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A Generalization of Planar Pascal's Triangle to Polynomial Expansion and Connection with Sierpinski Patterns

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***Abstract*—**The very well-known stacked sets of numbers referred to as Pascal's triangle present the coefficients of the binomial expansion of the form $(x+y)^n$. This paper presents an approach (the Staircase Horizontal Vertical, SHV-method) to the generalization of planar Pascal's triangle for polynomial expansion of the form $(x+y+z+w+r+\cdots)^n$. The presented generalization of Pascal's triangle is different from other generalizations of Pascal's triangles given in the literature. The coefficients of the generalized Pascal's triangles, presented in this work, are generated by inspection, using embedded Pascal's triangles. The coefficients of I-variables expansion are generated by horizontally laying out the Pascal's elements of (I-1) variables expansion, in a staircase manner, and multiplying them with the relevant columns of vertically laid out classical Pascal's elements, hence avoiding factorial calculations for generating the coefficients of the polynomial expansion. Furthermore, the classical Pascal's triangle has some pattern built into it regarding its odd and even numbers. Such pattern is known as the Sierpinski's triangle. In this study, a presentation of Sierpinski-like patterns of the generalized Pascal's triangles is given. Applications related to those coefficients of the binomial expansion (Pascal's triangle), or polynomial expansion (generalized Pascal's triangles) can be in areas of combinatorics, and probabilities.

***Keywords*—**Generalized Pascal's triangle, Pascal's triangle, polynomial expansion, Sierpinski's triangle, staircase horizontal vertical method.

I. Introduction

PASCAL'S triangle which may, at first, looks like a pile of stacked numbers, it carries many secrets and patterns. One of such secrets is its representation of the coefficients of the binomial expansion. In earlier work, the author introduced some polynomial expansion of the form $(\omega+\lambda_1)(\omega+\lambda_2)\dots.(\omega+\lambda_n)$ which, compactly, presented as $\sum_{k=0}^{n}\omega^{n-k}\sum_{T_{nk}}\lambda\dots^{k}\dots\lambda$, and is named the Guelph expansion [1]. The term $\sum_{T_{nk}}\lambda\dots^{k}\dots\lambda$ denotes the sum of the products of each and every possible combination of k elements of the set $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$, and T_{nk} is the binomial coefficient. Such sum of combinations of roots, also, represents the coefficients, a_{n-k}, of the expansion:

$$a_n\,\omega^n + a_{n-1}\,\omega^{n-1} + a_{n-2}\,\omega^{n-2} + \cdots + a_0 = \sum_{k=0}^{n} a_{n-k}\,\omega^{n-k}.$$

Therefore, $a_{n-k} = \sum_{T_{nk}}\lambda\dots^{k}\dots\lambda$ can be used to evaluate the coefficients of a polynomial knowing its roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$, or, inversely, one can find the roots knowing the coefficients of a polynomial [2].

In the case of single valueλ, the Guelph expansion reduces to the well-known binomial expansion

$$(\omega+\lambda)^n = \sum_{k=0}^{n}\omega^{n-k}\,T_{nk}\,\lambda^k.$$

The binomial coefficient T_{nk} (for n=0, 1, 2, 3,….., and k=0, 1, 2, 3,…n) can generate what is known to the most of the western world as Pascal's triangle [3], in China as Yang Hui's triangle, and in Iran as Khayyam's triangle [4].

Numerous studies are available in the literature about the Pascal's triangles, and binomial expansion [5]-[9], as well as polynomials expansion and their coefficients [10]–[12]. Generalizations of Pascal's triangles both planar and geometrical have also been discussed by many authors [13]–[15]. In geometrical representation, the generalization of Pascal's triangle is made for trinomials expansion (Pascal's pyramid), and hyper pyramids for higher orders of expansions.

The author approached a generalization of binomial expansion to polynomial expansion in the form of:

$$(x+y+z+w+\cdots)^n = \sum T_{nk}T_{kk'}\,T_{k'k''}\dots\dots x^{n-k}\,y^{k-k'}z^{k'-k''}w^{k''-k'''}\dots \text{ , with}$$

$$k= 0, 1, 2, .., n;\ \ k'= 0, 1, 2, .., k;\ \ k''=0, 1, 2, .., k'; \text{ ...etc.}$$

The coefficients of this formalization of the polynomial expansion can be generated by inspection using what is called the Embedded Pascal's Triangles (EPTs) inspection method. In such inspection method, one can generate the coefficients of I- variables expansion by multiplying horizontally laid out classical Pascal elements with the corresponding coefficients of the (I-1) variables expansion [16]. An efficient algorithm is available to generate the coefficients of those polynomials expansion, as well as the monomials accompanying such expansion which are in lexicographic order [17].

