Sebha University Journal of Pure & Applied Sciences Vol.21 No. 1 2022
DOI: 10.51984/JOPAS.v21i1.1582

مجلة جامعة سبها للعلوم البحتة والتطبيقية
Sebha University Journal of Pure & Applied Sciences

Journal homepage: www.sebhau.edu.ly/journal/index.php/jopas

On δ-Sequential Spaces

*Amal Emhemd Kornas [a], Khadiga Ali Arwini[b]

[a] Higher Institute of Science and Technology, Tripoli-Libya
[b] Department of Mathematics, Faculty of Science, Tripoli University, Tripoli-Libya

Keywords:
Convergence
Compactness
Countability axioms
Hausdorff spaces
Regular open sets
Sequential spaces

ABSTRACT

Sequence converges is an important research object in topology and analysis, since it is closely related to continuity, compactness and other related properties. In this article, we use the notion of regularly convergence, which is a generalization of the convergence notion, to define the regularly sequentially closed sets and the operator of regularly sequential closure; then we consider thier charactarizations and prove that the convergence and regularly convergence are coincide in regular spaces. Finally we introduce new axioms by involving δ-open sets and regular open sets with the concept of regularly convergence; namely δ-sequential space and r-sequential space, when we show that there are no general relations between these new spaces and the sequential space, in addition we prove some statements as; r-first countable space is δ-sequential, r-sequential space is stronger than δ-sequential space and the spaces δ-sequential and sequential are coincide both in regular spaces and in compact r-T_2 spaces.

الفضاءات المتسلسلة من نوع δ

*أمل امحمد كرناص [a] و خديجة على الرويني [b]

[a] القسم العام، المعهد العالي للعلوم و التقنية، طرابلس- ليبيا
[b] قسم الر ياضيات، جامعة طرابلس، طرابلس- ليبيا

الكلمات المفتاحية:
التقارب
التراص
مسلمات العد
الفضاء هاوسدورف
المجموعات المفتوحة المنتظمة
الفضاءات المتسلسلة

الملخص

تقارب المتتاليات هو مفهوم مهم فى بحوث التبولوجي و التحليل، لما له من علاقة بالاستمرارية و التراص و بعض المفاهيم الاخرى. فى هذا البحث، نستعمل مفهوم التقارب المنتظم و التي هى تعميم لمفهوم التقارب لتعريف المجموعات المغلقة المنتظمة و غلاقة المتسلسلة المنتظمة، تم ندرس خواصهما و نبرهن أن التقارب و التقارب المنتظم يتكافئان فى الفضاءات المنتظمة. أخيرا نعرف مسلمات جديدة باستخدام المجموعات المفتوحة من الصنف δ و المجموعات المفتوحة المنتظمة؛ تسمى: الفضاءات المتسلسلة من الصنف δ و الفضاءات المتسلسلة المنتظمة، وقد بينا أنه لاتوجد علاقة بشكل عام بين تلك الفضاءات و الفضاءات المتسلسة، بالاضافة نثبت بأن فضاء العد الاولى المنتظم هو فضاء متسلسل من الصنف δ، الفضاء المتسلسل المنتظم هو أقوى من الفضاء المتسلسل من الصنف δ، و أيضا الفضاءات المتسلسلة من الصنف δ و الفضاءات المتسلسة تكون متكافئة فى كل من الفضاءات المنتظمة و الفضاءات المتراصة و التى تحقق هاوسدورف المنتظم.

Introduction:

Sequential space originally was due to Franklin in 1965 [1], where he introduced sequential open sets and sequential closed sets, and used them to study the properties of sequential spaces, moreover he showed that this space is one of the weakest axioms of countability [2].

In 2014, Sudip [3] introduced I-sequential topological space, which is a strictly weaker than the first countable space, few years later, Shou Lin and Li Liu [4] defined G-sequential space via the concept of G-closed set, and they proved that every sequential space is G-sequential. Kornas and Arwini in 2020, used the concept of dense sets to defined the space of dense-sequential space; namely D-sequential space, where this space is strictly weaker than the sequential space [5].

The concept of regular open (r-open) set in topological spaces, was introduced in 1937 by Stone [6], this set was also considered in the semiregularization space [7, 8]. The complement of r-open set is called r-closed set, and the family of all r-open sets and r-closed sets

*Corresponding author:
E-mail addresses: amlkornas@gmail.com , (K. Arwini) Kalrawini@yahoo.com

Article History : Received 22 November 2021 - Received in revised form 19 April 2022 - Accepted 30 April 2022

in a space X are denoted by RO(X) and RC(X), respectively. In 2011, Abdullah and Radhy [9] defined the r-limit points, r-neighborhood and studied their properties, and introduced certain types of r-irresolute (r-strongly open and r-strongly closed) functions, in addition, they investigated the concepts of regularly converges via regular open sets. In 2020 [10], Kornas and Arwini used the concept of r-open sets to define a generalization class of countability axioms namely r-countability axioms. This class of axioms includes r-separable spaces, r-first countable spaces, r-Lindelöf spaces, r-σ-compact spaces and r-second countable spaces [10]. In this article, we extend this work to conclude new axioms of countability by involving regular open sets and δ-open sets with the notion of regularly convergence; namely r-sequential space and δ-sequenial space. Then we study their properties and investigate some inter-relations between these new spaces and sequential space in regular space and in compact r-T_2 space.
We divided our article into five main sections as; introduction, preliminaries, r-sequential space, δ-sequenial space and finaly conclusion.

