Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 152 (2021) 55-68 EISSN 2392-2192

Generalizations of Regular Closed Set

Mabrouka Mohammed Almarghani[1], **Khadiga Ali Arwini**[2,*]

[1]Mathematics Department, Higher Institute of Science and Technology, Tripoli, Libya

[2]Mathematics Department, Tripoli University, Tripoli, Libya

*E-mail address: Kalrawini@yahoo.com

ABSTRACT

In this article, we introduce several types of generalized regular closed sets in topological spaces, using the concepts of v-sets, the class of generalizations contains the sets; regular v-sets, generalized regular v-sets, generalized star v-set and generalized star regular v-set. We illustrate the inter-relations between these sets, then we study the characterization of this class, which relative to unions, intersections and subspaces, finally we investigate their behavior in regular spaces and extremely disconnected spaces.

***Keywords*:** Topological space and generalizations, subspaces, regular space, extremely disconnected space

AMS Subject Classification (2000): 54A05, 54B05, 54D10, 54G05

1. INTRODUCTION

In 1986 Maki [1] introduced a generalization of closed sets called v-sets, where a generalized v-sets in a topological space (X, τ) defined by considering the sets that can be represented by a union of closed sets, the complement of v-sets are called ʌ-sets, then Maki defined the notion of closure operator using ʌ-sets, namely ʌ-closure operator, when he obtained the relationship between the given topology τ and the topology $\tau^{\wedge}$, that generalized by

(Received 28 November 2020; Accepted 12 December 2020; Date of Publication 14 December 2020)

the family of generalized Λ-sets. Moreover, in [2] he studied the characterization of these sets and their relation with some low separation axioms, as $T_{1/2}$ - spaces. More details can be found in [3-5].

The purpose of this article is to use the notions of v-sets to define general forms of regular closed sets, we introduce a class of generalizations that contains the sets; v_r-sets, $g.v_r$-sets, $g^*.v_r$-sets and g^*.v-sets. We study the implications between these sets, and study their properties relative to unions, intersections and subspace, finally the characterizations of these sets are given, and we discuss some of their behavior in regular spaces and in extremely disconnected spaces.

2. PRELIMINARIES

Throughout this paper (X, τ) represented non-empty topological space, and will replaced by X if there is no chance of confusion, no separation axioms assumed unless otherwise mentioned. If A is a subset of a space X, the notions $\overline{A}$ and A^o denote the closure and the interior of A; respectively.

In the following two sections, we recall the definitions of regular-closed sets, v-sets and generalized v-sets, and present some of their properties that we need in the sequel.

2. 1. Regular Closed Sets

Regular closed sets were due to Stone [6], where a set A is regular closed set if A equals to the closure of its interior. Stone used regular closed sets to define the semiregularization space of a topological space. See more information on [7-11].

Definition 2.1.1. [6, 10] A subset N of a space X is called regular open (namely r-open set) if $N=\overline{N}^o$, while the set N is called δ-open set if N is the union of r-open sets. The complement of r-open set called regular closed (namely r-closed) set, while the complement of δ-open set called δ-closed. The family of all r-open sets in (X, τ) is denoted by $RO(X, \tau)$, while the family of all r-closed sets in X is denoted by $RC(X, \tau)$.

Remarks 2.1.1. [10]

1- A subset N of a space X is r-closed if $N=\overline{N^\circ}$.
2- Every r-closed set is δ-closed set but not conversely.
3- Every δ-closed set is closed set but not conversely.

Proposition 2.1.1. [6]

1- Intersection of r-closed sets is not necessarily r-closed.
2- Finite union of r-closed sets is r-closed.

Definition 2.1.2. [11] A subset A of a space X is said to be r-clopen if it is both r-open and r-closed in X.

Remarke 2.1.2. [11] In a topological space X, a subset A is r-clopen iff A is clopen.

Definition 2.1.3. [11] Let A be a subset of X then, the r-closure of A is defined as the intersection of all r-closed sets containing A, and is denoted $\overline{A}^{r}$.

Proposition 2.1.2. [11] Let X be a space and A, B⊆X, then:

1- $\overline{A}^{r}$ is δ-closed set but not r-closed set in general.
2- $x \in \overline{A}^{r}$ if and only if $A \cap W \neq \emptyset$, for any r-open set W containing x.
3- $A \subseteq \overline{A} \subseteq \overline{A}^{r}$.
4- If A is r-closed then $A = \overline{A}^{r}$.
5- A is δ-closed if and only if $A = \overline{A}^{r}$.

Definition 2.1.4. [11] Let A be a subset of a space X, then the r-interior of A is defined as the union of all r-open sets of contained, and is denoted by $A^{\circ r}$.

Proposition 2.1.3. [11] Let X be a space, and A⊆B⊆X, then:

1- $A^{\circ r}$ is δ-open set, but not r-open in general.
2- $x \in A^{\circ r}$ if and only if there exists an r-open set W such that $x \in W \subseteq A$
3- $A^{\circ r} \subseteq A^{\circ} \subseteq A$.
4- If A is r -open then $A^{\circ r} = A$.
5- A is δ-open if and only if $A^{\circ r} = A$.

Definition 2.1.5. [12] A space X is called regular-space if for any closed set F and x∉F there exist disjoint open sets U and V such that x∈U and F⊆V.

Theorem 2.1.1. [12] A space X is regular if for every x∈X, and each open set U in X such that x∈U there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.

Theorem 2.1.2. [11] In regular space, any open set can be expressed as a union of r-open sets, i.e. for any open set U, then $U = \bigcup_{\alpha \in I} W_{\alpha}$, where W_{α} are r-open sets for all α.