The EPTs-based generalization of Pascal's triangles is different from the generalization, given in the literature [18], of Pascal's triangles of s^{th} orders (s^{th} arithmetic triangle) of polynomials of the form:

$$(1+x+x^2+\cdots+x^{s-1})^n = \sum_{m=0}^{(s-1)n}\binom{n}{m}_s x^m \quad ,s\ge 2$$

where, $\binom{n}{m}_s$ are the generalized binomial coefficients of order s. The expansion $(1+x+x^2+\cdots+x^{s-1})^n$ is, actually, a

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special case of the expansion$(x+y+z+w+\cdots)^n$, if x=1, y=x, z=x^2, w=x^3, .., etc.

The horizontally laid out elements of the classical Pascal's triangle carry a meaning as the coefficients of the binomial expansion raised to the power of n, where n=0, 1, 2, 3, ... Moreover, they represent the number of non-combined roots, single combined roots, two combined roots, three combined roots, ….., k-combined roots of a polynomial of degree n of the form $\sum_{k=0}^{n} a_{n-k}\, \omega^{n-k} = 0$. Similarly, the vertically laid out elements of Pascal's triangle carry a meaning, simply; each column represents, successively, the number of the coefficients of monomials expansion x^n, binomial expansion $(x+y)^n$, trinomial expansion $(x+y+z)^n$, tetranomial expansion $(x+y+z+w)^n$, ……, etc., for n=0, 1, 2, 3, … . Those sets of numbers are called Waterloo numbers, and the exact values of these coefficients of the considered expansion (called the attached values of Waterloo numbers) are generated by the EPTs inspection method [19]. Those Waterloo numbers can be generated by a form of a polynomial, called Tripoli's polynomial which have two interesting properties, namely; the roots of a degree m Tripoli polynomial are at -1, -2, -3, …, -m, and the sum of its coefficients is m+1 [20].

In this paper, a new inspection method to generate planar Pascal's triangles of order 0, 1, 2, 3, 4, …, N is introduced. The introduced new inspection method, in this paper, is an improvement of the EPTs method with the advantage of being much straight forward and simple. The 0-order represents the expansion of zero-nomial; $(0)^n$ expansion which is simply 1, the 1st order represents monomials expansion $(x)^n$, the 2nd order corresponds to the classical Pascal's triangle representing the coefficients of the binomial expansion; $(x+y)^n$, whereas order 3 of Pascal's triangle represents the coefficients of the trinomial expansion $(x+y+z)^n$, and so forth. In other words, one blows up 1 to many planar Pascal's triangles using the proposed SHV-method to be explained in the Section II of this paper.

The pattern of the odd and even numbers in Pascal's triangle of order 2 (classical Pascal's triangle) is known as the Sierpinski's triangle. In this work, patterns of the odd and even numbers are evolved for the higher orders of the SHV-generated planar Pascal's triangles and are presented at the closing of this paper.

II. The Staircase Horizontal Vertical (SHV) Method

A set of stacked numbers presented in a triangular form was known since the 1st half of the 17th century of the current era (CE) as the Pascal's triangle Such pattern of stacked numbers were earlier known in India as the staircase of mount Meru with the support of the work of an Indian mathematician (Halayudha) in the late 5th century of (CE). In 2010, Ratemi [16] introduced what is called the Embedded Pascal's triangles (EPTs) method to extend the expansion of binomial expansion to polynomial expansion, and deduced an inspection method to generate the coefficients of I-variables polynomial expansion in terms of the previous I-1 variables polynomial expansion.

