset B_0$ and extend $u_i - f$ by zero outside of $\Omega$. It follows that $\|u_i - f\|_{L^1(B_0)} + \|g_{u_i-f}\|_{L^{p+\varepsilon_0}(B_0)}$ is bounded. Hence, the Rellich–Kondrachov theorem (see e.g. Theorem 4.1 in [22]) implies that there exist a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ and $u \in L^{p+\varepsilon_0}(B_0)$ such that

$$u_{i_k} - f \to u - f \quad \text{in } L^{p+\varepsilon_0}(B_0).$$

Since $\{g_{u_{i_k}}\}_{k=1}^{\infty}$ is bounded in $L^{p+\varepsilon_0}(\Omega)$, there exist $g$ and a subsequence again denoted by $\{g_{u_{i_k}}\}_{k=1}^{\infty}$ such that $g$ is a $(p+\varepsilon_0)$-weak upper gradient of $u$ and that

$$g_{u_{i_k}} \to g \quad \text{weakly in } L^{p+\varepsilon_0}(B_0).$$

Furthermore, $u \in N^{1,p+\varepsilon_0}(\Omega)$. See, for example, Lemma 3.2 in [3]. Notice also, that $g$ is a $q$-weak upper gradient of $u$ for all $q \le p + \varepsilon_0$.

Finally, we show that $u$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem. Clearly $u$ is admissible, and we start by showing that $u - f \in N_0^{1,p}(\Omega)$. Let $0 < \varepsilon \le \varepsilon_0$. Then for sufficiently large $i$ we have that $p - \varepsilon < p_i$ and hence $u_i - f \in N_0^{1,p-\varepsilon}(\Omega)$. It follows from Lemma 2.10 that

$$\|u_i - f\|_{N_0^{1,p-\varepsilon}(\Omega)} \le C\|g_{u_i-f}\|_{L^{p-\varepsilon}(\Omega)}.$$

When $\varepsilon > 0$ is small enough, we have $p-\varepsilon < p_0$ and therefore $X \setminus \Omega$ is uniformly $(p-\varepsilon)$-fat. Lemma 2.11 now implies that $u - f \in N_0^{p-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ small enough. Hence Lemma 2.12 shows that $u - f \in N_0^{1,p}(\Omega)$

Assume then that $v$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem and fix $0 < \varepsilon \le \varepsilon_0$. Then, for sufficiently large $i$, we have $p-\varepsilon < p_i < p+\varepsilon$ and hence $v \in \mathcal{K}^{p_i}_{\psi_1,\psi_2,f}(\Omega)$. Using that $u_i$ is a solution of the $\mathcal{K}^{p_i}_{\psi_1,\psi_2,f}(\Omega)$-problem we get

$$\int_\Omega g_{u_i}^{p_i}\,d\mu \le \int_\Omega g_v^{p_i}\,d\mu. \tag{3.13}$$

As $\{g_{u_{i_k}}\}_{k=1}^{\infty}$ converges weakly in $L^{p-\varepsilon}(B_0)$ (since $p-\varepsilon < p+\varepsilon_0$) to a weak upper gradient $g$ of $u$ we obtain, for sufficiently large $k$ such that $p_{i_k} > p-\varepsilon$, that

$$\begin{aligned}\int_\Omega g_u^{p-\varepsilon}\,d\mu \le \int_\Omega g^{p-\varepsilon}\,d\mu &\le \liminf_{k\to\infty}\int_\Omega g_{u_{i_k}}^{p-\varepsilon}\,d\mu \\ &\le \liminf_{k\to\infty}\left(\int_\Omega g_{u_{i_k}}^{p_{i_k}}\,d\mu\right)^{(p-\varepsilon)/p_{i_k}}\mu(\Omega)^{1-(p-\varepsilon)/p_{i_k}} \\ &\le \liminf_{k\to\infty}\left(\int_\Omega g_v^{p_{i_k}}\,d\mu\right)^{(p-\varepsilon)/p_{i_k}}\mu(\Omega)^{1-(p-\varepsilon)/p_{i_k}}.\end{aligned}$$

Here we also used the Hölder inequality and (3.13). By letting $k\to\infty$, the right-hand side converges by the dominated convergence theorem to

$$\left(\int_\Omega g_v^p\,d\mu\right)^{(p-\varepsilon)/p}\mu(\Omega)^{1-(p-\varepsilon)/p}.$$

Then by letting $\varepsilon\to 0$, we obtain

$$\int_\Omega g_u^p\,d\mu \le \int_\Omega g_v^p\,d\mu.$$

As $v$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem, we conclude that $u=v$ q.e. and therefore $u$ is a solution of the $\mathcal{K}^p_{\psi_1,\psi_2,f}(\Omega)$-problem. ■

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