2. Preliminaries

In this section we recall the basic definitions, theorems and some properties regarding regular closed sets, sequential spaces, r-countability axioms and r-compact spaces, needed in this work. Throughout this paper X or (X, τ) represents topological space, and for a subset A of a space X, $\overline{A}, A^{\circ}, X / A$ denote the closure of A, the interior of A, and the complement of the set A in X; respectively.

2.1. Regular Closed Sets

Definition 2.1.1. [6, 11] A subset B of a space (X, τ) is called regular closed (briefly r-closed) if $B = \overline{B^{\circ}}$, while the set B is called δ-closed set if B is the intersection of r-closed sets. The complement of r-closed set is called r-open, and the family of all r-closed sets and r-open sets in X are denoted by $RC(X, \tau)$ and $RO(X, \tau)$, respectively.

Theorem 2.1.2. [6] A subset W of a space X is r-open iff $W = \overline{W}^{\circ}$.

Remark 2.1.3. [11] In a topological space X, we have:

1. Every r-closed set is δ-closed set.
2. Every δ-closed set is closed set.

r-closed set $\Longrightarrow$ δ-closed set $\Longrightarrow$ closed set

Corollary 2.1.4. [6, 11]
1. Intersection of r-closed sets is not necessarily r-closed.
2. Finite union of r-closed sets is r-closed.
3. Finite intersection of r-open sets is r-open.
4. δ-open set is a union of r-open sets.

Theorem 2.1.5. [10] In regular space X, any open set can be expressed as a union of r-open sets.

Corollary 2.1.6. [10] In regular space X, if $A \subseteq X$, then:
1. A is open set if and only if A is δ-open.
2. A is closed if and only if A is δ-closed.

Definition 2.1.7. [9] Let A be a subset of X then, the r-closure of A defined as the intersection of all r-closed sets containing A, and is denoted $\overline{A}^r$.

Theorem 2.1.8. [9, 11] Let X be a space and $A, B \subseteq X$, then:
1. $\overline{A}^r$ is δ-closed set but not r-closed set in general.
2. $A \subseteq \overline{A} \subseteq \overline{A}^r$.
3. If A is r-closed then $A = \overline{A}^r$.
4. A is δ-closed if and only if $A = \overline{A}^r$.
5. $\overline{\overline{A}^r}^r = \overline{A}^r$.
6. If $A \subseteq B$, then $\overline{A}^r \subseteq \overline{B}^r$.

Definition 2.1.9. [9] Let X be a space, and $x \in X$. An r-neighborhood of x is any subset of X which contains an r-open set containing x. The collection of all r-neighborhoods of x is denoted by $N_r(x)$.

2.2. Sequential Spaces

Definition 2.2.1. [12] Let X be a topological space, a sequence of points $(x_n) \in X$ is said converge to a point $x \in X$ (written $x_n \longrightarrow x$) if for every neighborhood U of X which contains x there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ then $x_n \in U$. The point x is called a limit point of (x_n) written $\lim_{n\to\infty} x_n = x$.

Theorem 2.2.2. [1] If $A \subseteq X$, then $\overline{A}$ contains the limits of all its convergent sequences.

Corollary 2.2.3. [13] If A is closed subset of X, then A contains the limits of all its convergent sequences.

Theorem 2.2.4. [14] Let X be a T_2 space and (x_n) be a convergent sequence. Then the $\lim_{n\to\infty} x_n$ is unique.

Theorem 2.2.5. [15] If a space X satisfy that any converges sequence has a unique limit point, then X is T_1.

Lemma 2.2.6. [16] Let X be a first countable space. For each $x \in X$, there is a countable nested local basis $\{U_n\}$ of x such that $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots$ If $x_n \in U_n$ for all n, then x_n is a converges to x.

Theorem 2.2.7. [17] Let X be a first countable space and has the property that every convergent sequence has a unique limit then X is T_2 space.

Definition 2.2.8. [1] Let X be a topological space, and let A be a subset of X. We say that A is a sequential closed if it contains all the limits of all its sequences.

Corollary 2.2.9. [18] In any topological space every closed subset of X is sequentially closed, but not conversely.

closed set $\Longrightarrow$ sequentially closed

Theorem 2.2.10. [14] If X is a first countable space, a subset A of X is closed if and only if whenever a sequence (x_n) in A satisfies $x_n \to x$, then we have $x \in A$.

Theorem 2.2.11. [14] Let X be a first countable space and $Y \subseteq X$, and then $\overline{Y}$ is the set of all limits of sequences from Y.

Corollary 2.2.12. [14] Let X be a first countable space a subset $Y \subseteq X$ is closed if and only if Y is sequentially closed.

Definition 2.2.13. [18] Given a subset A of a topological space X, the sequential closure of A is denoted by $[A]_{seq}$ and defines as: the set of all points $x \in X$ for which there is a sequence in A that converges to x. That is $[A]_{seq} = \{ x \in X: \textit{there exists } (x_n) \in A: (x_n) \to x\}$

Theorem 2.2.14. [18] If A is a subset of a topological space X, then:
1. $A \subseteq [A]_{seq}$ for all $A \subseteq X$.
2. $[A]_{seq} \subseteq \overline{A}$.
3. A is sequentially closed if and only if $A = [A]_{seq}$.

