Chapter one

1.1. Background

Differential Equation is an equation with unknown function that contains one or more derivatives of the unknown function.

The order of the differential equation is the highest derivative in the equation, and the Differential equations can be classified based on the order:

I- First order: - just the first derivative appear in the equation.

For example : $y'^2 + y = \sin x$

II- Higher order: - derivatives higher than the first appear in the equation.

For example: $y'' + \sin\ (xy) = 0$

Differential equations can be classified as based on the number of functions that are involved.

(1)-A single differential equation is a single unknown function.

For example: $\frac{\mathrm{dy}}{\mathrm{dt}} + \ 4\mathrm{y} = \ln \mathrm{t}$

(2)- A system of differential equations -there is more than one unknown function. For example, $\frac{dx}{dt} + 4y = \ln t$ together with $\frac{dy}{dt} + 4x = e^t$,

Differential equations can be classified as based on the type of unknown function:-

(a)-*Ordinary* - unknown function is a function in a single variable. For exemple: $\frac{dy}{dx} + \sin y = \ln x$, $\frac{d^2p}{at^2} + \ p = te^t$,etc.

(b)-*Partial* - unknown function is a function in more than one variable. For example:

$$\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}= \sin x+\ln t \ \ , y_{xx}+ \ y_t = te^t \ \ , etc..$$

An ordinary differential equation $y'(t) = f(t, y(t))$ is called non-linear iff the function (t, u) $\rightarrow f(t,u)$ is non-linear in the second argument.

To see that the solutions of the nonlinear system near the origin resemble those of the linearized system.

1.2. Introduction: [11]

A nonlinear system refers to a set of nonlinear equations (algebraic, difference, differential, integral, functional, or abstract operator equations, or a combination of some of these) used to describe a physical device or process that otherwise cannot be clearly defined by a set of linear equations of any kind. Dynamical system is used as a synonym for mathematical or physical system when the describing equations represent evolution of a solution with time and, sometimes, with control inputs and/or other varying parameters as well.

The theory of nonlinear dynamical systems, or nonlinear control systems if control inputs are involved, has been greatly advanced since the nineteenth century. Today, nonlinear control systems are used to describe a great variety of scientific and engineering phenomena ranging from social, life, and physical sciences to engineering and technology. This theory has been applied to a broad spectrum of problems in physics, chemistry, mathematics, biology, medicine, economics, and various engineering disciplines.

Stability theory plays a central role in system engineering, especially in the field of control systems and automation, with regard to both dynamics and control. Stability of a dynamical system, with or without control and disturbance inputs, is a fundamental requirement for its practical value, particularly in most real-world applications. Roughly speaking, stability means that the system outputs and its internal signals are bounded within admissible limits (the so-called bounded-input/bounded-output stability) or, sometimes more strictly, the system outputs tend to an equilibrium state of interest (the so-called asymptotic stability). Conceptually, there are different kinds of stabilities, among which three basic notions are the main concerns in nonlinear dynamics and control systems: the stability of a system with respect to its equilibria, the orbital stability of a system output trajectory, and the structural stability of a system itself.

Illustrative Example: consider the system

$$x' = x + y^2$$
$$y' = -y \quad ; \quad x' = \frac{dx}{dt} \ , \ y' = \frac{dy}{dt}$$

There is a single equilibrium point at the origin. To picture nearby solutions y is small, we note that, when, y^2 is much smaller. Thus, near the origin at least, the differential equation $x' = x + y^2$ is very close to $x' = x$.

This suggests that we consider instead the linearized equation

$$x' = x$$
$$y' = -y,$$

Derived by simply dropping the higher-order term. We can, of course, solve this system immediately. We have a saddle at the origin with a stable line along the y-axis and an unstable line along the x-axis.

We assume new variables u and v via

$u = x + \frac{1}{3}y^2$

$v = y,$

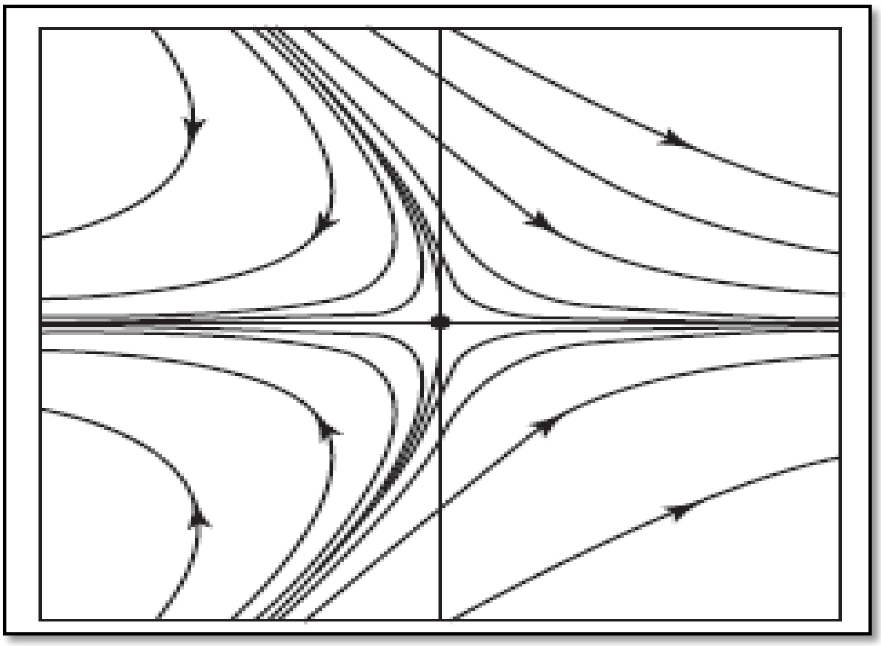


Figure (1.1)
"Phase plane for $x' = x + y^2,\quad y' = -y$ "

