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Axioms of Countability Via Preopen Sets

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ABSTRACT

We used the concept of preopen sets to introduce a particular form of the μ-countability axioms; namely pre-countability axioms, this class of axioms includes; pre-separable spaces, pre-first countable spaces and pre-second countable spaces. In this article, we study the topological properties of these spaces, as the hereditary property and their images by some particular functions; moreover we investigate the behavior of pre-countability axioms in some special spaces as; submaximal spaces, regular spaces and partition spaces.

***Keywords*:** Topological space and generalizations, generalized continuity, countability axioms, subspaces, separability, regular spaces

AMS Subject Classification (2000): 54A05, 54C08, 54D70, 54B05, 54D10, 54D65

1. INTRODUCTION

Various types of generalized countability axioms were introduced by many researchers as; g-countability axioms, b-countability axioms, D-countability axioms and recently r-countability axioms. In 1974, Siwiec [1] used the concept of weak base in topological space to define g-first countable space and g-second countable space, where he showed their relations with metrizability. One year later, Siwiec [2] wrote a survey of the concepts which generalize first countability, and he studied the relation among these generalizations. Gruenhage [3] in 1976, defined a class of spaces in terms of a simple two-person infinite game, namely W-spaces,

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he showed that this space is weaker than first countable space; in addition he studied the properties of this class of spaces and their relation with bi-sequential spaces. In 1991, Jian-Ping [4] considered some generalizations of first countable spaces; namely ω_k-spaces, when he illustrated the relations among Frechet spaces, ω-spaces and first countable spaces. Selvarani in 2013 [5], introduced b-countability axioms via b-open sets, after that Arwini and Kornas [6, 7] defined two generalization types of countability axioms, the first one is called D-countability axioms when they used the concept of dense sets to defined these axioms, and they proved that D-separable space and D-second countable space are equivalent to separable spaces, while the second class of generalizations called R-countability axioms, where they used the notion of regular open sets to defined this class, they studied thier properties and poved that r-countability axioms and countability axioms are coincide in regular spaces.

Different generlaizations of open sets have been considered, some of these notions were defined similary using the closure and the interior operations as; α-set, semi-open set, preopen set and b-open set. The concept of preopen (or locally dense) sets play a significant role in general topology, where Corson and Michael in 1964 [8] were first defined preopen set by the name "locally dense" set, then in 1982 Mashhour, Elmonsef and Eldeeb [9] used the name "preopen" set rather than locally dense set. More details on preopen sets can be found in [10-12]. A generalized topology (brifely GT) on a set X was introduced by Csaszar [13], and usually denoted by (X, μ), where the elements of the generalized topology μ are denoted by μ-open sets. In 2013, Ayawan and Canoy [14] defined a countability axioms on GT; namely μ-countability axioms, when they considered the properties of these concepts and characterized μ-first countability and μ-second countability of the product of GT's. Details on bases property in generalized topology can be found in [15, 16].

In this article, we used the notions of preopen sets to define a particular type of the μ-countability axioms, which related to Ayawan et al., we call this class of axioms pre-countability axioms, where this class consists the axioms of: pre-separable, pre-first countable space and pre-second countable space. We illustrate the relations between pre-countability axioms among themselves and between the classical countability axioms, then we study the hereditary properties for these spacses and thier images under some special function as; pre-irresoulte and M-open functions, finally we study the behavior of these spaces in some special spaces as; submaximal space, regular space and in partition space.

We divided our article into six main sections as; preopen sets, pre-separable spaces, pre-first countable spaces, pre-second countable spaces, properties of pre-countability axioms and finaly conclusion.

2. PREOPEN SETS

The preopen sets play significant role in general topology, here we recall the definitions and the properties of preopen sets that we need in the sequel. Throughout this paper (X,τ) simply X denotes topological space, and we shall denote closure and preclosure of a set A with respect to the topological space (X,τ) as $\overline{A}$ and $\overline{A}^p$; respectively.

Definition 2.1. [8] A subset N of a topological space (X,τ) is called preopen if $N \subseteq \overline{N}^o$, and the complement of preopen set is called preclosed. The family of all preopen sets and preclosed sets in X are denoted by PO(X,τ) and PC(X,τ), respectively.

