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QUALITATIVE PROPERTIES OF SOME HIGHER ORDER DIFFERENCE EQUATIONS

H. EL-METWALLY, E.M. ELABBASY AND A. ESHTIBA

Abstract. The main objective of this paper is to study the global attractivity, and the boundedness for the solutions of the rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^{p} + \gamma x_{n-m}^{q}}{A x_{n-k}^{p} + B x_{n-m}^{q}}, \quad n \geq 0,$$

where the parameters $\alpha, \beta, \gamma, A, B, p$ and $q \in (0, \infty)$ and the initial conditions $x_{-l}, x_{-l+1}, ..., x_{-1}, x_0$ where $l = \max\{k, m\}$ are positive real numbers.

1. INTRODUCTION

Recursive sequences are also often called difference equations, which are very important in mathematical theory and application [1-13]. Hence, it is very valuable to investigate the behavior of solutions of the system of difference equations and to present thestability character of equilibrium points.

In this paper we study the global attractivity, and the boundedness for the solutions of the rational difference equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^{p} + \gamma x_{n-m}^{q}}{A x_{n-k}^{p} + B x_{n-m}^{q}}, \quad n \geq 0, \tag{1}$$

where the parameters $\alpha, \beta, \gamma, A, B, p$, and $q \in (0, \infty)$ and the initial conditions $x_{-l}, x_{-l+1}, ..., x_{-1}, x_0$ where $l = \max\{k, m\}$ are positive real numbers.

Here, we recall some notations and results which will be useful in our investigation.

Let I be an interval real numbers and let $f : I^{k+1} \times I \to I$ be continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, 2, ..., \tag{2}$$

with $x_{-k}, x_{-k+1}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(2).

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The linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = p_1 y_n + p_2 y_{n-1} + ... + p_{k+1} y_{n-k},$$

where $p_i = \frac{\partial f}{\partial x_{n_i}}(\overline{x}, \overline{x}, ..., \overline{x})$, $i = 0, 1, ..., k$.

Theorem A [15]: Assume that $p_1, p_2, ..., p_{k+1} \in R$. *Then*

$$\sum_{i=1}^{k+1} | p_i | < 1,$$

is a sufficient condition for the locally stability of Eq.(2).

Theorem B [15]: Consider the difference equation

$$y_{n+1} = g(y_n, ..., y_{n-k}), \quad n = 0, 1, ..., \tag{3}$$

where $g \in C[(0,\infty)^{k+1}, (0,\infty)]$ is increasing in each of its arguments and where the initial conditions $y_{-k}, ..., y_0$ are positive. Assume that Eq.(3) has a unique positive equilibrium $\overline{x}$ and suppose that the function h defined by

$$h(y) = g(y, y, ..., y), \quad y \in (0, \infty),$$

satisfies

$$(h(y) - y)(y - \overline{y}) < 0 \quad \text{for} \quad x \neq \overline{x}.$$

Then $\overline{y}$ is a global attractor of all positive solutions of Eq.(3).

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \geq 0. \tag{4}$$

Theorem C [14]: Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y)$ is non-increasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is non-decreasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) Eq.(4) has no solutions of prime period two in $[a, b]$.

Then $Eq.(4)$ has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of $Eq.(4)$ converges to $\overline{x}$.

2. Main Results

The work of this paper dividel into two parts; Part **I** concerned with the special cases of Eq.(1) and Part **II** deals with the general Eq.(1).

Part I

Here, we consider the following cases of Eq.(1).

(1) Whenever $A = \gamma = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \quad n \geq 0. \tag{5}$$

(2) Whenever $A = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{B x_{n-m}^q},$$

or

$$x_{n+1} = C + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \quad n \geq 0, \tag{6}$$

where $C = \alpha + \frac{\gamma}{B}$.

3. Whenever $\beta = B = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p}, \quad n \geq 0. \tag{7}$$

4. Whenever $B = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{A x_{n-k}^p},$$

or

$$x_{n+1} = D + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p}, \quad n \geq 0, \tag{8}$$

where $D = \alpha + \frac{\beta}{A}$.