Pascal's triangle carries many secrets and applications, two of them are of the concern of this paper; namely; expansions, and patterns. The Pascal's triangle represents a binomial expansion of the form $(x+y)^n$, and the pattern of the odd and even numbers represents what is called The Sierpinski's triangle. In this section, the Staircase Horizontal Vertical (SHV) method which is deduced, solely, from the elements of Pascal's triangle is derived from the EPTs inspection method, and is a much easier straight forward method. Such method starts with 1 and explodes it, successively, to Pascal's triangles of different orders; starting from 0-variables expansion of the form $(0)^n$, and proceed to 1-variable expansion of the form $(x)^n$, 2-variables expansion of the form $(x+y)^n$, 3-variables expansion of the form $(x+y+z)^n$, …until I-variables expansion of the form $(x+y+z+w+r+\cdots.)^n$.

A. Pascal's Triangle and Waterloo Matrix

The elements of Pascal's triangle are generated by the binomial coefficients

$$T_{nk} = \binom{n}{k} = \frac{n!}{(n-k)!k!} \quad , \ k = 0, 1, 2, 3 \ldots, n \qquad (1)$$

are given in Table I.

TABLE I
PASCAL'S TRIANGLE

n / k	0	1	2	3	4	…..
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
…..						

If one pulls up each column so as to start from the row corresponding to n=0, then the result will be what is called the Waterloo matrix which presents the Waterloo numbers [7]. The elements of this matrix can be calculated using (2):

$$W_{nk} = \binom{n+k}{k} = \frac{(n+k)!}{n!k!} \qquad (2)$$

Table II presents the Waterloo matrix.

TABLE II
THE WATERLOO MATRIX

n / k	0	1	2	3	4	5	…..
0	1	1	1	1	1	1	
1	1	2	3	4	5	6	
2	1	3	6	10	15	21	
3	1	4	10	20	35	56	
4	1	5	15	35	70	126	
5	1	6	21	56	126	252	
…..							

B. Generating Pascal's Triangles of Different Orders Using SHV-Method

Starting with the expansion of 0-variables this corresponds to the expansion of $(0)^n$ which has a value of 1 for n=0, hence Pascal's triangle of order 0 is 1, (see Fig. 1 (a)). To generate the next monomial expansion of the form$(x)^n$, one draws one step and lay out the value of 1 corresponding to n=0, and to generate the other elements for n=1, 2, 3,….etc., one multiplies the element corresponding to n=0 by the elements of the first column of Waterloo matrix $W1 = [1\ 1\ 1\ 1\ 1\ 1\ \ldots\ldots]'$, consecutively. Therefore, Pascal's triangle of order 1 is a column of 1's, (see Fig. 1 (b)). Next, to generate the coefficients of a binomial of the form $(x+y)^n$, one lays out, horizontally, all of the 1's of the previous monomial expansion in a staircase manner of step size corresponding to the numbers of the 1st column of the Waterloo matrix, then, to complete for the rest of the elements, one multiplies those laid out elements for each step by the columns of the Waterloo matrix in a consecutive manner, i.e.;

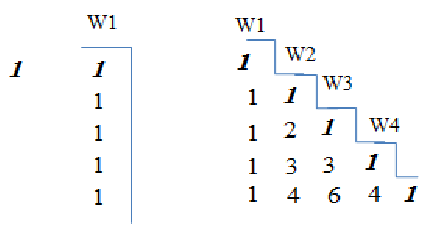
$$W1 = [1\ 1\ 1\ 1\ 1\ \ldots\ldots]', W2 = [1\ 2\ 3\ 4\ 5\ \ldots\ldots]',$$

$$W3 = [1\ 3\ 6\ 10\ 15\ \ \ldots]', W4 = [1\ 4\ 10\ 20\ \ \ldots]', \ldots, \text{etc.}$$

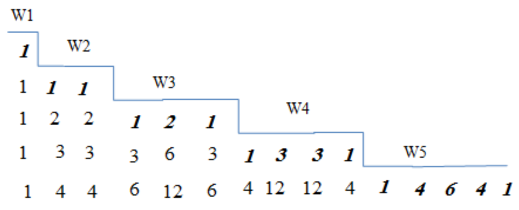
Hence, one gets the classical Pascal's triangle which is of order 2, (see Fig. 1 (c)). Now, to generate the coefficients of trinomials of the form $(x+y+z)^n$, one draws a staircase of varying step width, i.e.; one element width, two elements width, three elements width, etc. Such width size of the staircase steps corresponds to the elements of the 2nd column of the Waterloo matrix $W2 = [1\ 2\ 3\ 4\ 5\ \ldots]'$. Next, one lays out the previously generated binomial coefficients under the steps of the new staircase (bolded numbers in Fig. 1 (d)). Finally, to complete the rest of the elements, one multiplies those laid out elements for each step by the elements of the columns of the Waterloo matrix in a consecutive manner, i.e.; W1, W2, W3, W4, etc. The full coefficients of the trinomial expansion corresponding to planar Pascal's triangle of order 3 are depicted in Fig. 1 (d). To get further expansions, one follows the same algorithm.