Proposition 2.1. [8] A subset M of a space X is preclosed iff $\overline{M^{o}} \subseteq$ M.

Remarks 2.1. [8]

1- Every open set is preopen, but not conversely.
2- Every closed set is preclosed, but not conversely.
3- Every denes set is preopen, but not conversely.

Proposition 2.2. [17] Let X be a space, then a subset N of X is preopen if and only if N= U∩D, where U is open subset of X and D is dense in X.

Proposition 2.3. [17]

1- Arbitrary union of preopen sets is preopen.
2- Intersection of preopen sets need not be preopen.
3- The intersection of open set and preopen set is preopen.

Definition 2.2. [17] A subset A of a space X is said to be pre-regular if it is both preopen and preclosed in X.

Remarke 2.2. [17] If A is a clopen subset in a topological space, then A is pre-regular. The inverse is not true.

Examples 2.1.

1- If (ℝ,τ) is the trivial space on ℝ, then PO(X,τ) = P(ℝ). Note that any singleton in preopen (preclosed, pre-clopen) but not open (closed, colpen).
2- Let X= ℝ with τ={ ℝ, 𝕂, ℚ, ∅}, then PO(X,τ) = P(X). The set 𝕂 is preopen but not dense in ℝ.
3- In the usual topological space (ℝ,μ), the sets 𝕂∪{1} and ℚ are preopen sets but their intersection 𝕂∪{1} ∩ ℚ ={1} is not preopen, since $\overline{\{1\}}^{o} = \emptyset$.

Theorem 2.1. [17] Let A be subset of a space (X, τ). If A is preopen in X, then: PO(A,τ_A) = {N∩A: N ∈ PO(X, τ)}.

Theorem 2.2. [17] Let A ⊆ Y ⊆ X. Then: If A is a preopen (preclosed) set in Y, and Y is a preopen (preclosed) set in X, then A is a preopen (preclosed) set in X.

Theorem 2.3. [17] Let Y be a subspace of a space X, if Y is a preopen set in X and U ⊆ Y, then U is a preopen set in Y iff U is a preopen set in X.

Definition 2.3. [18] A function F: X ⟶ Y is said to be:

1- A pre-irresolute if the inverse image under F of a preopen set is preopen.
2- An M-preopen if the image under F of a preopen set is preopen.

Definition 2.4. [17] Let X be a topological space and A ⊆ X. The pre-closure of A is defined as the intersection of all preclosed sets in X containing A, and is denoted by $\overline{A}^p$. It is clear that $\overline{A}^p$ is preclosed set for any subset A of X.

Proposition 2.4. [11] Let X be a topological space and A, B ⊆ X, then:

1- $A \subseteq \overline{A}^p \subseteq \overline{A}$.

2- If A ⊆ B, then $\overline{A}^p \subseteq \overline{B}^p$.

3- A is preclosed if and only if $A = \overline{A}^p$.

3. PRE SEPARABLE SPACES

We use the notion of preopen sets to define a new class of countability axioms; called pre-countability axioms, this class consists the axioms of: pre-separable spaces, pre-first countable spaces and pre-second countable spaces. In this section we study the topological properties of pre-separable spaces, and we illustrate its relations with separable and b-separable spaces. We recall the definitions and some properties concerning separable spaces, b-dense sets and b-separable spaces which we need in the sequel. See [5], [19] and [20].

Definition 3.1. [19] A topological space X is said to be separable space if there exist countable dense subset of X.

Theorem 3.1. [19]

1- An open subspace of separable space is separable.
2- Image of separable space under continuous map is separable.

Definition 3.2. [5] A subset B of a topological space (X,τ) is called b-open if $B \subseteq \overline{B}^{o} \cup \overline{B^{o}}$, and the family of all b-open sets in X are denoted by BO(X,τ). While the subset B of X is called b-dense if $\overline{B}^b = X$.

Definition 3.3. [5] A topological space X is said to be b-separable space if there exist a countable b-dense subset of X.

Definition 3.4. A subset D of a topological space X is called pre-dense if $\overline{D}^p = X$.

Example 3.1. In the trivial topological space (X,τ), the family of all preopen sets is PO(X,τ) = P(X) , so any non-empty subset in X is dense, while the only pre-dense set is X.