4. If $A \subseteq B$, then $[A]_{seq} \subseteq [B]_{seq}$.
5. $[A \cup B]_{seq} = [A]_{seq} \cup [B]_{seq}$.
6. $[A \cap B]_{seq} \subseteq [A]_{seq} \cap [B]_{seq}$.

Definition 2.2.15. [5] A topological space is said to be sequential space if given any subset of it which is not closed, there is sequence of points in the subset having a limit, which lies outside the subset.

Theorem 2.2.16. [18] Let X be a topological space and A subset of X. Then the following conditions are equivalent:
1. X is sequential space.
2. $[A]_{seq} = \overline{A}$.
3. A is non closed $[A]_{seq} / A \neq \emptyset$.
4. A is sequentially closed then A is closed.

Corollary 2.2.17. [15] Every first countable space is sequential.

$$\text{first countable} \Longrightarrow \text{sequential}$$

Theorem 2.2.18. [10] Relation between countability axioms is given in the following diagram:

Diagram 1. Relation between countability axioms

2.3. *R*-Countability Axioms and *R*-Compactness

Definition 2.3.1. [10] In a topological space X, a collection $\mathfrak{B}_x$ of r-open sets that contains x is called r-local basis at x if for any r-open set W such that $x \in W$ there exists B_x in $\mathfrak{B}_x$ such that $x \in B_x \subseteq W$.

Definition 2.3.2. [10] A topological space X is called:
1. r-separable if there exist a countable r-dense subset of X.
2. r-first countable space if for every $x \in X$ there is a countable r-local base $\mathfrak{B}_x$ of X.
3. r-Lindelöf (nearly Lindelöf) space if every cover of X by regularly open sets has countable subcover.
4. r-second countable space if X has a countable r-base.

Theorem 2.3.3. [10] In regular space: r-countability axioms and countability axioms are equivalent.

$$r\text{-countability axioms} \overset{\text{regular}}{\Longleftrightarrow} \text{countability axioms}$$

Theorem 2.3.4. [10]
1. Every r-second countable space is r-separable, r-first countable and r-Lindelöf space.
2. r-countability axioms are generalization of countability axioms.

Definition 2.3.5. [10] A space X is called r-compact (nearly compact) space if every r-open cover of X has a finite subcover.

Theorem 2.3.6. [9]
1. Any compact space is r-compact.
2. Any closed subset of compact space is compact.

Theorem 2.3.7. [10] The implication of r-countability axioms among themselves and with the classical countability axiom is shown in the following diagram:

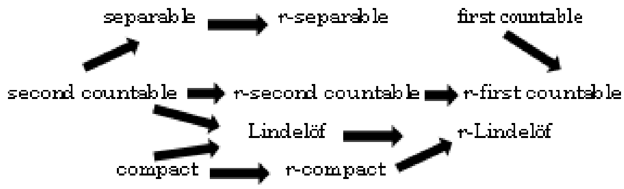


Diagram 2. Relations between r-countability axioms, countability axioms and r-compactness.

2.4. *R*-Separation Axioms

Definition 2.4.1. [19] A space X is said to be:
1. r-T_1 if whenever x, y are distinct points in X, there exist r-open sets U and V such that U containing x but not y, and V containing y but not x.
2. r-T_2 if for every pair of distinct points x, y of X, there exist disjoint r-open sets U and V such that $x \in U$ and $y \in V$.

Corollary 2.4.2. [19]
1. Every r-T_2 space is r-T_1.
2. Every r-T_i space is T_i, where i = 1, 2.

Theorem 2.4.3. The relation between r-separation axioms and the classical separation axioms is given in the following diagram:

Diagram 3. Relations between r-separation axioms and separation axioms.

3. *R*-Sequential Spaces

3.1. Regularly Convergence

Definition 3.1.1. [9] Let X be a topological space, and $(x_n) \in X$ be a sequence. We say that the sequence (x_n) is regularly converges (r–converges) to $x \in X$ (written $x_n \overset{r}{\rightarrow} x$) if for every r-neighborhood $W \subseteq X$ which contains x there exist $N \in \mathbb{N}$ such that for all $n \geq N$ the points x_n line in W. The point x is called a regularly limit point (r-limit point) of (x_n) written $\lim\limits_{n\to\infty}^{r} x_n = x$.

Examples 3.1.2.
1. In the cocountable topological space (X, τ) where $X = \mathbb{R}$, we have $RO(\mathbb{R}, \tau) = \{\mathbb{R}, \emptyset\}$. So the sequence $(n)_1^{\infty}$ is r-converges to 0, but not converges to 0, since $\mathbb{N}^c$ is open set around 0 but does not contain any element of the sequence.
2. In the cofinite topological space $(\mathbb{R}, \tau)$, the sequence $(n)_1^{\infty}$ is r-converges to any real number, since $RO(\mathbb{R}, \tau) = \{\mathbb{R}, \emptyset\}$.
3. In the discrete topological space (X, τ), the sequence that r-converge are ones with constant tails, since $RO(X, \tau) = \tau$.
4. In the trivial topological space on X, every sequence r-converges to any point in the space since $RO(X) = \{X, \emptyset\}$.
5. In the space (X, τ) where $X = \mathbb{R}$ and $\tau = \{\emptyset\} \cup \{U \subseteq X : 0 \in U\}$ we have $RO(\mathbb{R}, \tau) = \{\mathbb{R}, \emptyset\}$. So any sequence is r-converges to any point.