C. Polynomial Expansion

The SHV-method has been presented in Section II to generate the coefficients of different orders of Pascal's triangles. Those coefficients belong to the corresponding polynomial expansions. The monomials of the respected expansion can be generated in a lexicographic order.



(a) (b) (c)



(d)

Fig. 1 The Staircase Horizontal Vertical (SHV) method for generating different orders of Pascal's triangles for: (a) zeronomials $(0)^n$, (b) monomials $(x)^n$, (c) binomials $(x+y)^n$, and (d) trinomials$(x+y+z)^n$

A formula for polynomial expansion [17] is presented in (3) as:

$$(x+y+z+w+\cdots)^n = \sum T_{nk}T_{kk'}\,T_{k'k''}\ldots\ldots x^{n-k}\,y^{k-k'}z^{k'-k''}w^{k''-k'''} \quad , \tag{3}$$

with

$$k = 0,1,2,..,n\ ;\ k' = 0,1,2,..,k\ ;$$

$$k'' = 0,1,2,..k';..\text{etc.}$$

The monomials shown in (3) are in lexicographic order; hence, they can be used to generate the related expansion. Equation (3) reduces to the well-known formula for binomial expansion given in (4):

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \tag{4}$$

TABLE III
BINOMIAL EXPANSION OF $(x+y)^4$

n	$k = 0,1,2,\ldots,n$	Coefficients $\binom{n}{k}$ [a]	Monomials [b] $x^{n-k}y^k$
	0	1	x^4
	1	4	x^3y
4	2	6	x^2y^2
	3	4	xy^3
	4	1	y^4

[a] can be read directly from the corresponding row of Pascal's triangle of order 2.
[b]Monomials are in lexicographic order.

Table III presents the required coefficients and monomials for the expansion with n=4. The full expansion is given by (5):

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \tag{5}$$

Similarly, (3) reduces to (6) for trinomial expansion:

$$(x+y+z)^n = \sum_{\substack{k=0,..,n\\ k'=0,..,k}}\binom{n}{k}\binom{k}{k'}x^{n-k}y^{k-k'}z^{k'}\ . \tag{6}$$

Table IV presents the coefficients, and monomials of the following trinomial expansion:

$$(x+y+z)^3 = x^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3y^2z + 3yz^2 + z^3.$$

D. Graphical Distributions of the Elements of Pascal's Triangles

A graphical presentation of the distribution of the elements of the different orders of Pascal's triangles as a function of the degree of expansion n, and the index [k] $=W_{nk}$ (Waterloo number) representing the number of the coefficients in the corresponding expansion is demonstrated in this section. Referring to Table III which represents the binomial expansion for n=4, one notices that k=0, 1, 2, 3, 4, hence there are 5 coefficients for such expansion. This number ($W_{n\ k}=W_{4\ 1}$) can be read directly from the Waterloo matrix for n=4, and k=1 (2nd column; corresponding to a binomial expansion). On the other hand, for the trinomial expansion raised to power n, k=0, 1, 2, 3, …, n, and k'=0, 1, 2, 3,…k. Referring to Table IV for the trinomial expansion where n= 3, the number of coefficients are 10. This number can be read directly from the Waterloo matrix which corresponds to $W_{3\ 2}$ (n=3, k=2: 3rd column representing trinomials).

TABLE IV
BINOMIAL EXPANSION OF $(x+y+z)^3$

	Index [a]		Coefficients [b]	Monomials [c]
n	k	k'	$\binom{n}{k}\cdot\binom{k}{k'}$	$x^{n-k}y^{k-k'}z^{k'}$
3	0	0	1	x^3
	1	0	3	x^2y
		1	3	x^2z
	2	0	3	xy^2
		1	6	xyz
		2	3	xz^2
	3	0	1	y^3
		1	3	y^2z
		2	3	yz^2
		3	1	z^3

[a] $k=0,1,2,\ldots,n,\ \ and\ \ k'=0,1,2,\ldots,k.$
[b] $\binom{n}{k}\cdot\binom{k}{k'}=\frac{n!}{(n-k)!(k-k')!k'!}$ which can, also, be read directly from the corresponding row of Pascal's triangle of order 3.
[c] Monomials are in lexicographic order.