Corollary 3.1. In a topological space (X,τ), these statements are hold:

1) Every pre-dense set is dense.
2) Every b-dense set is pre-dense

Proof:

1) Let D be a pre-dense subset of a space X, i.e. $\overline{D}^p = X$, since $\overline{D}^p \subseteq \overline{D}$ we get $\overline{D} = X$.
2) Let B be a b-dense subset of a space X, i.e. $\overline{B}^b = X$, since $\overline{B}^b \subseteq \overline{B}^p$ we get $\overline{B}^p = X$. b-Dense ⟹ Pre-Dense ⟹ Dense ⟹ Preopen

Examples 3.2.

1) In the trivial topological space (X,τ) any non-empty proper subset of X is dense set but not pre-dense.
2) Let $X = \mathbb{R}$ with $\tau = \{\mathbb{R}\} \cup \{A \subseteq \mathbb{R} : 0 \notin A\}$ then $PO(X,\tau) = \tau$ while $BO(X,\tau) = P(X)$. Hence the set $\mathbb{R}/\{0\}$ is dense and pre-dense but not b-dense.
3) In the usual topological space $(\mathbb{R}, \mu)$, the set $\mathbb{Q}$ and $\mathbb{K}$ are dense sets but not pre-dense, since $\mathbb{Q}$ and $\mathbb{K}$ are preopen sets satisfy $\mathbb{Q} \cap \mathbb{K} = \emptyset$.

Corollary 3.2. The subset B of a topological space (X,τ) is pre-dense if and only if every non-empty preopen set in X contains points of B.

Proof:

⟹ Let B be a pre-dense subset in X, and let N be a non-empty preopen set. Since $N \neq \emptyset$ and $\overline{B}^p = X$ there is $x \in N$ and $x \in \overline{B}^p$, hence $N \cap B \neq \emptyset$.

⟸ Let x be an arbitrary element in X, then any preopen set that contains x intersect B, i.e. $x \in \overline{B}^p$, hence $\overline{B}^p = X$.

Corollary 3.3. In a topological space (X,τ), these statements are hold:

1) Any subset of X that contains a pre-dense set is pre-dense.
2) If A is pre-dense set in B, and B is pre-dense in X, then A is pre-dense in X.

Proof:

1) Direct since $\overline{A}^p \subseteq \overline{B}^p$ for any sets A and B satisfy $A \subseteq B$.
2) Suppose N is a non-empty preopen set in X, then $N \cap B \neq \emptyset$ since B is pre-dense in X, and from corollary (3.1) and remark (2.1(3)) we have $N \cap B$ is a non-empty preopen set in B. Since A is pre-dense in B, then we have $(N \cap B) \cap A \neq \emptyset$, i.e. $(N \cap B) \cap A = N \cap A \neq \emptyset$. Therefore, A is pre-dense in X.

Example 3.3. The intersection of pre-dense sets need not be pre-dense, for example: In the usual topological space $(\mathbb{R}, \mu)$, the set $\mathbb{Q}$ and $\mathbb{K}$ are dense but not pre-dense (since $\mathbb{K}$ and $\mathbb{Q}$ are both pre-open sets). Now if we choose: $A = \mathbb{K} \cup \mathbb{Z}^+ \cup (\mathbb{Q}^-/\mathbb{Z}^-)$ and $B = \mathbb{K} \cup \mathbb{Z}^- \cup (\mathbb{Q}^+/\mathbb{Z}^+)$, then A and B are pre-dense sets, while $A \cap B = \mathbb{K}$ is not pre-dense.

Definition 3.5. A topological space X is called pre-separable space if there exist a countable pre-dense subset of X.

Corollary 3.4. In a topological space (X,τ), these statements are hold:

1) Every pre-separable space is separable space, but not conversely.
2) Every b-separable space is pre-separable space, but not conversely.

Proof:

1) Direct since every pre-dense set is dense set.
2) Direct since every b-dense set is pre-dense set.

Examples 3.4.