Definition 2.1.6. [13] A space X is called extremely disconnected (namely e.d) if, the closue of every open set in x is also open.

Proposition 2.1.4. [14] In extremely disconnected space (X, τ); we have:

1- Any r-closed set is open, so clopen.
2- Any r-open set is clopen.
3- $RO(X, \tau) = RC(X, \tau) = \tau \cap \mathcal{F}$, where $\mathcal{F}$ is the collection of all closed sets in X.

2. 2. V-Sets and Generalized V-Sets

Maki in 1986 [1] defined a generalization of closed sets, called v-sets by considering the sets that can be represented by a union of closed sets, the complement of v-sets called ʌ-sets, after that he introduced a generalization of v-sets and ʌ-sets called; generalized v-sets and generalized ʌ-sets; respectively. More details can be found in [2, 5].

Definition 2.2.1. [5] Let B be a subset of a topological space (X, τ), then B^v is defined as $B^v = \cup\{F: F \subseteq B, F \text{ is closed}\}$.

Theorem 2.2.1. [5] Let A, B and $\{B_i : i \in I\}$ be subsets of a topological space (X, τ); then the following properties hold:

1- $B^v \subseteq B$.
2- If $A \subseteq B$ then $A^v \subseteq B^v$.
3- $B^{vv} = B^v$.
4- $\bigcup_{i \in I} B_i^v \subseteq (\bigcup_{i \in I} B_i)^v$.
5- $(\bigcap_{i \in I} B_i)^v = \bigcap_{i \in I} B_i^v$.

Definition 2.2.2 [5] A subset B of a topological space (X, τ) is called v-set if $B^v = B$.

Theorem 2.2.2 [5] Let B and $\{B_i : i \in I\}$ be subsets of a topological space (X, τ), then the following hold:

1- If B is closed then B is v-set.
2- If B_i is v-set (for any $i \in I$), then $\bigcup_{i \in I} B_i$ is v-set.
3- If B_i is v-set (for any $i \in I$), then $\bigcap_{i \in I} B_i$ is v-set.

Theorem 2.2.3. [5] If X is a topological space and A, Y are subsets in X such that A is v-set in X. Then $A \cap Y$ is v-set in Y.

Definition 2.2.3. [5] Let B be a subset of a topological space (X, τ), then $B^\wedge$ is defined as $B^\wedge = \cap\{U: B \subseteq U, U \text{ is open}\}$.

Proposition 2.2.1. [5] Let B be a subset of a topological space (X, τ), then B is a ʌ-set if and only if B^c is a v-set.

Proposition 2.2.2. [5] Let A, B and $\{B_i : i \in I\}$ be subsets of a topological space (X, τ), then the following properties are hold:

1- $B \subseteq B^\wedge$.
2- If $A \subseteq B$ then $A^\wedge \subseteq B^\wedge$.
3- $B^{\wedge\wedge} = B^\wedge$
4- $(\bigcup_{i \in I} B_i)^\wedge = \bigcup_{i \in I} B_i^\wedge$.
5- $(\bigcap_{i \in I} B_i)^\wedge \subseteq \bigcap_{i \in I} B_i^\wedge$.

Definition 2.2.4. [5] A subset B of a topological space (X, τ) is called ʌ-set if $B = B^\wedge$.

Proposition 2.2.3. [5] Let B and $\{B_i : i \in I\}$ be subsets of a topological space (X, τ), then the following hold:

1- If $B\in\tau$ then B is $\wedge$-set.
2- If B_i is $\wedge$-set (for any $i\in I$), then $\bigcup_{i\in I} B_i$ is $\wedge$-set.
3- If B_i is $\wedge$-set ($i\in I$), then $\bigcap_{i\in I} B_i$ is $\wedge$-set.

Proposition 2.2.4. [5] If X is a topological space and A, Y are subsets in X such that A is $\wedge$-set in X. Then $A\cap Y$ is $\wedge$-set in Y.

Definition 2.2.5. [2] A subset B of a topological space (X,τ) is called a generalized v-set (namely g.v-set) if $U\subseteq B^{v}$whenever $U\subseteq B$ and U is open set. A subset B is called generalized $\wedge$-set (namely g.$\wedge$-set) of X if, B^{c} is a g.v-set.

Theorem 2.2.4. [2] A subset B of a topological space (X,τ) is a g.$\wedge$-set if and only if $B^{\wedge}\subseteq F$ whenever $B\subseteq F$ and F is closed set.

Proposition 2.2.5. [2] In a topological space (X,τ), the following properties hold:

1- Every v-set (rasp $\wedge$-set) is g.v-set (rasp g.$\wedge$-set).
2- For each $x\in X$, $\{x\}$ is an open set or a g.v-set.

Theorem 2.2.5. [2] A subset B of topological space is g.v-set if and only if for every closed set F such that $B^{v}\cup B^{c}\subseteq F$, F=X holds.

3. REGULAR V-SETS AND REGULAR Λ-SETS

The concepts of v_δ-sets, v_{pre}sets, $v_\propto$-sets, v_b-sets and v_m-sets have been studied by many researchers, see [15-21]. In this section we use the concept of v-sets and $\wedge$-sets to define new class of generalizations which includes; regular v-sets (namely v_r-sets) and regular $\wedge$-sets (namely $\wedge_r$-sets), then we discuss thier properties and study the behaviour of these generalizations in regular spaces and in e.d spaces.