Proposition 3.1.3. In a space X, if a sequence (x_n) converges to a point x, then (x_n) is r-converges to x, but the converse is not true see example (3.1.2 (1)).

Theorem 3.1.4. Let A be a subset in a space X, and let (x_n) be a sequence such that $(x_n) \subseteq A$ and (x_n) is r-converges to x, then $x \in \overline{A}^{r}$.

Proof. Suppose $x \notin \overline{A}^r$, then there exists an r-open set W such that $x \in$ W and W ∩ A = ∅. Since $(x_n) \subseteq$ A, we have W does not have any of (x_n), which is impossible since $(x_n) \xrightarrow{r} x$. Therefore $x \in \overline{A}^r$.

Corollary 3.1.5. If X is a topological space and A ⊆ X is an r-closed subset of X, then A contains the r-limits of all its r-convergent sequences.

Proof. Since A is r-closed subset of a space X, we have $\overline{A}^r = A$, and by the theorem (3.1.4), then A contains the r-limits of all its r-converges sequences.

Theorem 3.1.6. If X is regular space, and the sequence (x_n) is r-converges to a point x in X, then (x_n) converges to x.

Proof. Suppose U is an open set that contains x, then by the theorem (2.1.5), U can be written as; U $=\cup W_\alpha$; where W_α is r-open set for all α. Since $x \in$ U then there exists α such that $x \in W_\alpha$, and since $(x_n) \xrightarrow{r} x$ then there is N ∈ ℕ such that $x_n \in W_\alpha$ for all $n \geq$ N, so we have $x_n \in W_\alpha \subseteq$ U for all $n \geq$ N, thus $(x_n) \to x$.

Corollary 3.1.7. In regular spaces, the concepts of r-converges and converges are equivalent.

Theorem 3.1.8. In r-T_2 space, any r-convergent sequence has a unique limit point.

Proof. Suppouse X is r-T_2 space, and let (x_n) be a sequence in X has two r-limits, say x and y. Since X is r-T_2 space, then there exist disjoint r-open sets W and V with $x \in$ W and $y \in$ V. Since $(x_n) \xrightarrow{r} x$ and $x \in$ W, there is N ∈ ℕ such that $(x_n) \in$ W for all $n \geq$ N, but W ∩ V = ∅, so V contains almost finite numbers of (x_n) members, i.e. $(x_n) \overset{r}{\nrightarrow} y$. This is a contraduction, hence $x = y$.

Theorem 3.1.9. If a space X satisfy that any r-converges sequence has a unique r-limit points, then X is r-T_1.

Proof. Suppose x and y are distinct points in X, then the sequence (x) and (y) are r-converges to x and y; respectively. Hence $(x) \overset{r}{\nrightarrow} y$ and $(y) \overset{r}{\nrightarrow} x$, so there exist r-open sets W and V such that $y \in$ W $\not\ni x$ and $x \in$ V $\not\ni y$. Therefore X is r-T_1 space.

Lemma 3.1.10. Let X be an r-first countable space. For each $x \in$ X, there is a countable nested r-local basis $\{U_n\}$ of x such that $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots$ If $x_n \in U_n$ for all n, then (x_n) is an r-converges to x.

Proof. Let X be an r-first countable space, and $x \in$ X, let $\{V_n: n \in N\}$ be a countable r-local basis at x, and let $U_1 = V_1, U_2 = V_1 \cap V_2, \ldots, U_n = V_1 \cap V_2 \cap \ldots \cap V_n, \ldots,$ then $\{U_n: n \in \mathbb{N}\}$ is the desired nested r-local basis at x. If $x_n \in U_n$ for all n, then since the U_n are nested and form an r-local basis, i.e. x_n is eventually in any r-open set U. Thus (x_n) is an r-converges to x.

Theorem 3.1.11. Let X be an r-first countable space. Then X is r-T_2 if and only if every r-convergent sequence has an unique r-limit.

Proof.
⟹ Direct from theoem (3.1.8).
⟸ Suppose that X is not r-T_2 space, there exist two distinct points $x, y \in$ X, that can not be separated by disjoint r-open sets. Now since X is r-first countable space, there are r-local bases $\{U_n: n \in N\}$ and $\{V_n: n \in N\}$ at x and y; respectively. The r-bases members satisfy $U_n \cap V_n \neq \emptyset$ for all n, since x and y can not be separated by r-open sets. Let $x_n \in U_n \cap V_n$, then (x_n) is r-converges to x and y, this contraduct the assumption. Hence X is r-T_2.

3.2. Regularly Sequentially Closed

Definition 3.2.1. Let X be a topological space, and let A be a subset of X. We say that A is a regularly sequentially closed (brifely r-sequentially closed) if it contains all the r-limits of all its sequences.