5. Whenever $\beta = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\gamma x_{n-m}^q}{A x_{n-k}^p + B x_{n-m}^q}, \quad n \geq 0. \tag{9}$$

6. Whenever $\gamma = 0$ then Eq.(1) has the form

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{A x_{n-k}^p + B x_{n-m}^q}, \quad n \geq 0. \tag{10}$$

In the following we investigate the behavior of the solutions to the special cases of Eq.(1).

Case 1. Study of Eq.(5)

In this section, we study the local stability, the boundedness, global attractivity, oscillatery, and periodicity for the solutions of the equation

$$x_{n+1} = \alpha + \frac{\beta x_{n-k}^p}{B x_{n-m}^q}, \quad n \geq 0.$$

Local Stability and boundedness of Eq.(5)

It is easy to see that Eq.(5) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta \overline{x}^p}{B \overline{x}^q}.$$

Let $f : (0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(u, v) = \alpha + \frac{\beta u^p}{B v^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = E\frac{pu^{p-1}}{v^q}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -E\frac{qv^{q-1}u^p}{(v^q)^2},$$

where $E = \frac{\beta}{B}$. Set

$$p_1 = Ep\overline{x}^{p-q-1}, \quad \text{and} \quad p_2 = -Eq\overline{x}^{p-q-1}.$$

Then the linearized equation of Eq.(5) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x}, \overline{x})$, and $p_1 = -f_v(\overline{x}, \overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2\lambda^{k-m} + p_1 = 0.$$

Theorem 1. *If* $\overline{x} < \frac{1}{\sqrt[p-q-1]{E(p+q)}}$, *then the positive equilibrium point* $\overline{x}$ *of Eq.(5) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = Ep\overline{x}^{p-q-1}$, and $p_2 = -Eq\overline{x}^{p-q-1}$. Then

$$|p_1| + |p_2| < 1 \Leftrightarrow Ep\overline{x}^{p-q-1} + Eq\overline{x}^{p-q-1} < 1,$$

which is valid iff

$$\overline{x}^{p+q-1} < \frac{1}{E(p+q)}.$$

So by Theorem A $\overline{x}$ is locally asymptotically stable when $\overline{x} < \frac{1}{\sqrt[p-q-1]{E(p+q)}}$. □

Here, we investigate the bounded character of Eq.(5).

Theorem 2. *If* $0 < p < 1$, *then the Eq.(5) is bounded and persists.*

Proof. Assume that $\{x_n\}$ be a solution of Eq.(5). We obtain from Eq.(5) that

$$x_{n+1} > \alpha, \quad \text{for } n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(5) that

$$x_{n+1} \leq \alpha + Lx_{n-k}^p,$$

where $L = \frac{\beta}{B\alpha^q}$. Now we consider the difference equation

$$y_{n+1} = \alpha + Ly_n^p, \quad \text{for } n \geq 0. \tag{11}$$

Let $\{y_n\}$ be a solution of Eq.(11) with $y_0 = x_0$. Then obviously

$$x_{n+1} \leq y_{n+1}, \quad \text{for } n = 0, 1, \dots .$$

We shall prove that the sequence $\{y_n\}$ is bounded. Let

$$f(x) = \alpha + Lx^p.$$

Then

$$f^{'}(x) = Lpx^{p-1} > 0, \quad \text{and} \quad f^{''}(x) = Lp(p-1)x^{p-2} < 0.$$

Therefore the function f is increasing and concave . Thus we obtain that there is a unique fixed point y^* of the equation $f(y) = y$. Also the function f satisfies

$$(f(y) - y)(y - y^*) < 0, \quad y \in (0, \infty).$$

It follows by Theorem C that y^* is a global attractor of all positive solutions of Eq.(11) and so $\{y_n\}$ is bounded. Therefore from Eq.(5) the sequence $\{x_n\}$ is also bounded. This completes the proof of the theorem. □

Global attractivity for Eq.(5)

Here we study the global asymptotic stability of the positive solutions of Eq.(5).