1) The trivial topology on uncountable is separable space, but not pre-separable since the only pre-dense subset is X.
2) The usual topological space $(\mathbb{R}, \mu)$ is separable but not pre-separable, since if D is a nonempty countable subset of ℝ then D^C is preopen set and $D \cap D^C = \emptyset$, so D is not pre-dense.
3) If X= ℝ and $\tau = \{\mathbb{R}, \mathbb{Q}, \mathbb{K}, \emptyset\}$, then X is separable space but BO(X, τ) = PO(X, τ) = P(X), so X is not pre-separable nor b-separable space.
4) Let X= ℝ with τ=$\{\mathbb{R}, \mathbb{K}, \emptyset\}$, then PO(X,τ)=$\{A\subseteq \mathbb{R}: A\cap \mathbb{K} \neq \emptyset\}\cup\{\emptyset\}$, so X is not pre-separable space.
5) Let X= ℝ with τ =$\{\mathbb{R}, \mathbb{Q}, \emptyset\}$, then PO(X,τ)=$\{A\subseteq \mathbb{R}: A\cap \mathbb{Q} \neq \emptyset\}\cup\{\emptyset\}$, so X is pre-separable space.

Theorem 3.2. A preopen subspace of pre-separable space is pre-separable space**.**

Proof: Let Y is a preopen subspace of pre-separable space X, then X has a countable pre-dense subset D. Now if M is a non-empty preopen subset in Y, then form theorem (2.2) M is a nonempty preopen set in X, since D is pre-dense we have M ∩ D≠ ∅, therefore; M ∩ (D∩Y) = (M ∩ Y) ∩ D= M ∩ D ≠ ∅, i.e. D∩Y is a countable pre-dense set in Y, hence Y is pre-separable space.

Remark 3.1. Any open subspace of pre-separable space is pre-separable space.

Proof: Direct since any open set is preopen.

Example 3.5. A subspace of pre-separable space need not pre -separable space, for example: Let X be uncountable set, and τ =$\{A\subseteq X: x \in A\}\cup\{\emptyset\}$, where x is a point in X. Then PO(X,τ)= τ, since $\overline{A}^{o}$ = X where x ∈ A, while $\overline{A}^{o}$ = ∅ where x ∉ A for any subset A of X. The set {x} is pre-dense, so X is pre-separable space while the subspace $\{x\}^C$ is the discrete space, so it is not pre-separable space.

Theorem 3.3. A pre-irresolute image of pre-separable space is pre-separable space.

Proof: Let F: X ⟶Y be a pre-irresolute function from a pre-separable space X, then X has a countable pre-dense subset A, so F(A) is countable. Now suppose N is a non-empty preopen set in F(X), since F is pre-irresolute F^{-1}(N) is a non-empty preopen set in X, so F^{-1}(N) ∩ A≠ ∅, hence N ∩ F(A) ≠ ∅. We have F(A) is a countable pre-dense subset of F(X).

Example 3.6. Continuous image of a pre-separable space need not be pre-separable space, for example: Let X = Y = ℝ and τ_1={∅, ℝ, {1}} and let τ_2 be the trivial topology on ℝ. Then PO(ℝ, τ_1)={A ⊆ ℝ: 1 ∈ A}, while PO(ℝ, τ_2)= P(ℝ). Then the identity map from (ℝ, τ_1) onto the space (ℝ, τ_2) is continuous, however the space (ℝ, τ_1) is pre-separable, since {1} is a countable pre-dense subset , while (ℝ, τ_2) is not pre-separable space.

4. PRE FIRST COUNTABLE SPACES

We introduce the concept of pre-local base in topological spaces, and use it to define the axiom of pre-first countable spaces. We study the hereditary property and the image of these spaces.

Definition 4.1. [19] In a topological space X, a collection $\mathfrak{B}_x$ of open sets that contains x is called basis at x if for any open set U such that x ∈ U there exists B_x in $\mathfrak{B}_x$ such that x ∈ B_x ⊆ U.

Definition 4.2. [19] A topological space X is said to be first countable space if for every x ∈ X there is a countable local base $\mathfrak{B}_x$ at x.

Theorem 4.1. [19]

1- A subspace of first countable space is first countable.
2- Image of first countable space under continuous and open map is first countable.

Definition 4.3. In a topological space X, a collection $\mathfrak{N}_x$ of preopen sets that contains an element x is called pre-local basis at x if for any preopen set B such that x ∈B there is N_x in $\mathfrak{N}_x$ such that x ∈ N_x ⊆ B.