Examples 3.2.2.
1. In the discrete topological space X, every subset of X is r-sequential closed and sequentially closed.
2. In the cofinite topological space X, any non-empty finite subset of X is sequentially closed but not r-sequential closed.
3. In the cocountable topological space X, any non-empty set is sequentially closed, because if A ⊆ X, and $(x_n) \subseteq$ A. Let $y \notin$ A, then $\{x_n\}^c$ is open in X and contains y but does not contain any of (x_n) members, so $(x_n) \overset{r}{\nrightarrow} y$ where $y \notin$ A. Now $RO(X, \tau) = \{X, \emptyset\}$. , hence the only r-sequentially closed set is X, i.e. any non-empty proper subset in X is sequentially closed, but not r-sequentially closed.
4. In the usual topology on ℝ the singelton $\{0\}$ is r-sequentially closed but not r-closed.

Corollary 3.2.3. In any topological space, every δ-closed subset of X is r-sequentially closed.

Proof. Let A be a δ-closed subset of X, then $A = \overline{A}^r$, since every r-limits of a sequence is part of $\overline{A}^r$ then A contains the r-limits of all it is sequence, hence A is r-sequentially closed.

Corollary 3.2.4. Every r-closed subset of X is r-sequentially closed, but the converse is not true, see example (3.2.2 (4)).

Corollary 3.2.5. Every r-sequentially closed set is sequentially closed, but the converse is not true, see example (3.2.2 (3)).

Proof. Let A be an r-sequentially closed subset in a space X, and let (x_n) be a sequence in A that converges to x, then (x_n) is r-converges to x, since A is r-sequentially closed subset, then x belongse to A. Hence A is sequentially closed.

Theorem 3.2.6. If X is a regular space then, every sequentially closed set is r-sequentially closed.

Proof. Let A be a sequentially closed subset of X, and let (x_n) be a sequence in A that r-converges to x, since X is regular space, then (x_n) is converges to x and since A is sequentially closed, then x is belongs to A. Hence A is r-sequentially closed.

Theorem 3.2.7. If X is an r-first countable space, then a subset A of X is δ-closed iff whenever a sequence (x_n) in A satisfies $(x_n) \xrightarrow{r} x$, then $x \in$ A.

Proof.
⟹ Direct from theorem (2.1.8 (4)) and theorem (3.1.4).
⟸ Suppose that A is not δ-closed subset of X, and let $x \in \overline{A}^r / A$ and $B_x = \{U_i \subseteq X: i = 1, 2, \ldots\}$ be a countable r-local base at x, by theorem (3.1.9) we define a new family V_i of r-open sets as $V_i = U_1 \cap U_2 \cap \ldots \cap U_i$. From corollary (2.1.4 (3)) the sets V_i are r-open that contain x, and $V_j \subseteq U_i$ for all $j \geq i$. Pick an arbitrary element $x_i \in V_i \cap$ A, and by theorem (3.1.9) x_i is an r-convergent sequence at x, since $x_i \in$ A and $\lim^r_{i\to\infty} x_i = x$, we conclude that $x \in$ A, which is a contradiction. Thus $\overline{A}^r = A$, i.e. A is δ-closed.

Example 3.2.8. The usual topology on $\mathbb{R}$ is r-first countable space (since it is first countable), and if $A = \{0\}$, then the only sequence in A is the constant sequence $(0)_1^{\infty}$, Which is r-converges to $0 \in A$, but A is not r-closed set.

Corollary 3.2.9. If X is an r-first countable space, then a subset A of X is δ-closed iff A is r-sequentially closed.

3.3. Regularly Sequential Closure

Definition 3.3.1. Let A be a subset of a topological space X, the regularly sequential closure (brifely r-sequential closure) of A is denoted by $[A]_{r-seq}$ and defines as: the set of all points $x \in X$ for which there is a sequence in A that r-converges to x: that is $[A]_{r-seq} = \{x \in X: there\ exists\ a\ sequence\ (x_n) \in A: (x_n) \xrightarrow{r} x\}$.

Examples 3.3.2.
1. In the discrete topological space X, $[A]_{r-seq} = \overline{A}^r = A$, for every subsets A of X.
2. In the cofinite topological space X, $[A]_{r-seq} = \overline{A}^r = X$, for every non-empty subsets A of X.

Theorem 3.3.3. Let X be a topological space and A, B are subset of X, then:
1. $A \subseteq [A]_{r-seq}$ for all $A \subseteq X$.
2. $[A]_{r-seq} \subseteq \overline{A}^r$.
3. $[A]_{seq} \subseteq [A]_{r-seq}$.
4. A is r-sequential closed iff $A = [A]_{r-seq}$.
5. If $A \subseteq B$, then $[A]_{r-seq} \subseteq [B]_{r-seq}$.
6. $[A \cup B]_{r-seq} = [A]_{r-seq} \cup [B]_{r-seq}$.
7. $[A \cap B]_{r-seq} \subseteq [A]_{r-seq} \cap [B]_{r-seq}$.