Theorem 3. *Assume that* $0 < p < 1 < q$, $\alpha > E(p+q-1)^{\frac{1}{q-p+1}}$. *Then every positive solution of Eq.(5) converges to the unique positive equilibrium point* $\overline{x}$ *of Eq.(5).*

Proof. Note that when $0 < p < 1 < q$, it was shown in Theorem 2 that every positive solution of Eq.(5) is bounded. Then we have the following

$$s = \lim_{n\to\infty} \inf x_n, \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_n.$$

It is clear that $s \leq S$. We want to proof that $s \geq S$. Now it is easy to see from Eq.(5) that

$$s \geq \alpha + E\frac{s^p}{S^q}, \quad \text{and} \quad S \leq \alpha + E\frac{S^p}{s^q}.$$

Thus we have

$$sS^q \geq \alpha S^q + Es^p, \quad \text{and} \quad s^q S \leq as^q + ES^p.$$

Thus

$$\alpha s^{q-1}S^q + Es^p s^{q-1} \leq \alpha s^q S^{q-1} + ES^p S^{q-1}.$$

Then we get

$$\alpha S^{q-1}s^{q-1}(S-s) \leq E(S^{p+q-1} - s^{p+q-1}).$$

So

$$\alpha S^{q-1}s^{q-1} \leq E\frac{S^{p+q-1} - s^{p+q-1}}{S-s}. \tag{12}$$

If we consider the function x^{p+q-1}, then there exists a $c \in (s, S)$ such that

$$\frac{S^{p+q-1} - s^{p+q-1}}{S-s} = (p+q-1)c^{p+q-2} \leq (p+q-1)S^{p+q-2}. \tag{13}$$

Theen from (12) and (13) we get

$$\alpha S^{q-1}s^{q-1} \leq E(p+q-1)S^{p+q-2}.$$

or

$$\alpha S^{1-p}s^{q-1} \leq E(p+q-1).$$

Since $S \geq \alpha$ and $s \leq \alpha$. Then we obtain

$$\alpha\alpha^{1-p}\alpha^{q-1} = \alpha^{q-p+1} \leq E(p+q-1).$$

which contradicts to $0 < p < 1 < q$. Which implies that $s = S$. Thus the proof is complete. □

Example 1. *Figure (1) shows the global attractivity of the equilibrium point* $\overline{x} = 1.1837$ *of Eq.(5) whenever* $x_{-1} = 5.6487$, $x_0 = 1.0231$, $p = 0.5$, $q = 0.9$, $\alpha = 0.7$, $\beta = 0.19$, *and* $B = 0.52$.

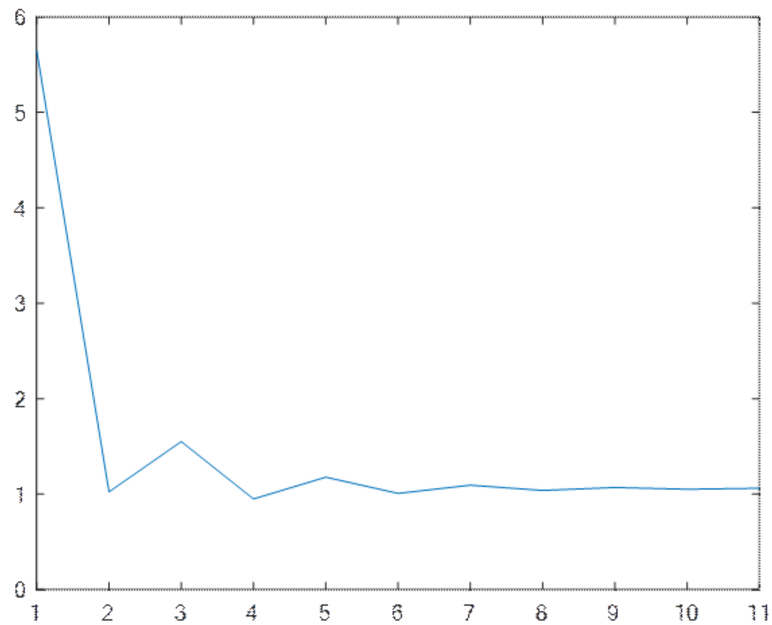

Figure (1)

Oscillatery of the solutions for Eq.(5)

In the next theorem, we study the oscillatery character of Eq.(5).