Although r-closed sets different (stronger form) from δ-closed sets, the closure and interior operators with respect to r-closed set are coincide with the closure and interior operators with respect to δ-closed sets; respectively (i.e. $\overline{A}^{r}=\overline{A}^{\delta}$ and $A^{\circ r}=A^{\circ\delta}$ for a subset A of a space), while the notions of $\wedge_r$-sets and $\wedge_\delta$ -sets are different, where $\wedge_r$-sets are strictly stronger than $\wedge_\delta$-sets.

Definition 3.1. Let B be a subset of a topological space (X,τ), then B^{vr} is defined as: $B^{vr}=\cup\{N: N\subseteq B, N \text{ is r-closed}\}$.

Theorem 3.1. Let A, B and $\{B_\alpha: \alpha\in I\}$ be subsets of a topological space (X,τ), then the following properties are hold:

1- If N is r-closed set such that $N\subseteq A$, then $N\subseteq A^{vr}$.
2- $B^{vr}\subseteq B^{v}\subseteq B$.
3- If $A\subseteq B$ then $A^{vr}\subseteq B^{vr}$.

4- $(A^{v_r})^{v_r} = A^{v_r}$.
5- $(\bigcap_{\alpha\in I} B_\alpha)^{v_r} \subseteq \bigcap_{\alpha\in I} {B_\alpha}^{v_r}$.
6- $\bigcup_{\alpha\in I} {B_\alpha}^{v_r} \subseteq (\bigcup_{\alpha\in I} B_\alpha)^{v_r}$.

Proof:

1- Direct from definition (2.2.1).
2- Direct from definitions (2.2.1) and (3.1).
3- Suppose that $x\in A^{v_r}$, then there exists an r-closed set N such that $x\in N$ and $N\subseteq A$. Since $A\subseteq B$, then $x\in N\subseteq B$, N is r-closed, so $x\in B^{v_r}$. Hence $A^{v_r} \subseteq B^{v_r}$.
4- From (2) we get $A^{v_r} \subseteq A$, and from (3) we have $(A^{v_r})^{v_r} \subseteq A^{v_r} \rightarrow$ (1). Now suppose $x\notin(A^{v_r})^{v_r}$ then for any r-closed set N such that $N \subseteq A^{v_r}$ we have $x\notin N$, and since any r-closed set contained in A is also contained in A^{v_r} (from definition (3.1)) we get $x\notin N$ for any r-closed set N such that $N\subseteq A$, hence $x\notin A^{v_r}$, thus $A^{v_r} \subseteq (A^{v_r})^{v_r} \rightarrow$ (2). From (1) and (2) we have $(A^{v_r})^{v_r} = A^{v_r}$.
5- Since $(\bigcap_{\alpha\in I} B_\alpha) \subseteq B_\alpha$ for all $\alpha \in I$, then $(\bigcap_{\alpha\in I} B_\alpha)^{v_r} \subseteq {B_\alpha}^{v_r}$, so we get $(\bigcap_{\alpha\in I} B_\alpha)^{v_r} \subseteq \bigcap_{\alpha\in I} {B_\alpha}^{v_r}$.
6- Since $B_\alpha \subseteq (\bigcup_{\alpha\in I} B_\alpha)$ for all $\alpha \in I$, then ${B_\alpha}^{v_r} \subseteq (\bigcup_{\alpha\in I} B_\alpha)^{v_r}$, so we get $\bigcup_{\alpha\in I} {B_\alpha}^{v_r} \subseteq (\bigcup_{\alpha\in I} B_\alpha)^{v_r}$.

Example 3.1. Let $X = \{a, b, c, d\}$, and $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then $RO(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}\}$, so if $A = \{a, c\}$ and $B = \{b, d\}$, then $A^{v_r} = B^{v_r} = \emptyset$ while $(A \cup B)^{v_r} = X^{v_r} = X$, so $(A \cup B)^{v_r} \neq A^{v_r} \cup B^{v_r}$.
Now choose $C = \{a, d\}$ and $D = \{b, c, d\}$, then C and D are v_r-sets while $(C \cap D)^{v_r} = \{d\}^{v_r} = \emptyset$ i.e. $(C \cap D)^{v_r} \neq C^{v_r} \cap D^{v_r} = \{d\}$.

Definition 3.2. A subset B of a topological space (X, τ) is called v_r-set if $B^{v_r}=B$.

Theorem 3.2. Every v_r-set is v-set, but not conversely.

Proof: Let B be a v_r-set in (X, τ), and from theorem (3.1(2)) we have $B^{v_r} \subseteq B^{v} \subseteq B$, hence $B^{v_r}=B^{v}=B$.

Example 3.2. In example (3.1), if $B = \{d\}$ then $B^{v_r}= \emptyset$, while $B^{v} = B$. Hence B is v-set but not v_r-set.

Theorem 3.3. Let A be a subset of a topological space (X, τ) then the following hold:

1- If A is r-closed then A is v_r-set.
2- If B_α is v_r-set for all $\alpha \in I$, then $\bigcup_{\alpha\in I} B_\alpha$ is v_r-set.

Proof:

1- If A is r-closed then from definition (3.2) we get $A^{v_r}=A$.
2- Since $B_\alpha \subseteq \bigcup_{\alpha\in I} B_\alpha$ for all $\alpha \in I$ then ${B_\alpha}^{v_r} \subseteq (\bigcup_{\alpha\in I} B_\alpha)^{v_r}\ \forall \alpha \in I$, so $\bigcup_{\alpha\in I} B_\alpha \subseteq (\bigcup_{\alpha\in I} B_\alpha)^{v_r} \rightarrow$ (1)

From theorem (3.1.(2)) we have $(\bigcup_{\alpha\in I} B_\alpha)^{v_r} \subseteq \bigcup_{\alpha\in I} B_\alpha \rightarrow$ (2), so we obtian $(\bigcup_{\alpha\in I} B_\alpha)^{v_r} = \bigcup_{\alpha\in I} B_\alpha$.