Proof.
1. Let $x \in A$, then the constant sequence (x) r-converges to x, hence $x \in [A]_{r-seq}$.
2. Every r-limit of a sequence is part of r-closure.
3. Suppose $x \notin [A]_{r-seq}$, i.e. any r-converges sequence (x_n) in A can not r-converges to x, then any converges sequence (x_n) in A can not converges to x, thus $x \notin [A]_{seq}$.
4. $\Longrightarrow$ From number (1) we have $A \subseteq [A]_{r-seq} \longrightarrow (1)$. Now let $x \in [A]_{r-seq}$, then there exist $(x_n) \in A$ such that (x_n) r-converges to x, since A is r-sequential closed, then $x \in A$, hence $[A]_{r-seq} \subseteq A \longrightarrow (2)$, by (1) and (2) we have $A = [A]_{r-seq}$.
$\Longleftarrow$ Let (x_n) be a sequence in A, and (x_n) is r-converges to x, $x \in X$ then $x \in [A]_{r-seq}$, so $A = [A]_{r-seq}$, then $x \in A$. Hence A is r-sequentially closed.
5. Suppose $x \notin [B]_{r-seq}$, so any r-converges sequence in B can not r-converges to x, and since $A \subseteq B$, then any r-converges sequence in A can not r-converges to x, then $x \notin [A]_{r-seq}$. Hence $[A]_{r-seq} \subseteq [B]_{r-seq}$.
6. $\Longleftarrow$ Since $A \subseteq A \cup B$, and $B \subseteq A \cup B$, we obtain $[A]_{r-seq} \subseteq [A \cup B]_{r-seq}$, and $[B]_{r-seq} \subseteq [A \cup B]_{r-seq}$, then $[A]_{r-seq} \cup [B]_{r-seq} \subseteq [A \cup B]_{r-seq}$.
$\Longrightarrow$ Let $x \in [A \cup B]_{r-seq}$, then there exist $(x_n) \in A \cup B$ r-converges to x, then infinite numbers of the sequence (x_n) contained in A or B, so we obtain a subsequence in A or B that r-converges to x, i.e. $x \in [A]_{r-seq}$, or $x \in [B]_{r-seq}$, hence $x \in [A]_{r-seq} \cup [B]_{r-seq}$.
7. Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$, then $[A \cap B]_{r-seq} \subseteq [A]_{r-seq}$ and $[A \cap B]_{r-seq} \subseteq [B]_{r-seq}$. Hence $[A \cap B]_{r-seq} \subseteq [A]_{r-seq} \cap [B]_{r-seq}$.

Example 3.3.4. Let $X = \mathbb{N}$, $\tau = \{\mathbb{N}, \emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \dots\}$, then $RO(X, \tau) = \{\mathbb{N}, \emptyset\}$. If $A = \{1, 3, 5, 7, \dots\}$, and $B = \{2, 4, 6, 8, \dots\}$, then $[A]_{r-seq} = \mathbb{N}$, and $[B]_{r-seq} = \mathbb{N}$, $[A]_{r-seq} \cap [B]_{r-seq} = \mathbb{N}$, but $[A \cap B]_{r-seq} = \emptyset$, then $[A \cap B]_{r-seq} \neq [A]_{r-seq} \cap [B]_{r-seq}$.

3.4. *R*−Sequential Space

Definition 3.4.1. A topological apace X is said to be regularly sequential (brifely r-sequential) space if given any subset of it which is not r-closed there is sequence of points in the subset having an r-limits, which lies outsid the subset.

Examples 3.4.2.
1. The cocountable topological space X is r-sequential, but it is not sequential (because any uncountable proper set is sequentially closed but not closed).
2. The discrete topological space on uncountable X is r-sequential, but it is not r-separable, r-Lindelöf, r-σ-compact nor r-second countable.
3. The cofinite topological space X is r-sequential and sequential space.
4. The usual topology is sequential space but not r-sequential space, since $\{0\}$ is r-sequential closed but not r-closed.

Theorem 3.4.3. Let X be a topological space and A subset of X. Then the following conditions are equivalent:
1. X is r-sequential space.
2. If A is r-sequentially closed, then A is r-closed.

Proof. 1⟹2) Let X be an r-sequential space, and let A be an r-sequentially closed subset of X, suppose A is not r-closed then from definition r-sequential space there is a sequence $(x_n) \subseteq A$, and $(x_n) \xrightarrow{r} x$ where $x \notin A$, but this contraduct, that A is r-sequentially closed, hence A is r-closed. 2⟹1) Suppose A is a non r-closed subset of X, then by assumption A is non r-sequentially closed, i.e. there exist a sequence $(x_n) \subseteq A$ and $(x_n) \xrightarrow{r} x$, and $x \notin A$. Thus X is r-sequential space.

Theorem 3.4.4. Let X be a topological space and A subset of X. Then the following conditions are equivalent:
1. X is r-sequential space.
2. If A is non r-closed, then $[A]_{r-seq} / A \neq \emptyset$.

Proof.
1⟹2) Let X be a r-sequential space, and a subset A of X is non r-closed, then there is $(x_n) \subseteq A$ with $(x_n) \xrightarrow{r} x$ where $x \notin A$, i.e. $x \in [A]_{r-seq}$, hence $[A]_{r-seq} / A \neq \emptyset$.

2⟹1) Suppose A is a non r-closed subset of X, then since $[A]_{r-seq} / A \neq \emptyset$ there is $x \in [A]_{r-seq}, x \notin A$, i.e. there is $(x_n) \subseteq A, (x_n) \xrightarrow{r} x$, but $x \notin A$. Thus X is r-sequential space.

4. δ-Sequential Spaces

Definition 4.1. A topological apace X is said to be δ-sequential space if given any subset of it which is not δ-closed there is sequence of points in the subset having an r-limits, which lies outsid the subset.

Corollary 4.2. Every r-sequential space is δ-sequential space, but the converse is not true, see example (3.4.3(4)).