Theorem 4. *Assume that k is odd and m is even and $m < k$, then Eq.(5) has oscillatory solutions.*

Proof. Case (1) let $\{x_n\}$ be a solution of Eq.(5)with

$$x_{-k}, x_{-k+1}, ..., x_{-1} \geq \overline{x}, \quad \text{and} \quad x_{-m+1}, x_{-m+1}, ..., x_0 < \overline{x}.$$

We get from Eq.(5) that

$$x_1 = \alpha + \frac{\beta x_{-k}^p}{B x_{-m}^q} \geq \alpha + \frac{\beta \overline{x}^p}{B \overline{x}^q} = \overline{x},$$

and

$$x_2 = \alpha + \frac{\beta x_{-m+1}^p}{B x_{-k+1}^q} < \alpha + \frac{\beta \overline{x}^p}{B \overline{x}^q} = \overline{x}$$

Then, the result follows by induction.
Case (2) let

$$x_{-m}, x_{-m+1}, ..., x_0 \geq \overline{x}, \quad \text{and} \quad x_{-k+1}, x_{-k+1}, ..., x_{-1} < \overline{x}.$$

is similary the case (1). Then it will be omitted. □

Example 2. *Figure (2) shows the oscillatory solutions of Eq.(5) whenever* $x_{-1} = 1.6487$, $x_0 = 2.0231$, $\alpha = 0.23$, $p = 0.2$, $q = 2$, $\beta = 0.9$, *and* $B = 0.5$.

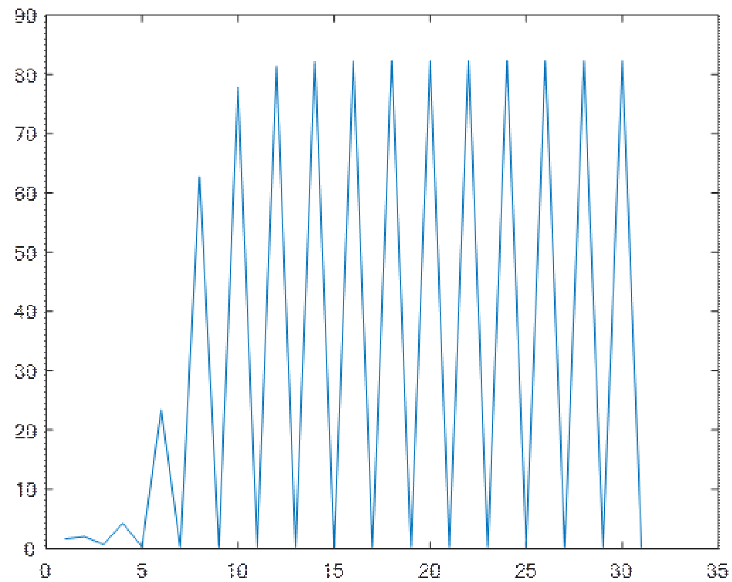

Figure (2)

Periodicity for Eq.(5)

The next theorem deals with the existence of periodic solutions to Eq.5).

Theorem 5. *Let k is odd and m is even. If $< 0 < p < 1 < q$, then a solution of Eq.(5) is a periodic solution of priod period two.*

Proof. Let $\{x_n\}$ be a solution of Eq.(5), with the initial values, we must find some positive numbers x_{-1}, x_0 such that

$$x_{-1} = \frac{\alpha B x_0^q + \beta x_{-1}^p}{B x_0^q}, \quad \text{and} \quad x_0 = \alpha + \frac{\alpha B x_{-1}^q + \beta x_0^p}{B x_{-1}^q}. \tag{14}$$

Let $x_{-} = x$, and $x_0 = y$, then we obtain from (14)

$$x = \frac{\alpha B y^q + \beta x^p}{B y^q}, \quad \text{and} \quad y = \frac{\alpha B x^q + \beta y^p}{B x^q}. \tag{15}$$

Now we want to prove that (15) has a solution (x, y), $x > 0$, $y > 0$. From the first relation of (15) we have

$$y = \frac{\beta^{\frac{1}{q}} x^{\frac{p}{q}}}{B^{\frac{1}{q}} (x - \alpha)^{\frac{1}{q}}}. \tag{16}$$