Examples 3.3.

1- In the usual topology on $\mathbb{R}$, the set $\mathbb{R}$ /{3} is v_r-set but not r-closed.
2- In example (3.1), if A= {a, d} and B = {b, c, d}, then A and B are v_r-sets while A∩B = {d} is not v_r–set, since $\{d\}^{V_r} = \emptyset$.

Definition 3.3. Let B be a subset of a topological space (X, τ), then $B^{\wedge r}$ is defined as: $B^{\wedge r} = \cap\{W: B \subseteq W, W \in RO(X,\tau)\}$.

Theorem 3.4. Let A, B and $\{B_\alpha : \alpha \in I\}$ be subsets of a topological space (X, τ), then the following properties are hold:

1- $B \subseteq B^{\wedge} \subseteq B^{\wedge r}$.
2- If $A \subseteq B$ then $A^{\wedge r} \subseteq B^{\wedge r}$.
3- $(B^{\wedge r})^{\wedge r} = B^{\wedge r}$.
4- $(\bigcap_{\alpha \in I} B_\alpha)^{\wedge r} \subseteq \bigcap_{\alpha \in I} B_\alpha{}^{\wedge r}$.
5- $\bigcup_{\alpha \in I} B_\alpha{}^{\wedge r} \subseteq (\bigcup_{\alpha \in I} B_\alpha)^{\wedge r}$.
6- $(B^{V r})^c = (B^c)^{\wedge r}$.
7- $(B^{\wedge r})^c = (B^c)^{V r}$.

Proof:

1- Direct from definitions (2.2.3) and (3.3).
2- Suppose that $x \notin B^{\wedge r}$, then there exists an r-open set W such that $B \subseteq W$ and $x \notin W$. Since $A \subseteq B$, then $x \notin A^{\wedge r}$, hence $A^{\wedge r} \subseteq B^{\wedge r}$.
3- From (1) and (2) we have $B \subseteq B^{\wedge r}$ so $B^{\wedge r} \subseteq (B^{\wedge r})^{\wedge r} \rightarrow (1)$. Now suppose $x \notin B^{\wedge r}$ then there exists an r-open set W such that $B \subseteq W$ and $x \notin W$, and since $B^{\wedge r} \subseteq W$ we have $x \notin (B^{\wedge r})^{\wedge r}$, so $(B^{\wedge r})^{\wedge r} \subseteq B^{\wedge r} \rightarrow (2)$. From (1) and (2) we have $B^{\wedge r} = (B^{\wedge r})^{\wedge r}$.
4- Since $(\bigcap_{\alpha \in I} B_\alpha) \subseteq B_\alpha$ for all $\alpha \in I$ then $(\bigcap_{\alpha \in I} B_\alpha)^{\wedge r} \subseteq B_\alpha{}^{\wedge r}$ for all $\alpha \in I$, hence $(\bigcap_{\alpha \in I} B_\alpha)^{\wedge r} \subseteq \bigcap_{\alpha \in I} B_\alpha{}^{\wedge r}$.
5- Since $B_\alpha \subseteq (\bigcup_{\alpha \in I} B_\alpha)$ for all $\alpha \in I$, then $B_\alpha{}^{\wedge r} \subseteq (\bigcup_{\alpha \in I} B_\alpha)^{\wedge r}$ for all $\alpha \in I$, hence $\bigcup_{\alpha \in I} B_\alpha{}^{\wedge r} \subseteq (\bigcup_{\alpha \in I} B_\alpha)^{\wedge r}$.
6- Since $B^{Vr} = \cup\{F: F \text{ is r-closed}, F \subseteq B\}$ then $(B^{Vr})^c = \cap\{F^c: F^c \text{ is r-open}, B^c \subseteq F^c\} = (B^c)^{\wedge r}$.
7- Since $B^{\wedge r} = \cap\{W: W \text{ is r-open}, B \subseteq W\}$ then $(B^{\wedge r})^c = \cup\{W^c: W^c \text{ is r-closed}, W^c \subseteq B^c\} = (B^c)^{Vr}$.

Remark 3.1. In general $(A \cap B)^{\wedge r} \neq A^{\wedge r} \cap B^{\wedge r}$, for example:
In example (3.1) if A = {a, c} and B = {b, d}, then $A^{\wedge r} = X$, $B^{\wedge r} = X$ and $(A \cap B)^{\wedge r} = \emptyset^{\wedge r} = \emptyset$, but $A^{\wedge r} \cap B^{\wedge r} = X$. While if C = {a} and D = {b}, then $C^{\wedge r} = \{a\}$ and D = {b, c}, so $(C \cup D)^{\wedge r} = X$ but $A^{\wedge r} \cup B^{\wedge r} = \{a, b, c\}$.

Remark 3.2. If W is r-open set such that $A \subseteq W$, then $A^{\wedge r} \subseteq W$.

Definition 3.4. A subset B of a topological space (X, τ) is called $\wedge_r$-set if $B = B^{\wedge r}$.

Theorem 3.5. Every $\wedge_r$- set is $\wedge$-set.