Theorem 4.3. Let X be a topological space and A subset of X. Then the following conditions are equivalent:
1. X is δ-sequential space.
2. If A is r-sequentially closed, then A is δ-closed.

Theorem 4.4. Let X be a topological space and A subset of X. Then the following conditions are equivalent:
1. X is δ-sequential space.
2. If A is non δ-closed, then $[A]_{r-seq} / A \neq \emptyset$.

Theorem 4.5. In regular space X, the following conditions are equivalent:

1. X is δ-sequential space.
2. X is sequential space.

Proof.
1⟹2) Suppose A is a sequentially closed subset of X, then from theorem (3.2.6) A is r-sequentially closed, so A is δ-closed set, we obtain A is closed set. Thus X is sequential space.
2⟹1) Suppose A is r-sequentially closed subset of X, then A is sequentially closed, since X is sequenital space then A is closed set in a regular space, we obtain A is δ-closed set. Thus X is δ-sequential space.

Corollary 4.6. Every regular r-sequential space is sequential.

Proof. Direct since any r-sequential space is δ-sequential.

Theorem 4.7. If A is a subset of an r-first countable space X, then $[\mathrm{A}]_{r-seq} = \overline{\mathrm{A}}^{r}$.

Proof. From the theorem (3.3.3 (2)) we have $[\mathrm{A}]_{r-seq} \subseteq \overline{\mathrm{A}}^{r} \longrightarrow (1)$. Now suppose $x \in \overline{\mathrm{A}}^{r}$, since X is r-first countable space, we may choose a countable nested r-local basis $\{\mathrm{U}_n\}$ of x such that $\mathrm{U}_1 \supseteq \mathrm{U}_2 \supseteq \cdots \supseteq \mathrm{U}_n \supseteq \cdots$, since U_n is r-open set and $x \in \mathrm{U}_n, x \in \overline{\mathrm{A}}^{r}$, each U_n intersect A, so choose $x_n \in \mathrm{U}_n \cap \mathrm{A}$, hence (x_n) is r-converges to x, i.e. $\overline{\mathrm{A}}^{r} \subseteq [\mathrm{A}]_{r-seq} \longrightarrow (2)$. From (1) and (2) we get $[\mathrm{A}]_{r-seq} = \overline{\mathrm{A}}^{r}$.

Corollary 4.8. In r-first countable space, any r-sequentially closed set is δ-closed.

Corollary 4.9. Every r-first countable space is δ-sequential.

Example 4.10. The usual topology is r-first countable space but not r-sequential, since $\{0\}$ is r-sequential closed but not r-closed.

Lemma 4.11. Every closed subset of a compact, r-T_2 space is δ-closed.

Proof. Let A be a closed subspace of a compact, r-T_2 space X. Since X is compact, then A is compact, so it is r-compact from theorem (2.3.6 (1)). Now fix $y \in \mathrm{A}^c$, since X is r-T_2 then for each $x \in \mathrm{A}$ there are disjoint r-open sets U_x and V_x such that $x \in U_x$ and $y \in V_x$. We obtain an r-open cover $\{U_x : x \in \mathrm{A}\}$ for A. Since A is r-compact then it has a finite subcover $\{U_x : x \in \mathrm{I} \subseteq \mathrm{A} \text{ and } \mathrm{I} \text{ is finite}\}$. Clearly $\mathrm{U} = \bigcup_{x\in \mathrm{I}} U_x$ and $\mathrm{V} = \bigcap_{i\in \mathrm{I}} V_x$ are disjoint, and V is r–open set satisfy $y \in \mathrm{V} \subseteq \mathrm{A}^c$. Since y is an arbitrary point of A^c, $\mathrm{A}^c = \bigcup_{y\in \mathrm{A}^c} V$, where V is r-open for any $y \in \mathrm{A}^c$, so A is δ-closed.

Remark 4.12. In compact, r-T_2 space X, a subset A of X is open if and only if A is δ-open.

Corollary 4.13. Every sequentially closed subset of a compact, r-T_2 space is r-sequentially closed.

Proof
⟸ Direct since any r-sequentially closed set is sequentially closed.
⟹ Suppose A is an sequentially closed subset of X, and a sequence $(x_n) \subseteq \mathrm{A}$ with $(x_n) \overset{r}{\rightarrow} x$. If V is an open set such that $x \in \mathrm{V}$, then form remark (4.12) and corollary (2.1.2 (4), V is δ-open set then there is $\mathrm{N} \in \mathbb{N}$ such that $x_n \in \mathrm{V}$ for all $n \geq \mathrm{N}$, thus $(x_n) \rightarrow x$, so $x \in \mathrm{A}$ i.e. A is r-sequentially closed set.

Theorem 4.14. Let X be a compact, r-T_2 space. Then the following conditions are equivalent:
1. X is δ-sequential space.
2. X is sequential space.

Proof
1⟹2) Suppose A is an sequentially closed subset of X, then by corollary (2.1.6), A is a r-sequentially closed set and since X is δ-sequential space, i.e. A is δ-closed set, then A is closed set. Hence X is sequential space.