From (16) and the second relation of (15) we get

$$\frac{\beta^{\frac{1}{q}} x^{\frac{p}{q}}}{B^{\frac{1}{q}}(x-\alpha)^{\frac{1}{q}}} - \frac{\beta^{\frac{p+q}{q}} x^{\frac{p^2-q^2}{q}}}{B^{\frac{p+q}{q}}(x-\alpha)^{\frac{p}{q}}} - \alpha = 0.$$

Now define the function

$$f(x) = \frac{1}{(x-\alpha)^{\frac{1}{q}}}((\frac{\beta}{B})^{\frac{1}{q}} x^{\frac{p}{q}} - (\frac{\beta}{B})^{\frac{p+q}{q}} x^{\frac{p^2-q^2}{q}} (x-\alpha)^{\frac{1-p}{q}}) - \alpha, \quad x > \alpha. \tag{17}$$

Then

$$\lim_{x\to\alpha^+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -\alpha.$$

Hence Eq.(17) has at least one solution $x > \alpha$. Then if $\overline{y} = \frac{\beta^{\frac{1}{q}} \overline{x}^{\frac{p}{q}}}{B^{\frac{1}{q}}(\overline{x}-\alpha)^{\frac{1}{q}}}$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. Thus the proof is complete. □

Example 3. *Figure (3) shows the periodicity solutions of Eq.(5) whenever* $x_{-1} = 1.737$, $x_0 = 2.423$, $\alpha = 0.7$, $p = 0.2$, $q = 4$, $\beta = 0.5$, *and* $B = 0.32$.

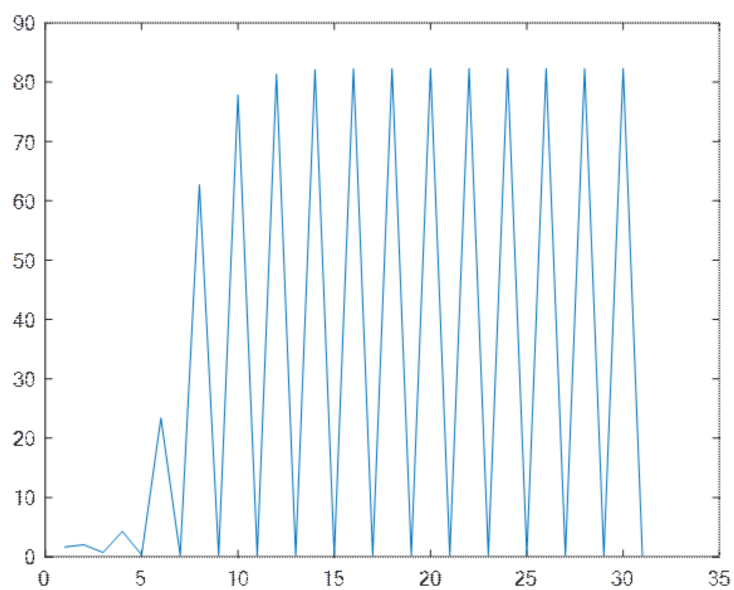

Figure (3)

Case 2. Study of Eq.(6)

This equation is similar of Eq.(5) and its investigation is similar to Eq.(5) and so will be omitted.

Case 3. Study of Eq.(7)

The proofs of the theorems in this section are similar to the proofs of the theorems in Section 3 and will be left to the reader.

Theorem 6. *If* $\overline{x} < \frac{1}{\sqrt[q-p-1]{F(p+q)}}$, *then the positive equilibrium point* $\overline{x}$ *of Eq.(7) is locally asymptotically stable, and is called a sink.*

Theorem 7. *If* $0 < q < 1$, *then the Eq.(7) is bounded and persists.*

Theorem 8. *Assume that* $0 < q < 1 < p$, $\alpha > F(q+p-1)^{\frac{1}{p-q+1}}$. *Then every positive solution of Eq.(7) converges to the unique positive equilibrium point* $\overline{x}$ *of Eq.(7).*

Theorem 9. *Assume that* m *is odd and* k *is even and* $k < m$, *then Eq.(7) has oscillatory solutions.*

Theorem 10. *Let* m *is odd and* k *is even. If* $< 0 < q < 1 < p$, *then Eq.(7) has periodic solutions of priod two.*

Case 4. Study of Eq.(8)

This equation is similar of Eq.(7) and its investigation is similar to Eq.(7) and so will be omitted.