The distribution of the binomial expansion up to the expansion of $(x+y)^{19}$ is shown in Fig. 2 (the number of coefficients corresponding to n=19 is $[k]=W_{n\ k}=W_{19\ 1}=20$). The distribution approaches the normal distribution as n (the degree of the expansion) gets larger. Similarly, the distribution for trinomial expansion up to $(x+y+z)^5$ is shown in Fig. 3 (the number of coefficients corresponding to n=5 is $[k]=W_{n\ k}=W_{5\ 2}=21$).

III. The Sierpinski's Like Patterns

It is quite interesting that when shading odd, and even numbers of the classical Pascal's triangle, a pattern is evolved and is known as the Sierpinski's triangle [21]. Fig. 4 (a) presents such pattern for a right angled Pascal's triangle. In a similar manner, one can apply such odd-even shading to generate patterns for higher orders of planar Pascal's triangles. Figs. 4 (b)-(d) present such patterns for Pascal's triangles of orders 3, 4, and 5, respectively.

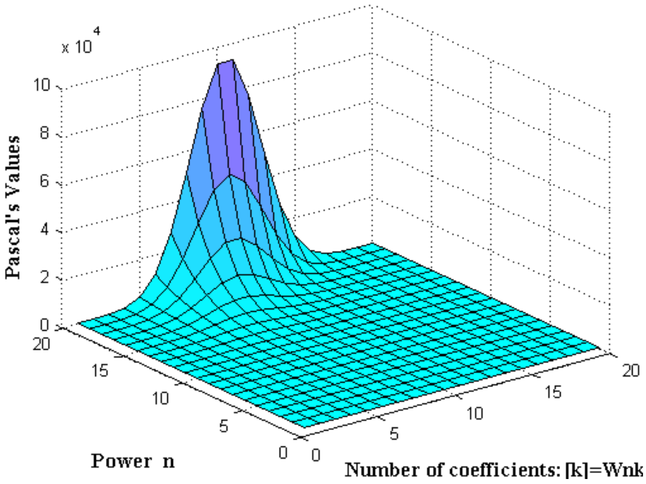


Fig. 2 $(x+y)^n$ distribution function (Pascal's elements of order 2, for n=0, 1, 2, 3, …, 20)

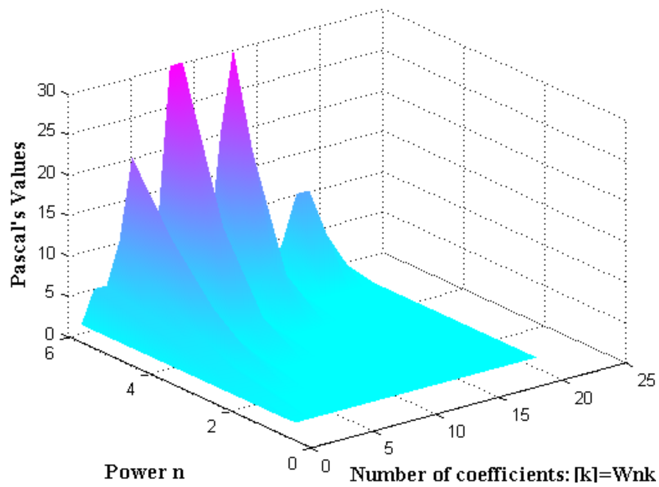


Fig. 3 $(x+y+z)^n$ distribution function (Pascal's elements of order 3, for n=0, 1, 2, ..., 5)

IV. Conclusion

In this work, an inspection simple straight forward method named as SHV- method is introduced to get a generalization of planar Pascal's triangle to extend it from binomial expansion to polynomial expansion.

Furthermore, graphical distributions of the elements of Pascal's triangles are presented. The pattern made by the odd and even numbers of the classical Pascal's triangle which is known as the Sierpinski's triangle is extended for the higher orders of Pascal's triangles. The simple generating method for the generalization of Pascal's triangle can be well used for education. The patterns generated from the odd-even distributions of the elements in Pascal's triangle can be used for the design of carpets, textiles, and other industrial applications.