Proof: If a subset B of a topological space (X, τ) is $\wedge_r$-set, then $B=B^{\wedge r}$ from (3.4.(1)) we have $B \subseteq B^{\wedge} \subseteq B^{\wedge r}$ (=B), so $B^{\wedge} = B^{\wedge r} = B$. i.e. B is $\wedge$-set.

Example 3.4. In the previous example, if A= {a, b, c}, then A is $\wedge$-set but not $\wedge_r$-set since $A^{\wedge r} = X$.

Theorem 3.6. Let B and $\{B_\alpha : \alpha \in I\}$ be subsets of a topological space(X, τ), then the following hold:

1- If B is r-open, then B is $\wedge_r$-set.
2- If B_α are $\wedge_r$-set (for all $\alpha \in I$), then $(\bigcap_{\alpha \in I} B_\alpha)$ is $\wedge_r$-set.
3- B is $\vee_r$-set iff B^c is $\wedge_r$-set.

Proof:

1- Direct from definitions (3.3) and (3.4).
2- From theorem (3.4(4)) we have $(\bigcap_{\alpha\in I} B_\alpha)^{\wedge r} \subseteq \bigcap_{\alpha\in I} B_\alpha{}^{\wedge r}$, then $(\bigcap_{\alpha\in I} B_\alpha)^{\wedge r} \subseteq \bigcap_{\alpha\in I} B_\alpha \to (1)$ since B_α is $\wedge_r$-set $\forall \alpha \in I$ from theorem (3.4(1)), we have $\bigcap_{\alpha\in I} B_\alpha \subseteq (\bigcap_{\alpha\in I} B_\alpha)^{\wedge r} \to (2)$. From (1) and (2) we have $(\bigcap_{\alpha\in I} B_\alpha)^{\wedge r} = \bigcap_{\alpha\in I} B_\alpha$, so $\bigcap_{\alpha\in I} B_\alpha$ is $\wedge_r$-set.
3- If B is $\vee_r$-set, then $B = B^{\vee r} = \cup\{ N: N$ is r-closed, $N \subseteq B\}$, we get $B^c = (B^{\vee r})^c = \cap\{ N^c: N^c$ is r-open, $B^c \subseteq N^c\} = (B^c)^{\wedge r}$. Now, if B^c is $\wedge_r$ set then $B^c = (B^c)^{\wedge r} = \cap\{ W: W$ is r-open, $B^c \subseteq W\}$so $B = \cup\{ W^c: W^c$ is r-closed , $W^c \subseteq B\} = B^{\vee r}$.

Examples 3.5.

1- In the usual topological space $(\mathbb{R}, \mu)$, the set {1} is $\wedge_r$-set but not r-open since $\overline{\{1\}}^{o} = \{1\}^{o} = \emptyset$.
2- In example (3.1) if A= {a} and B= {b, c} then A, B are $\wedge_r$-sets but A∪B={a, b, c} is not $\wedge_r$-set since $(A \cup B)^{\wedge r} = X$.

Remark 3.3. No general relation between $\wedge_r$-set and open set, for example:-
In usual topology $\mathbb{R}$, the singlet {1} is $\wedge_r$-set but not open, while the set $(0, 1) \cup (1, 2)$ is open but not $\wedge_r$-set since $\big((0,1) \cup (1,2)\big)^{\wedge r} = (0, 2)$.

Theorem 3.7. If X is a topological space and A, Y subset in X, Y is r-open, A is $\wedge_r$-set in X. Then $A \cap Y$ is $\wedge_r$-set in Y.

Proof: $A = \cap\{W: A \subseteq W$, W is r-open }, then $A \cap Y = \cap\{W: A \subseteq W$, W is r-open$\} \cap Y = \{W \cap Y: A \cap Y \subseteq W \cap Y$, $W \cap Y$ is r-open in Y}. From proposition (2.2.4), $A \cap Y$ is $\wedge_r$ set in Y.

Lemma 3.1. In a topological space (X, τ), if $x \in X$ then {x} is r-closed iff {x} is clopen.

Proof: ⇒ Suppose {x} is r-closed set, then {x} is closed. If {x} is not open, then we have $\overline{\{x\}}^{\circ} = \emptyset$, hence {x} is not r-closed. Which this is contradiction. Hence {x} is closed and open thus clopen.

$\Leftarrow$ If $\{x\}$ is open and closed, abvoisly $\overline{\{x\}^{\circ}} = \overline{\{x\}} = \{x\}$, hence $\{x\}$ is r-closed.

Lemma 3.2. In a topological space (X, τ), if $x \in X$, then $\{x\}$ is r-closed iff $\{x\}$ is v_r-set.

Proof: $\Rightarrow$ Direct

$\Leftarrow$ Suppose $\{x\}$ is v_r-set, then $\{x\}=\{x\}^{v_r}$. Since $\{x\}^{v_r} = \begin{cases} \{x\}, & \{x\} \text{ is r} - \text{closed} \\ \emptyset, & \{x\} \text{ is not r} - \text{closed} \end{cases}$
so $\{x\}=\{x\}^{v_r}$ imply $\{x\}$ is r-closed.

Corollary 3.1. In a topological space(X, τ), if $x \in X$, then these stetmemts are equivalent:

1- $\{x\}$ is r-closed.
2- $\{x\}$ is v_r-set.
3- $\{x\}$ is clopen.
4- $\{x\}$ is r-clopen.
5- $\{x\}^c$ is clopen.
6- $\{x\}^c$ is r-open.
7- $\{x\}^c$ is $\wedge_r$-set.

Corollary 3.2. Let (X, τ) be a topological space and let $A \subseteq X$ such that the number of r-closed sets conttained in A is finite, then A is v_r-set iff A is r-closed.