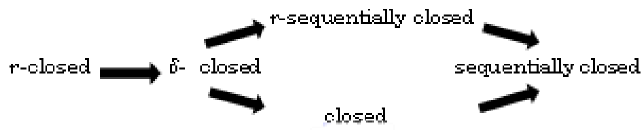
2⟹1) Suppose A is an r-sequentially closed subset of X, then by corollary (3.2.4), A is a sequentially closed set and since X is sequential space, i.e. A is closed set, then by theorem (2.1.2) A is δ-closed set. Hence X is δ-sequential space.

5. Conclusion

The purpose of this paper is to introduce the notions of regularly sequentially closed set and the operator of regularly sequential closure, when we illustrate their properties. Then we use these notions to define two spaces, namely r-sequential space and δ-sequenial space, we investigate the characterizations of these spaces, and their behavior in regular space and in compact r-T_2 space. Our results are summarize in the following diagrams:

A. convergence ⟹ r-convergence.

B. Here we give the implication of r-sequentially closed, sequentially closed, δ-closed, r-closed and closed sets:



C. r-sequential space ⟹ δ-sequential space.

D. r-closed set $\overset{r-\text{sequential space}}{\Longleftrightarrow}$ r-sequentially closed set.

E. δ-closed set $\overset{\delta-\text{sequential space}}{\Longleftrightarrow}$ r-sequentially closed set.

F. In regular space we have:

- Convergence $\overset{\text{regular}}{\Longleftrightarrow}$ r-convergence.
- r-sequentially closed set $\overset{\text{regular}}{\Longleftrightarrow}$ sequentially closed set.
- δ-sequential space $\overset{\text{regular}}{\Longleftrightarrow}$ sequential space.
- r-sequential space $\overset{\text{regular}}{\Longrightarrow}$ sequential space.

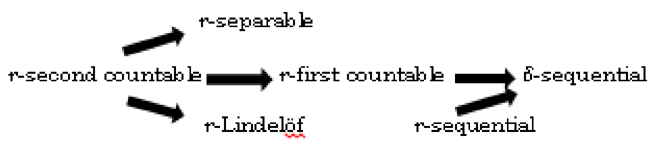
G. In r-first countable space, we have:

- r-T_2 $\overset{r-\text{first countable}}{\Longleftrightarrow}$ unique r-limit.
- δ-closed set $\overset{r-\text{first countable}}{\Longleftrightarrow}$ r-sequentially closed set.

H. In compact r-T_2 space, we have:

- closed set $\overset{\text{compact } r-T_2}{\Longleftrightarrow}$ δ-closed set.
- Sequentially closed set $\overset{\text{compact } r-T_2}{\Longleftrightarrow}$ r-sequentially closed set.
- sequential space $\overset{\text{compact } r-T_2}{\Longleftrightarrow}$ δ-sequential space.

I. We illustrate the implication of δ-sequential space with the class of r-countability axioms:



References

[1]- S. P. Franklin, Spaces in Which Sequences Suffice I, Fund. Math. 57 (1965) 107-115.
[2]- S. P. Franklin, Spaces in Which Sequences Suffice II, Fund. Math. 61 (1967) 51-56.
[3]- S. K. Pal, I -Sequential Topological Spaces, Applied Mathematics E-Notes, 14 (2014) 236-241.
[4]- S. Lin, L. Liu, G-Methods G-sequential Spaces and G-Continuity in topological Spaces, Topology and its Applications. 212 (2016) 29-48.
[5]- K. A. Arwini, A. E. Kornas, D -Countability Axioms, an International Scientific Journal. 143 (2020) 28-38.
[6]- M. H. Stone, Applications of the Theory of Boolean Rings to General Topology, Trans. Am. Math. Soc. 41 (1937) 375-481.
[7]- C. Ronse, Regular Open or Closed Sets, Philips Research Laboratory Brussels, A v. E van Beclaere 2 B (1990) 1170.
[8]- C. Dorsett, Regularly Open Sets and R-Topological properties, Nat. Acod. Sci. Letters, 10 (1987) 17-21.
[9]- H. K. Abdullah, F. K. Radhy, Strongly Regular Proper Mappings, Journal of Al-Qadisiyah for Computer Science and Mathematics 3(1) (2011) 185-204.
[10]- A. E. Kornas, K. A. Arwini, R -Countability Axioms, an International Scientific Journal. 149 (2020) 92-109.
[11]- N. V. Velicko, H-Closed Topological Spaces, Amer. Mat. Soc. Transl., 78 (1968) 103-118.
[12]- N. Bourbaki, General Topology, Addison-Wesley, Part 1, Reading, Massachusetts, (1966).
[13]- A. Goreham, Sequential Convergence in Topological Spaces, Oxford University, (2001).
[14]- M. Frankland, General Topology additional Notes, University of Illinois at Urbana- Champaign, (2012).
[15]- T. P. Kremsater, Sequential Space Methods, The University of British Columbia (1972).
[16]- C. Wendl, F. Schmaschke, Notes on Nets and Convergence in Topology, Humboldt-Universitat, ZU Berlin, (2017).
[17]- H. F. Cullen, Unique Sequential Limits, Boll. Un. Mat. Ital., Ser. 3-20 (1965) 123-124.
[18]- J. Dereziński, General Topology, Department of Mathematical Methods in Physics, Warsaw University (2006).
[19]- S. P. Arya, T. M. Nour, Separation Axioms for Bitopological Spaces, Indian J. Pure appl. Math., 19(1) (1988) 42-50.