Case 5. Study of Eq.(9)

Local Stability and boundedness for Eq.(9)

Eq.(9) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta \overline{x}^p}{A\overline{x}^p + B\overline{x}^q}.$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = \alpha + \frac{\beta u^p}{Au^p + aBv^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = \frac{A\beta p v^q u^{p-1}}{(Au^p + Bv^q)^2}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -\frac{\beta B q v^{q-1} u^p}{(Au^p + Bv^q)^2},$$

Set

$$p_1 = \frac{A\beta p \overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}, \quad \text{and} \quad p_2 = -\frac{B\beta q \overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}.$$

Then the linearized equation of Eq.(9) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x}, \overline{x})$, and $p_1 = -f_v(\overline{x}, \overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2\lambda^{k-m} + p_1 = 0.$$

Theorem 11. *If* $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}$, *then the positive equilibrium point* $\overline{x}$ *of Eq.(9) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = \frac{A\beta p\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}$, and $p_2 = -\frac{B\beta q\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2}$. Therefore

$$|p_1| + |p_2| < 1 \Leftrightarrow \frac{A\beta p\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2} + \frac{B\beta q\overline{x}^{q+p-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < 1.$$

which is valid iff

$$\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}.$$

So by Theorem A $\overline{x}$ is locally asymptotically stable when $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{\beta B(p+q)}$. □

Theorem 12. *If* $0 < p < 1$, *then the Eq.(9) is bounded and persists.*

Proof. Assume that $\{x_n\}$ be a solution of Eq.(9). We obtain from Eq.(9) that

$$x_{n+1} > \alpha, \quad \text{for } n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(9) that

$$x_{n+1} \leq \alpha + \frac{\beta x_{n-k}^p}{A\alpha^p + B\alpha^q} \leq \alpha + \frac{\beta x_{n-k}^p}{B\alpha^q}, \quad for \quad n \geq 0.$$

The rest of the proof is similar to the proof of the Theorem 2 and will be omitted. □

Global Stability of Eq.(9)

In this section we investigate the global asymptotic stability of Eq.(9).

Theorem 13. *The positive equilibrium point* $\overline{x}$ *is a global attractor of Eq.(9). If*

$$(AM^P + Bm^q)(Am^p + BM^q) \neq \beta B(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}), \quad (18)$$

where M *is given by* $M = \alpha + \frac{\beta M^p}{A\alpha^p + B\alpha^q}$.

Proof. We can see that the function

$$f(u, v) = \alpha + \frac{\beta u^p}{Au^p + Bv^q},$$

is increasing in u and decreasing in v. Since Eq.(9) is bounded by Theorem 2. Suppose that (m, M) is a solution of the system

$$M = f(M, m), \quad \text{and} \quad m = f(m, M).$$

We obtain from Eq.(1) that

$$M = \alpha + \frac{\beta M^p}{AM^P + Bm^q}, \quad \text{and} \quad m = \alpha + \frac{\beta m^p}{Am^P + BM^q}.$$

Thus

$$(M - m)(AM^P + Bm^q)(Am^p + BM^q) - B\beta(M^{p+q} - m^{p+q}) = 0.$$

Then we obtain

$$(M-m)[(AM^P+Bm^q)(Am^p+BM^q)-B\beta(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i}+\sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1})]=0.$$

Scien the condition (18) holds, then we get

$$M=m$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(9), and then the proof is complete. □

Example 4. *Figure (4) shows the global attractivity of the equilibrium point of Eq.(9) whenever* $x_{-1}=5.4235$, $x_0=8.987$, $p=0.2$, $q=0.3$, $\alpha=0.6$, $\beta=0.4$, $A=0.4521$, *and* $B=1.563$.