Examples 4.1.

1) The collection $\mathfrak{B}_0$= {(-n, n): n ∈ ℕ} is a local base at 0 in the usual topology on ℝ, but not pre-local base at 0, since 𝕂 ∪{0} is preopen set containing 0 but (-n, n) ⊈ 𝕂 ∪ {0} for any n ∈ ℕ.
2) In the trivial space on a set X with more than one element, the collection $\mathfrak{N}_x$={{x}} is a pre-local base at x , but not local base at x, since {x} is not an open set.

Definition 4.4. A topological space X is called pre-first countable space if for every x ∈ X there is a countable pre-local base at x.

Examples 4.2.

1) The cofinite topological space on uncountable set X is not first countable space nor pre-first countable.
2) The usual topology on ℝ is first countable space but not pre-first countable space, since (a,b) ∩ 𝕂 is preopen set for any open interval (a, b).
3) The trivial space on uncountable is pre-first countable space but not pre-separable space.

Theorem 4.2. A preopen subspace of pre-first countable space is pre-first countable.

Proof: Let Y be a preopen subspace of a pre-first countable space X, then any y∈Y (⊆ X) has a countable pre-local base $\mathfrak{N}_y$ for X. Now we need to prove that the collection $\{N_y \cap Y: N_y \in \mathfrak{N}_y\}$ is a countable pre-local base at y for Y. Suppose M is a preopen set in Y that contains y, then form theorem (2.2) we get M is preopen in X, since $\mathfrak{N}_y$ is a pre-local base at y, there exists a preopen set $N_y \in \mathfrak{N}_y$ such that $y \in N_y \subseteq M$, then $y \in N_y \cap Y \subseteq M \cap Y = M$, therefore $\{N_y \cap Y\}$ is a countable pre-local base at y in the subspace Y.

Example 4.3. A subspace of pre-first countable space need not be pre-first countable space, for example: Let X = (0,∞) with the cofinite topology. Then $PO(X,\tau)=\{A\subseteq X : A \text{ infinite}\}\cup\{\emptyset\}$, therefor X is not pre-first countable space. Now if Y=[0,∞), with $\sigma =\{V\cup\{0\}: V\in \}\cup\{\emptyset\}$, then (Y,σ) is a topological space and (X,τ) is a subspace of Y. Note that if B⊆Y and 0∈B then $\overline{B}^{o}=Y$ so B is a preopen in Y, while if 0∉B then $\overline{B}^{o}=\emptyset$ so B is not preopen in this case, hence $PO(Y,\sigma)=\{B\subseteq Y: 0\in B\}\cup\{\emptyset\}$, therefore {{0,x}} is a countable pre-local base at any point x in the space Y, therefore Y is pre-first countable but the subspace X is not.

Theorem 4.3. Image of pre-first countable space under pre-irresolute and M-preopen map is pre-first countable space.

Proof: Suppose F: X⟶ Y is a pre-irresolute, M-preopen map from a pre-first countable space X onto a topological space Y. Then for any y ∈ F(X) there is a countable pre-local base $\mathfrak{N}_{F^{-1}(y)}$ at $F^{-1}(y)$ for X, since F is M-preopen map the collection $F(\mathfrak{N}_{F^{-1}(y)})$ is countable collection of preopen sets in F(X), and since F is pre-irresolute, then $F(\mathfrak{N}_{F^{-1}(y)})$ is a countable pre-local base at y.

Example 4.4. Pre-irresolute image of a pre-first countable space need not be pre-first countable space, for example: Let X = Y = ℝ with τ_1=P(ℝ) and τ_2= cofinite topology on ℝ. Then the identity map from $(\mathbb{R},\tau_1)$ onto the space $(\mathbb{R},\tau_2)$ is pre-irresolute, however the space $(\mathbb{R},\tau_1)$ is pre-first countable while $(\mathbb{R},\tau_2)$ is not pre-first countable.

5. PRE SECOND COUNTABLE SPACES

In the present section, we introduce the axiom of pre-second countable space, and we show that this axiom is stronger than the axioms of pre-separable and pre-first countable.