Proof: $\Leftarrow$ Direct
$\Rightarrow$ Suppose $A^{v_r} = A$, so $A = \cup\{N, N \text{ is r-closed and } N \subseteq A\}$, then from proposition (2.1.1(2)) we have the finite union of r-closed sets is r-closed, then A is r-closed.

Remarks 3.4.

1- If A is any finite subset of a space X, then A is r-closed iff A is v_r-set.
2- If A is a subset of a space(X, τ), where $|RO(X, \tau)| < \infty$, then A is r-closed iff A is v_r-set.

Theorem 3.8. In e.d space (X, τ), if $A \subseteq X$ then:
1- $A^{\wedge_r} = \overline{A}^r$.
2- $A^{v_r} = A^{\circ r}$.

Proof:

1- $A^{\wedge_r} = \cap\{W: W \text{ is r-open set and } A \subseteq W\}$. Since X is e.d, then $RO(X, \tau) = RC(X, \tau)$, so we have $A^{\wedge_r} = \cap\{W: W \text{ is r-closed set and } A \subseteq W\} = \overline{A}^r$.
2- $A^{v_r} = \cup\{N: N \text{ is r-open}, N \subseteq A\} = A^{\circ r}$.

Corollary 3.3. In regular space (X, τ), if $A \subseteq X$, then:

1- $\overline{A}^r = \overline{A}$.
2- $A^{\circ r} = A^{\circ}$.

Proof:

1- From proposition (2.1.2(3)) we have $\overline{A} \subseteq \overline{A}^{r} \rightarrow (1)$. Now suppose $x \notin \overline{A}$, then there exists a closed set F such that $A \subseteq F$ and $x \notin F$. From theorem (2.1.2) F can be expressed as an intersection of r-closed sets, say $F = \bigcap_{\alpha \in I} N_\alpha$, where N_α are r-closed, since $x \notin F$ then there is $\alpha \in I$ such that $x \notin N_\alpha$ and $A \subseteq F \subseteq N_\alpha$. So we have an r-closed set N_α such that $A \subseteq N_\alpha$ but $x \notin N_\alpha$, hence $x \notin \overline{A}^{r}$, thus $\overline{A}^{r} \subseteq \overline{A} \rightarrow (2)$. From (1) and (2) we get $\overline{A}^{r} = A$.
2- From proposition (2.1.3(3)), we have $A^{\circ r} \subseteq A^{\circ} \rightarrow (1)$. Now let $x \in A^{\circ}$, then there exists an open set U_x such that $x \in U_x \subseteq A$. Since X is regular, then $U_x = \bigcup_{\alpha \in I} W_\alpha$, where W_α are r-open sets, then there is $\alpha \in I$ such that $x \in W_\alpha \subseteq U_x \subseteq A$, so $x \in A^{\circ r}$, hence $A^{\circ} \subseteq A^{\circ r} \rightarrow (2)$. From (1) and (2) we get $A^{\circ r} = A^{\circ}$.

Corollary 3.4. In e.d regular space X, if $A \subseteq X$, then:

1- $A^{\wedge r} = \overline{A}^{r} = \overline{A}$.
2- $A^{\vee r} = A^{\circ r} = A^{\circ}$.

Proof: Direct from theorem (3.8) and corollary (3.3).

4. GENERALIZED REGULAR V-SETS AND GENERALIZED REGULAR Λ-SETS

New classes of generalizations are define in this section, the first class contains the sets; generalized regular v-set, generalized star v-set and generalized star regular v-set, while the second class contains; generalized regular ʌ-set, generalized star ʌ-set and generalized star regular ʌ-set. We investigate their properties and study their behavior in regular spaces and in e.d spaces.

Definition 4.1. A subset B of a topological space (X, τ) is called:

1- Generalized regular v-set (namely $g.v_r$-set) if $U \subseteq B^{\vee r}$ whenever $U \subseteq B$ and U is open set.
2- Generalized star v-set (namely $g^*.v$-set) if $W \subseteq B^{\vee}$ whenever $W \subseteq B$ and W is r-open set.
3- Generalized star regular v-set (namely $g^*.v_r$-set) if $W \subseteq B^{\vee r}$ whenever $W \subseteq B$ and W is r-open set.

Corollary 4.1. In a space (X, τ), we have:

1- Any v_r-set is $g.v_r$-set.
2- Any $g.v_r$-set is g.v-set.
3- Any $g^*.v_r$-set is $g^*.v$-set.
4- $g.v_r$-set is strictly between v_r-set and $g^*.v_r$-set.
5- g.v-set is strictly between v-set and g^*.v-set.

Definition 4.2. A subset B of a topological space (X, τ) is called:

1- generalized regular ʌ-set (namely $g.\wedge_r$-set) if B^c is $g.v_r$-set.
2- generalized star ʌ-set (namely $g^*.\wedge$-set) if B^c is g^*.v-set.

3- generalized star regular $\wedge$-set (namely $g^*.\wedge_r$-set) if B^c is $g^*.v_r$-set.

Corollary 4.2. If X is a space, then the inter-relations between g.$\wedge$-sets, g.$\wedge_r$-sets, $g^*.\wedge$-sets and $g^*.\wedge_r$-sets are given by:

1- Any $\wedge_r$-set is g.$\wedge_r$-set.
2- Any g.$\wedge_r$-set is g.$\wedge$-set.
3- Any $g^*.\wedge_r$set is g^*.v-set.
4- g.$\wedge_r$-set is strictly between $\wedge_r$-set and $g^*.\wedge_r$-set.
5- g.$\wedge$-set is strictly between $\wedge$-set and $g^*.\wedge$-set.