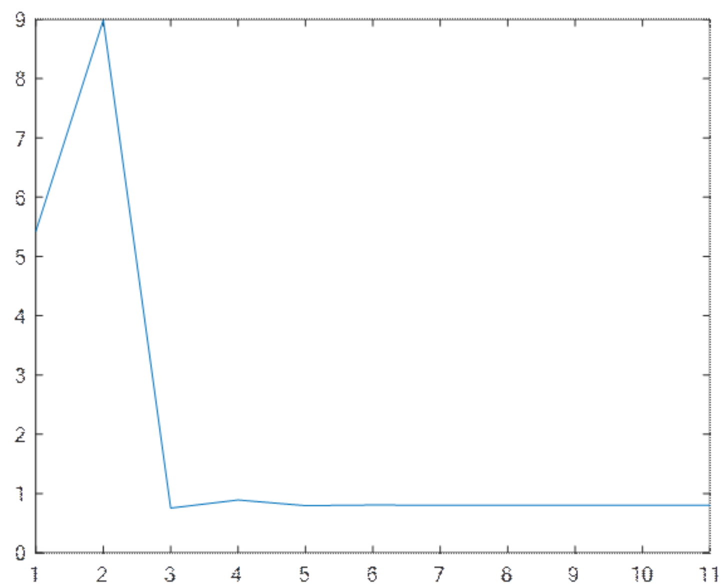

Figure (4)

Case 6: Study of Eq.(10)

This equation is the same of Eq.(9) and its investigation is similar to Eq.(9) and so will be omitted.

Part II

Now we investigate the behavior of the solutions of Eq.(1).

Local Stability of the Equilibruim Points the boundedness of Eq.(1).

In this section we study the local stability character of the positive equilibrium points of Eq.(1). Eq.(1) has a unique positive equilibrium point and is given by

$$\overline{x} = \alpha + \frac{\beta\overline{x}^p + \gamma\overline{x}^q}{A\overline{x}^p + B\overline{x}^q}.$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = \alpha + \frac{\beta u^p + \gamma v^q}{Au^p + Bv^q}.$$

Therefore,

$$\frac{\partial f(u,v)}{\partial u} = \frac{pv^q u^{p-1}(B\beta - A\gamma)}{(Au^p + Bv^q)^2}, \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = -\frac{qu^p v^{q-1}(B\beta - A\gamma)}{(Au^p + Bv^q)^2}.$$

Set

$$p_1 = \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}, \quad \text{and} \quad p_2 = -\frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + p_2 y_{n-m} + p_1 y_{n-k} = 0,$$

where $p_2 = -f_u(\overline{x},\overline{x})$, and $p_1 = -f_v(\overline{x},\overline{x})$. whose characteristic equation is

$$\lambda^{k+1} + p_2\lambda^{k-m} + p_1 = 0.$$

Theorem 14. *If $\frac{A}{B} < \frac{\beta}{\gamma}$ and $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}$, then the positive equilibrium point $\overline{x}$ of Eq.(1) is locally asymptotically stable, and is called a sink.*

Proof. We set $p_1 = \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}$, and $p_2 = -\frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2}$. So by Theorem A

$$|p_1| + |p_2| < 1 \Leftrightarrow \frac{p\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2} + \frac{q\overline{x}^{p+q-1}(B\beta - A\gamma)}{(A\overline{x}^p + B\overline{x}^q)^2} < 1.$$

which is valid iff

$$\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}.$$

So $\overline{x}$ is locally asymptotically stable when $\frac{\overline{x}^{p+q-1}}{(A\overline{x}^p + B\overline{x}^q)^2} < \frac{1}{(p+q)(B\beta - A\gamma)}$. □

Theorem 15. *Every solution of Eq.(1) is bounded and persists.*

Proof. Let $\{x_n\}$ be a positive solution of Eq.(1). We obtain from Eq.(1) that

$$x_{n+1} > \alpha, \quad \text{for } n \geq 0.$$

Hence $\{x_n\}$ persists. It follows again from Eq.(1) that

$$\begin{aligned} x_{n+1} &= \alpha + \frac{\beta x_{n-k}^p + \gamma x_{n-m}^q}{Ax_{n-k}^p + Bx_{n-m}^q} \\ &\leq \alpha + \frac{\max\{\beta,\gamma\}(x_{n-k} + x_{n-m})}{\min\{A,B\}(x_{n-k} + x_{n-m})} = \alpha + \frac{\max\{\beta,\gamma\}}{\min\{A,B\}} = M. \end{aligned}$$