1															
1	1														
1	2	1													
1	3	3	1												
1	4	6	4	1											
1	5	10	10	5	1										
1	6	15	20	15	6	1									
1	7	21	35	35	21	7	1								
1	8	28	56	70	56	28	8	1							
1	9	36	84	126	126	84	36	9	1						
1	10	45	120	210	252	210	120	45	10	1					
1	11	55	165	330	462	462	330	165	55	11	1				
1	12	66	220	495	792	924	792	495	220	66	12	1			
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1		
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1	
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

(a)

1																				
1	1	1																		
1	2	2	1	2	1															
1	3	3	3	6	3	1	3	3	1											
1	4	4	6	12	6	4	12	12	4	1	4	6	4	1						
1	5	5	10	20	10	10	30	30	10	5	20	30	20	5	1	5	10	10	5	1
1	6	6	15	30	15	20	60	60	20	15	60	90	60	15	6	30	60	60	30	6
1	7	7	21	42	21	35	105	105	35	35	140	210	140	35	21	105	210	210	105	21
1	8	8	28	56	28	56	168	168	56	70	280	420	280	70	56	280	560	560	280	56
1	9	9	36	72	36	84	252	252	84	126	504	756	504	126	126	630	1260	1260	630	126
1	10	10	45	90	45	120	360	360	120	210	840	1260	840	210	252	1260	2520	2520	1260	252
1	11	11	55	110	55	165	495	495	165	330	1320	1980	1320	330	462	2310	4620	4620	2310	462
1	12	12	66	132	66	220	660	660	220	495	1980	2970	1980	495	792	3960	7920	7920	3960	792
1	13	13	78	156	78	286	858	858	286	715	2860	4290	2860	715	1287	6435	12870	12870	6435	1287
1	14	14	91	182	91	364	1092	1092	364	1001	4004	6006	4004	1001	2002	10010	20020	20020	10010	2002
1	15	15	105	210	105	455	1365	1365	455	1365	5460	8190	5460	1365	3003	15015	30030	30030	15015	3003
1	16	16	120	240	120	560	1680	1680	560	1820	7280	10920	7280	1820	4368	21840	43680	43680	21840	4368
1	17	17	136	272	136	680	2040	2040	680	2380	9520	14280	9520	2380	6188	30940	61880	61880	30940	6188
1	18	18	153	306	153	816	2448	2448	816	3060	12240	18360	12240	3060	8568	42840	85680	85680	42840	8568
1	19	19	171	342	171	969	2907	2907	969	3876	15504	23256	15504	3876	11628	58140	116280	116280	58140	11628
1	20	20	190	380	190	1140	3420	3420	1140	4845	19380	29070	19380	4845	15504	77520	155040	155040	77520	15504
1	21	21	210	420	210	1330	3990	3990	1330	5985	23940	35910	23940	5985	20349	101745	203490	203490	101745	20349
1	22	22	231	462	231	1540	4620	4620	1540	7315	29260	43890	29260	7315	26334	131670	263340	263340	131670	26334
1	23	23	253	506	253	1771	5313	5313	1771	8855	35420	53130	35420	8855	33649	168245	336490	336490	168245	33649

(b)