Definition 5.1. [19] Let (X,τ) be a topological space, then sub collection $\mathfrak{B}$ of τ is said to be a base for τ if each member of τ can be expressed as a union of members of $\mathfrak{B}$.

Definition 5.2. [19] A topological space X is satisfies second countable axiom if X has a countable base $\mathfrak{B}$.

Theorem 5.1. [19]

1- Every second countable space is first countable and separable.
2- A subspace of second countable space is second countable.
3- Image of second countable space under continuous and open map is second countable.

Definition 5.3. A collection Ɲ of preopen sets in a topological space (X,τ) is called pre-base for X if each preopen set can expressed as a union of members of Ɲ.

Examples 5.1.

1) The space X = ℝ with $\tau = \{\mathbb{R}, \mathbb{Q}, \mathbb{K}, \emptyset\}$ the collection $\{\mathbb{Q}, \mathbb{K}\}$ is base for ℝ but not pre-base, while the collection $\{\{x\}\}_{x\in\mathbb{R}}$ is pre-base for ℝ but not base.
2) In the trivial space on ℝ the collection $\{\{x\}\}_{x\in\mathbb{R}}$ is pre-base but not base.

Definition 5.4. A topological space (X,τ) is called pre-second countable space if X has a countable pre-base.

Examples 5.2.

1) The space X = ℝ with $\tau = \{\mathbb{R}, \mathbb{Q}, \mathbb{K}, \emptyset\}$ is second countable space but not pre-second countable space.
2) The trivial space on uncountable is second countable space but not pre-second countable space.
3) The space X = ℝ with $\tau = \{\mathbb{R}, \mathbb{Q}, \emptyset\}$ is pre-second countable space, since $\{\mathbb{R}\} \cup \{\{x\}\}_{x\in\mathbb{Q}}$ is a pre-local base for ℝ.

Definition 5.5. A topological space (X,τ) is called pre-countable space if the collection PO(X,τ) is countable.

Corollary 5.1. Every pre-countable space is pre-second countable space.

Proof: Direct since the collection PO(X,τ) is a countable pre-local base for X.

Remark 5.1. Every pre-countable space is pre-first countable space and pre-separable space.

Examples 5.3.

1) The space X= ℝ with $\tau = \{\phi\} \cup \{A \subseteq \mathbb{R}: 1 \in A\}$ is pre-separable space (since $\{1\}$ is countable pre-dense set in ℝ), but X is not pre-second countable space since $\{1, x\}$ is preopen set for any x ∈ ℝ, x ≠ 1.
2) If X = ℝ and $\tau = \{\phi, \mathbb{R}, \{1\}\}$ then X is pre-first countable space (since {{1,x}} is a countable pre-local base at any point x ∈ ℝ), but not pre-second countable space. Note that, the space X is countable but not pre-second countable.

Theorem 5.2. A preopen subspace of pre-second countable space is pre-second countable space.

Proof: Let Y be a preopen subspace of a pre-second countable space X, then X has a countable pre-base Ɲ. Now we need to prove that the collection $Ɲ_Y$ ={N ∩Y: N ∈ Ɲ} is a countable pre-base for Y. Suppose M is a preopen set in Y, then form theorem (2.2) we get M is preopen in X, since Ɲ is a pre-base for X, there exist preopen sets N_α ∈ Ɲ such that M =∪ N_α, then M = M∩Y =∪ (N_α ∩Y), therefore {N_α ∩Y} is a countable pre-base for the subspace Y.

Theorem 5.3. Image of pre-second countable space under pre-irresolute and M-preopen map is pre-second countable space.

Proof: Suppose F: X⟶ Y is a pre-irresolute and M-preopen map from a pre-second countable space X onto a topological space Y. Then X has a countable pre-base Ɲ, since the map F is M-preopen map the collection F(Ɲ) is a countable collection of preopen sets in F(X), and since F is pre-irresolute, then F(Ɲ) is a countable pre-base for F(X).

Example 5.4. M-preopen image of a pre-second countable space need not be pre-second countable space, for example: Let X = Y = ℝ with τ_1={ℝ, ℚ, ∅}, then PO(X,τ_1) = P(ℚ)∪{ℝ}, and τ_2= {ℝ, ℚ, 𝕂, ∅}, then PO(X,τ_2) = P(ℝ). Then the identity map from (ℝ, τ_1) onto the space (ℝ, τ_2) is M-preopen map, however the space (ℝ, τ_1) is pre-second countable while (ℝ, τ_2) is not pre-second countable.