Theorem 4.1. A subset B of a topological space (X,τ) is:

1- g.$\wedge_r$-set iff $B^{\wedge r} \subseteq F$ whenever $B \subseteq F$ and F is closed set.
2- $g^*.\wedge$-set iff $B^{\wedge} \subseteq N$ whenever $B \subseteq N$ and N is r-closed set.
3- $g^*.\wedge_r$-set iff $B^{\wedge r} \subseteq N$ whenever $B \subseteq N$ and N is r-closed set.

Proof: Here we prove number (3), and the proves of (1) and (2) are similar.
$\Leftarrow$ Let W be an r-open set such that $W \subseteq B^c$, then $B \subseteq W^c$, and since W^c is r-closed by assumption we have $B^{\wedge r} \subseteq W^c$, so $W \subseteq (B^{\wedge r})^c = (B^c)^{\vee r}$. Thus B^c is $g^*.v_r$-set, this imply B is $g^*.\wedge_r$-set.
$\Rightarrow$ Suppose B is $g^*.\wedge_r$-set and let $B \subseteq N$ where N is r-closed set, then $N^c \subseteq B^c$, since B^c is $g^*.v_r$-set we have $N^c \subseteq (B^c)^{\vee r}$, so $N^c \subseteq (B^{\wedge r})^c$, this imply $B^{\wedge r} \subseteq N$.

Theorem 4.2. If (X,τ) is a topological space, then any singleton is open or g.v_r-set.

Proof: Suppose $x \in X$ and $\{x\}$ is not open set, then the only open set contained in $\{x\}$ is $\emptyset$, so if $U \subseteq \{x\}$ where U is open then $U=\emptyset$, hence $U \subseteq \{x\}^{\vee r}$, i.e. $\{x\}$ is g.v_r-set.

Theorem 4.3. If (X,τ) is a topological space, then any singleton is r-open or $g^*.v_r$-set.

Proof: Suppose $x \in X$ and $\{x\}$ is not r-open set, then the only r-open set contained in $\{x\}$ is $\emptyset$, so if $W \subseteq \{x\}$ where W is r-open then $W=\emptyset$, hence $W \subseteq \{x\}^{\vee r}$, i.e. $\{x\}$ is $g^*.v_r$-set.

Theorem 4.4. If a space (X,τ) is regular space, and $A \subseteq X$, then:
1- A is g.v_r-set iff A is $g^*.v_r$-set.
2- A is g.v-set iff A is g^*.v-set.

Proof:

1- $\Rightarrow$ Direct.
$\Leftarrow$ Suppose A is $g^*.v_r$-set, and let $U \subseteq A$ where U is open set. Since X is regular, then $U = \bigcup_{\alpha \in I} W_\alpha$, where W_α is r-open set for all α, so $W_\alpha \subseteq A$ for all α, since A is $g^*.v_r$-set we have $W_\alpha \subseteq A^{\vee r}$ for all α, hence $\cup W_\alpha = U \subseteq A^{\vee r}$, thus A is g.$v_r$-set.
2- The prove is similar to number (1).

Corollary 4.3. In regular space (X,τ), if $A \subseteq X$ then:

1- A is g.$\wedge_r$-set iff A is $g^*.\wedge_r$-set.

2- A is g.Λ-set iff A is g^*.Λ-set.

Remark 4.1. If (X, τ) is a topological space, and $x \in X$ then:

1- $\{x\}^c$ is closed or g.Λ_r-set.
2- $\{x\}^c$ is r-closed or $g^*.v_r$-set.

Theorem 4.5. If B is a $g^*.v_r$-set in a topological space (X, τ) then for every r-closed set N such that $B^{v_r} \cup B^c \subseteq N$, N=X holds.

Proof: Suppose B is a $g^*.v_r$-set and N is r-closed set that satisfy $B^{v_r} \cup B^c \subseteq N$ then $N^c \subseteq (B^{v_r})^c \cap B \subseteq B$ since B is $g^*.v_r$-set and $N^c \subseteq B$ we get $N^c \subseteq B^{v_r}$, so $N^c \subseteq \emptyset$, hence N=X.

5. CONCLUSIONS

Our aim in this paper is to define new generalizations of regular closed sets and regular open sets in topological space; the first class of generalizations contains the sets: regular v-sets (namely v_r-sets), generalized regular v-sets (namely g.v_r-sets), generalized star v-set (namely g^*.v-set) and generalized star regular v-set (namely $g^*.v_r$-set), while the second class of generalizations contains the sets: regular Λ-sets (namely Λ_r-sets) and generalized regular Λ-sets (namely g.Λ_r-sets), generalized star Λ-set (namely $g^*.\Lambda$-set) and generalized star regular Λ-set (namely $g^*.\Lambda_r$-set). We investegat the implications between these sets, and study their properties relative to unions, intersections and subspace, finally the characterizations of these sets have been given, and we discuss some of their behavior in regular spaces and in extremely disconnected spaces.

Here we give a summarization of our results:

- The following diagrams show the implications between the classes of generalizations:

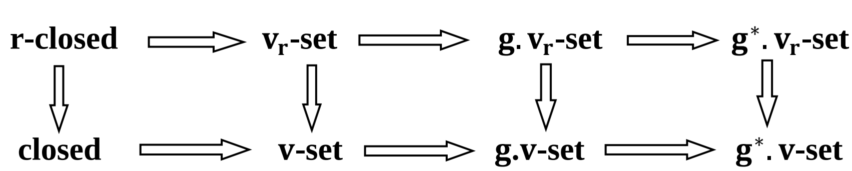

Diagram 1. Generalizations of regular closed set using v-sets.