Thus we get

$$0 < \alpha \leq x_n < \alpha + \frac{\max\{\beta,\gamma\}}{\min\{A,B\}} = M < \infty, \quad \text{for all } n \geq 1.$$

Therefore every solution of Eq.(1) is bounded and persists. Hence the result holds. □

Global Stability of Eq.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Theorem 16. *If* $(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) \neq (AM^P + Bm^q)(Am^p + BM^q)$*, and* $\frac{A}{B} < \frac{\beta}{\gamma}$*, then the positive equilibrium point* $\overline{x}$ *is a global attractor of Eq.(1).*

Proof. We can see that the function

$$f(u,v) = \alpha + \frac{\beta u^p + \gamma v^q}{Au^p + Bv^q},$$

is increasing in u and decreasing in v. Suppose that (m, M) is a solution of the system

$$M = f(M,m), \quad \text{and} \quad m = f(m,M).$$

We obtain from Eq.(1) that

$$M = \alpha + \frac{\beta M^p + \gamma m^q}{AM^P + Bm^q}, \quad \text{and} \quad m = \alpha + \frac{\beta m^p + \gamma M^q}{Am^P + BM^q}.$$

Thus

$$(M-m)[(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) - (AM^P + Bm^q)(Am^p + BM^q)] = 0.$$

Since $B\beta > A\gamma$,
$(B\beta - A\gamma)(\sum_{i=1}^{\infty}\alpha^{i-1}M^{p+q-i} + \sum_{i=1}^{\infty}\alpha^{p+q-i}M^{i-1}) \neq (AM^P + Bm^q)(Am^p + BM^q)$
hold. Then we obtain

$$m = M.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(1), and then the proof is complete. □

Example 5. *Figure (5) shows the global attractivity of the equilibrium point of Eq.(1) whenever* $x_{-1} = 2.4235$, $x_0 = 1.987$, $p = 0.7$, $q = 0.9$, $\alpha = 0.6$, $\beta = 0.4$, $\gamma = 0.2$, $A = 0.4521$, *and* $B = 0.52$.

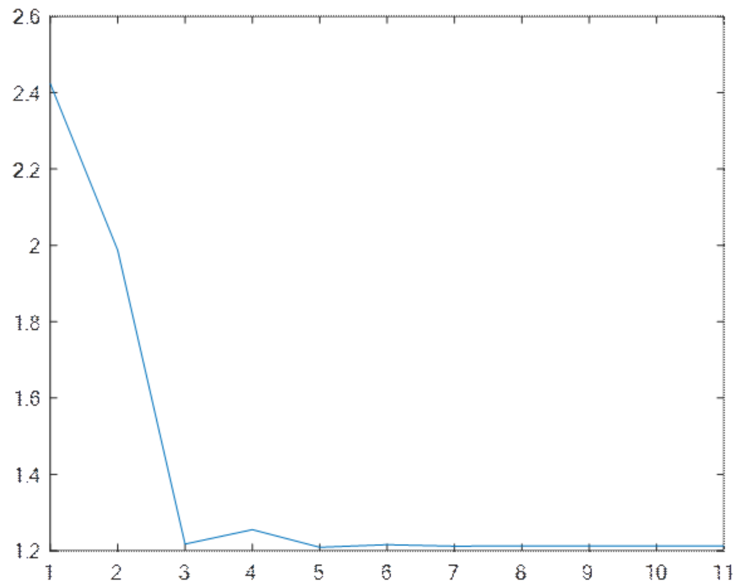

Figure (5)

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H. El-Metwally, Mathematics Department, Faculty of Science, Mansoura University Egypt

E-mail address: eaash69@yahoo.com, helmetwally@mans.edu.eg

E. M. Elabbasy, Mathematics Department, Faculty of Science, Mansoura University Egypt

E-mail address: emelabbasy@mans.edu.eg

A. Eshtiba, Mathematics Department, Faculty of Education, Tripoli University, Libya

E-mail address: amnaeshtiba@gmail.com.