6. PROPERTIES OF PRE COUNTABILITY AXIOMS

In this section, we study the behavior of the axioms of pre-countability in some spaces as; submaximal spaces, regular spaces and partition spaces.

6. 1. In Submaximal Spaces

Definition 6.1.1. [20] A topological space X is called submaximal space if each dense set in X is open.

Lemma 6.1.1. In submaximal space (X,τ), any preopen set is open, i.e. PO(X,τ) = τ.

Proof: Let N be a preopen subset in a space X, then from propostion (2.2) N can be written as N =V∩D, where V is open and D is dense set in X, since X is submaximal space, each dense set is open, we have N is open set.

Corollary 6.1.1 In a submaximal space X, a subset D is pre-dense in X iff D is dense.

$$\text{Dense} \overset{\text{Submaximal}}{\Longleftrightarrow} \text{Pre-Dense}$$

Corollary 6.1.2 In submaximal space; pre-countability axioms equivalent to the countability axioms, therefore:

1- Separable space and pre-separable space are equivalent.
2- First countable space and pre-first countable space are equivalent.

3- Second countable space and pre-second countable are equivalent.

6. 2. In Regular Spaces

Definition 6.2.1. [20] A space X is called regular-space if for any closed set F and x∉F there exist disjoint open sets U and V such that x∈U and F⊆V.

Theorem 6.2.1. [20] A space X is regular iff for every x∈X, and each open set U in X such that x∈U there exists an open set V such that x ∈V⊆ $\overline{V}$ ⊆U.

Theorem 6.2.2. In regular topological space X, if $\mathcal{N}_x = \{N_x\}$ is a pre-local base at a point x then $\{\overline{N_x}^o\}$ is a local base at x.

Proof: Let $\mathcal{N}_x$ be a pre-local base at x, and suppose V is an open set such that x ∈ V, since X is regular space then there is an open set U such that $x \in U \subseteq \overline{U} \subseteq V$. Since x ∈ U and U is preopen set, then there exists a preopen set N_x in $\mathcal{N}_x$ such that $x \in N_x \subseteq U \subseteq \overline{U} \subseteq V$, now since N_x is preopen set, we obtain $x \in N_x \subseteq \overline{N_x}^o$ and $\overline{N_x} \subseteq \overline{U} \subseteq V$. So $x \in N_x \subseteq \overline{N_x}^o \subseteq \overline{U}^o \subseteq V^o = V$, then $x \in N_x \subseteq \overline{N_x}^o \subseteq V$. Therefore, for any open set V such that x ∈V there is N_x in $\mathcal{N}_x$ such that $x \in \overline{N_x}^o \subseteq V$ where $\overline{N_x}^o$ is open set. Hence the new collection $\{\overline{N_x}^o\}$ of open sets is a local-base at x.

$$\text{From Pre-Local Base at x} \xRightarrow{\text{Regular Space}} \text{There is Local Base at x.}$$

Corollary 6.2.1. Any regular pre-first countable topological space X is first countable space.

Proof: By the previous theorem if $\{N_x\}$ is a countable pre-local base at a point x in X, then $\{\overline{N_x}^o\}$ is a countable local base at x.

Theorem 6.2.3. In a regular space X, from any pre-base for X there is a base for X.

Proof: Suppose $\mathcal{N}$ is a pre-base for a topological space X, then the collection $\{\overline{N}^o\}$ is a base, because if V is an open set in X, then V is preopen set so for any x ∈ V there is open set U such that $x \in U \subseteq \overline{U} \subseteq V$ (from theorem (6.2.1)) and since U is preopen set, then there is $N \in \mathcal{N}$ such that $x \in N \subseteq U \subseteq \overline{U} \subseteq V$, so $x \in N \subseteq \overline{N}^o \subseteq V$, i.e. $V = \cup \overline{N}^o$.

$$\text{From Pre-Base For X} \xRightarrow{\text{Regular Space}} \text{There is Base for X.}$$

Corollary 6.2.2. Any regular pre-second countable space is second countable space.