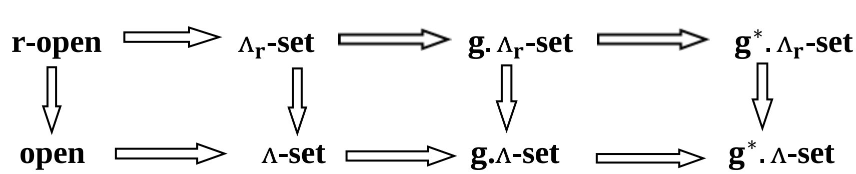

Diagram 2. Generalizations of regular open sets using Λ-sets.

- Any union of v_r-sets is v_r-set, while the intersection of v_r-sets need not be v_r-set.
- The complement of v_r-set is $\wedge_r$-set, and the complement of g.v_r-set is g. $\wedge_r$-set.
- A finite subset B of a topological space is r-closed if and only if B is v_r-set.
- Any singleton in a topological space is open or g. v_r-set.
- Any singleton in a topological space is r-open or g^*. v_r-set.
- In regular space X, if A⊆X then:
  - $\overline{A}^r = \overline{A}$ and $A^{\circ r} = A^\circ$.
  - A is g.v_r-set, iff A is g^*. v_r-set.
  - A is g. v-set, iff A is g^*. v-set.
- In e.d space X, if A⊆X then:
  - $\overline{A}^r = A^{\wedge_r}$ and $A^{\circ r} = A^{v_r}$.
  - A is g^*. v_r-set.
- In regular e.d space X, if A⊆X then:
  - $\overline{A}^r = \overline{A} = A^{\wedge_r}$ and $A^{\circ r} = A^\circ = A^{v_r}$.
  - Any subset of X is g. v_r-set (which is coincide with g^*. v_r-set).

References

[1] H. Maki, Generalized. ∧ -Sets And The Associated Closure Operator. *The Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement*, 1 (1986) 139-146

[2] H. Maki, J. Umehara and K. Yamamura, Characterizations Of $T_{\frac{1}{2}}$-Spaces Using Generalized v- Sets. *Indian J. Pure Appl. Math* 9 (7) (1988) 634-640

[3] N. Levine, Generalized Closed Sets In Topology. *Rend. Circ. Mat. Palermo*, 19 (2) (1970) 89-96

[4] W. Dunham, A New Closure Operator For Non T_1-Toplogies. *Kyungpook Math. J*, 22 (1982) 55-60

[5] M. Caldas, S. Jafari and G. Navalagi, More On λ-Closed Sets In Topological Spaces. *Revista Columbiana de Mathematical*,41 (2) (2007) 355-369

[6] M. H. Stone, Applications Of The Theory Of Boolean Rings To General Topology. *Trans. Am. Math. Soc.* 41 (1937) 375-481

[7] C. Ronse. Regular Open And Closed Sets. *Philips Research Laboratory Brissels, Ar. E Van Baclaera*, 2 B (1990) 1170

[8] Charles Dorsett, Regularly Open Sets And R-Topological Properties. *Nat. Acod. Sci-Letters*, 10 (1987) 17-21

[9] H. K. Abdullah and F. K. Radhy, Strongly Regular Proper Mappings. *Journal of Al-Qadisiyah for Computer Science and Mathematics* 3 (1) (2011)

[10] Zanyar A. Ameen, Baravan A. Asaad and Ramadhan A. Muhammed, On Super Classes Of -Open Sets In Topological Spaces. *International Journal of Applied Mathematics,* 32 (2) (2019) 259-277

[11] A. E. Kornas and K. A. Arwini, R-Countability Axioms. *An International Scientific Journal* 149 (2020) 92-109

[12] S. Willard, General Topology. Addison-Wesley Publishing Company, United States of American (1970).

[13] Dona Papert Strauss, Extremally Disconnected Spaces. *Proc. Amer. Math. Soc*. 18 (1967) 305-309

[14] Majid Mirmiran, A Note On Extremally Disconnected Spaces. *Research Open Journal Of Information Science*, 1 (1) (2013) 01-03

[15] M. Caldas and J. Dontchev, G.Λ_s-Sets And G.v_s-Sets. *Mem. Fac. Sci. Kochi Univ. Math.* 21 (2000) 21-30

[16] M. Caldas, D. N. Georgiou and S. Jafari, Study Of (Λ, α)-Closed Sets And Related Notions In Topoloical Spaces. *Bull. Malays. Math. Sci. Soc* 30 (2007) 23-36

[17] M. Caldas, S. Jafari and T. Noiri, On Λ_b-Sets And The Associated Topology $\tau\Lambda_b$. *Acta Math. Hungar,* 110 (2006) 337-345

[18] M. Caldas and S. Jafari, Generalized Λ_δ-Sets And Related Topics. *Georgian Math. J.* 16 (2009) 247-256

[19] F. Cammaroto and T. Noiri, On Λ_m-Sets And Related Topological Spaces. *Acta Math. Hungar*, 109 (2005) 261-279

[20] M. Ganster, S. Jafari and T. Noiri, On Pre- Λ -Sets And Pre-v –Sets. *Acta Math. Hungar*, 95 (2002) 337-343

[21] E. Hatir and T. Noiri, Λ_{sp}-Sets And Some Weak Separation Axioms. *Acta Math. Hungar*, 103 (2004) 225-232