1																			
1	1	1	1																
1	2	2	2	1	2	2	1	2	1										
1	3	3	3	3	6	6	3	6	3	1	3	3	3	6	3	1	3	3	1
1	4	4	4	6	12	12	6	12	6	4	12	12	12	24	12	4	12	12	4
1	5	5	5	10	20	20	10	20	10	10	30	30	30	60	30	10	30	30	10
1	6	6	6	15	30	30	15	30	15	20	60	60	60	120	60	20	60	60	20
1	7	7	7	21	42	42	21	42	21	35	105	105	105	210	105	35	105	105	35
1	8	8	8	28	56	56	28	56	28	56	168	168	168	336	168	56	168	168	56
1	9	9	9	36	72	72	36	72	36	84	252	252	252	504	252	84	252	252	84
1	10	10	10	45	90	90	45	90	45	120	360	360	360	720	360	120	360	360	120
1	11	11	11	55	110	110	55	110	55	165	495	495	495	990	495	165	495	495	165
1	12	12	12	66	132	132	66	132	66	220	660	660	660	1320	660	220	660	660	220
1	13	13	13	78	156	156	78	156	78	286	858	858	858	1716	858	286	858	858	286
1	14	14	14	91	182	182	91	182	91	364	1092	1092	1092	2184	1092	364	1092	1092	364
1	15	15	15	105	210	210	105	210	105	455	1365	1365	1365	2730	1365	455	1365	1365	455
1	16	16	16	120	240	240	120	240	120	560	1680	1680	1680	3360	1680	560	1680	1680	560
1	17	17	17	136	272	272	136	272	136	680	2040	2040	2040	4080	2040	680	2040	2040	680
1	18	18	18	153	306	306	153	306	153	816	2448	2448	2448	4896	2448	816	2448	2448	816
1	19	19	19	171	342	342	171	342	171	969	2907	2907	2907	5814	2907	969	2907	2907	969
1	20	20	20	190	380	380	190	380	190	1140	3420	3420	3420	6840	3420	1140	3420	3420	1140
1	21	21	21	210	420	420	210	420	210	1330	3990	3990	3990	7980	3990	1330	3990	3990	1330
1	22	22	22	231	462	462	231	462	231	1540	4620	4620	4620	9240	4620	1540	4620	4620	1540
1	23	23	23	253	506	506	253	506	253	1771	5313	5313	5313	10626	5313	1771	5313	5313	1771

(c)

1																		
1	1	1	1	1														
1	2	2	2	2	1	2	2	2	1	2	2	1	2	1				
1	3	3	3	3	3	6	6	6	3	6	6	3	6	3	1	3	3	3
1	4	4	4	4	6	12	12	12	6	12	12	6	12	6	4	12	12	12
1	5	5	5	5	10	20	20	20	10	20	20	10	20	10	10	30	30	30
1	6	6	6	6	15	30	30	30	15	30	30	15	30	15	20	60	60	60
1	7	7	7	7	21	42	42	42	21	42	42	21	42	21	35	105	105	105
1	8	8	8	8	28	56	56	56	28	56	56	28	56	28	56	168	168	168
1	9	9	9	9	36	72	72	72	36	72	72	36	72	36	84	252	252	252
1	10	10	10	10	45	90	90	90	45	90	90	45	90	45	120	360	360	360
1	11	11	11	11	55	110	110	110	55	110	110	55	110	55	165	495	495	495
1	12	12	12	12	66	132	132	132	66	132	132	66	132	66	220	660	660	660
1	13	13	13	13	78	156	156	156	78	156	156	78	156	78	286	858	858	858
1	14	14	14	14	91	182	182	182	91	182	182	91	182	91	364	1092	1092	1092
1	15	15	15	15	105	210	210	210	105	210	210	105	210	105	455	1365	1365	1365
1	16	16	16	16	120	240	240	240	120	240	240	120	240	120	560	1680	1680	1680
1	17	17	17	17	136	272	272	272	136	272	272	136	272	136	680	2040	2040	2040
1	18	18	18	18	153	306	306	306	153	306	306	153	306	153	816	2448	2448	2448
1	19	19	19	19	171	342	342	342	171	342	342	171	342	171	969	2907	2907	2907
1	20	20	20	20	190	380	380	380	190	380	380	190	380	190	1140	3420	3420	3420
1	21	21	21	21	210	420	420	420	210	420	420	210	420	210	1330	3990	3990	3990
1	22	22	22	22	231	462	462	462	231	462	462	231	462	231	1540	4620	4620	4620
1	23	23	23	23	253	506	506	506	253	506	506	253	506	253	1771	5313	5313	5313
1	24	24	24	24	276	552	552	552	276	552	552	276	552	276	2024	6072	6072	6072
1	25	25	25	25	300	600	600	600	300	600	600	300	600	300	2300	6900	6900	6900
1	26	26	26	26	325	650	650	650	325	650	650	325	650	325	2600	7800	7800	7800

(d)

Fig. 4 Odd-even patterns of Pascal's triangles; (a) pattern for order 2: $(x+y)^n$ (Sierpinski's triangle), (b) pattern for order 3: $(x+y+z)^n$, (c) pattern for order 4: $(x+y+z+w)^n$, and (d) pattern of order 5: $(x+y+z+w+r)^n$

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