Proof: Direct since if {N} is a countable pre-local base for a space X, then from the the previous theorem the collection $\{\overline{N}^o\}$ is a countable-base for X.

6. 3. In Partition Spaces

Definition 6.3.1. [21] A topological space X is called partition space if every open subset of X is closed.

Lemma 6.3.1. In partition space (X,τ), any subset of X is preopen, i.e PO(X,τ)= P(X).

Proof: Suppose A is a subset of X, then $A \subseteq \bar{A}$, since any closed set in partition space is aslo open, we get $\bar{A}$ is open set, i.e. $\bar{A}^{o} = \bar{A}$, so $A \subseteq \bar{A}^{o}$, thus A is preopen set.

Corollary 6.3.1 Partition space X is pre-separable (pre-second countable) space iff the space X is countable space.

Corollary 6.3.2 Every partition space X is pre-first countable space.

Proof: Direct since {{x}} is a countable pre-local base at any point x in X.

7. CONCLUSIONS

Using the concept of preopen sets we introduce new axioms of countability; namely pre-countability axioms, this class of axioms includes; pre-separable spaces, pre-first countable spaces and pre-second countable spaces. In this paper, we study the topological properties of these spaces, as the hereditary property and their images by some particular functions. Moreover, we investigate the behavior of pre-countability axioms in some special spaces as; submaximal spaces, regular spaces and partition spaces.

Outline some of our results:

A. The implication of pre-countability axioms among themselves is shown in the following Diagram 1:

Pre-Second Countable $\Longrightarrow$ Pre-First Countable

$\Downarrow$

Pre-Separable $\Longrightarrow$ Separable

Diagram 1. Implication of pre-countability axioms among themselves.

B. Implication of pre-separable spaces with b-separable and separable spaces is given:

b-Separable $\Longrightarrow$ Pre-Separable $\Longrightarrow$ Separable

where the relation between pre-dense sets with b-dense sets and dense sets is given by:

b-Dense $\Longrightarrow$ Pre-Dense $\Longrightarrow$ Dense $\Longrightarrow$ Preopen

C. Pre-first countable, pre-second countable and pre-separable spaces satisfy the preopen subspace hereditary property.

D. Pre-irresolute map preserves pre-separable spaces, while pre-first countable and pre-second countable spaces are preserved under pre-irresolute and M-preopen maps.

E. In submaximal spaces; pre-countability axioms and countability axioms are equivalent; therefore:

$$\text{Separable} \overset{\text{Submaximal}}{\Longleftrightarrow} \text{Pre-Separable}$$

$$\text{Pre-First countable} \overset{\text{Submaximal}}{\Longleftrightarrow} \text{First Countable}$$

$$\text{Pre-Second Countable} \overset{\text{Submaximal}}{\Longleftrightarrow} \text{Second Countable}$$

F. In regular space we have:

$$\text{Pre-First Countable} \overset{\text{Regular space}}{\Longrightarrow} \text{First Countable}$$

$$\text{Pre-Second Countable} \overset{\text{Regular space}}{\Longrightarrow} \text{Second Countable}$$

Diagram 2. Shows the relation between pre-second countable, pre-first countable, second countable and first countable spaces, in regular spaces:

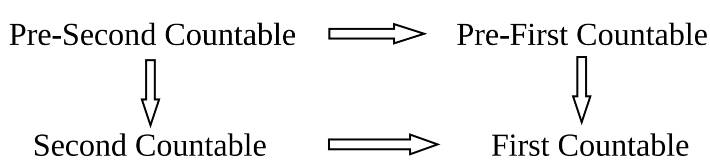

Diagram 2. Implication of pre-countability axioms in regular spaces.

G. Every partition space is pre-first countable, while the partition space is pre-separable (pre-second) spaces iff the space is countable; therefore:

$$\text{Partition Space} \Longrightarrow \text{Pre-First Countable}$$

$$\text{Pre-separable} \overset{\text{Partition}}{\Longleftrightarrow} \text{Countable Spaces}$$

$$\text{Pre-Second Countable} \overset{\text{Partition}}{\Longleftrightarrow} \text{Countable Space}$$

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