Then, in these new coordinates, the system becomes

$u' = x' + \frac{2}{3}yy' = x + \frac{1}{3}y^2 = u$

$v' = y' = -y = -v,$

That is to say, the nonlinear change of variables $F(x, y) = (x + \frac{1}{3}y^2, y)$ converts the original nonlinear system to a linear one, in fact, to the preceding linearized system.

1.2.1. Nonlinear Sinks and Sources

Solutions of planar nonlinear systems equilibrium points resemble those of their linear parts only in the case where the linearized system is hyperbolic; that is, when neither of the eigenvalues near of the system has a zero real part. We consider the special case of a sink. For simplicity, we will prove the results in the following planar case, although all of the results hold in R^n.

Let $X' = F(X)$ and suppose that F $(X_0) = 0$ Let DF_{X_0} denote the Jacobian matrix of F evaluated at X_0. Then, the linear system of differential equations

$Y' = DF_{X_0}Y$

Is called the linearized system near X_0. Note that, if $X_0 = 0$, the linearized system is obtained by simply dropping all of the nonlinear terms in F.

As we know from linear systems , we can say that an equilibrium point X_0 of a nonlinear system is hyperbolic if all of the eigenvalues of DF_{X_0} have nonzero real parts.

Suppose our system is

$x' = f(x, y)$

$y' = g(x, y)$, has (x_0, y_0) as an equilibrium point

With $f(x_0, y_0) = 0 = g(x_0, y_0)$.if we make the change of coordinates $u = x - x_0, v = y - y_0$ then the new system has an equilibrium point at (0,0).

Thus we may as well assume that $x_0 = y_0 = 0$ at the outset.

Let us assume at first that the linearized system has distinct eigenvalues $-\lambda < -\mu < 0$ .Thus, after these changes of coordinates, our system becomes:

$x' = -\lambda x + h_1(x, y)$

$y' = -\mu y + h_2(x, y),$

Where $h_j(x, y)$ contains all of the "higher order terms." Each h_j contains terms that are quadratic or higher order in x and y.

The linearized system is now given by

$x' = -\lambda x$

$y' = -\mu y.$

Thus we are justified in calling this type of equilibrium point a sink. In similar fashion, nonlinear systems near a hyperbolic source are also conjugate to corresponding linearized system.

1.2.2. The Linearization Theorem.

Suppose the n-dimensional system $X' = F(X)$ has an equilibrium point at X_0 that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of X_0.

1.2.3. Saddles

The equilibrium of which the linearized system has a saddle at the origin in R^2. We may assume that this system is in the form

$$x' = \lambda x + f_1(x, y)$$
$$y' = -\mu y + f_2(x, y),$$

Where $-\mu < 0 < \lambda$ and $f_j(x, y)/r$ tends to 0 as $r \to o$. As in the case of a linear system, we call this type of equilibrium point a saddle.

For the linearized system, the y-axis serves as the stable line, with all solutions on this line tending to 0 as $t \to \infty$. Similarly, the x-axis is the unstable line. We cannot expect these stable and unstable straight lines to persist in the nonlinear case. However, there does exist a pair of curves through the origin that have similar properties.

Let $W^s(0)$ denote the set of initial conditions with solutions that tend to the origin as $t \to \infty$. Let $W^u(0)$ denote the set of points with solutions that tend to the origin as $t \to -\infty$. $W^s(0)$ and $W^u(0)$ are called the stable curve and unstable curve, respectively.

1.2.4. The Stable Curve Theorem.

Suppose the system

$x' = \lambda x + f_1(x, y)$

$y' = -\mu y + f_2(x, y),$

Satisfies $-\mu < 0 < \lambda$ and $f_j(x, y)/r \to 0$ as $r \to 0$. Then there is an $\epsilon > 0$ and a curve $x = h^s(y)$ that is defined for $|y| < \epsilon$ and satisfies $h^s(0) = 0$.

Example. Consider the system

$x' = -x$

$y' = -y$

$z' = z + x^2 + y^2,$

The linearized system at the origin has eigenvalues 1 and -1 (repeated). The change of coordinates

$u = x$

$v = y$

$w = z + \frac{1}{3}(x^2 + y^2),$

Converts the nonlinear system to the linear system

$u' = -u$

$v' = -v$

$w' = w,$

The plane w = 0 for the linear system is the stable plane. Under the change of coordinates this plane is transformed to the surface

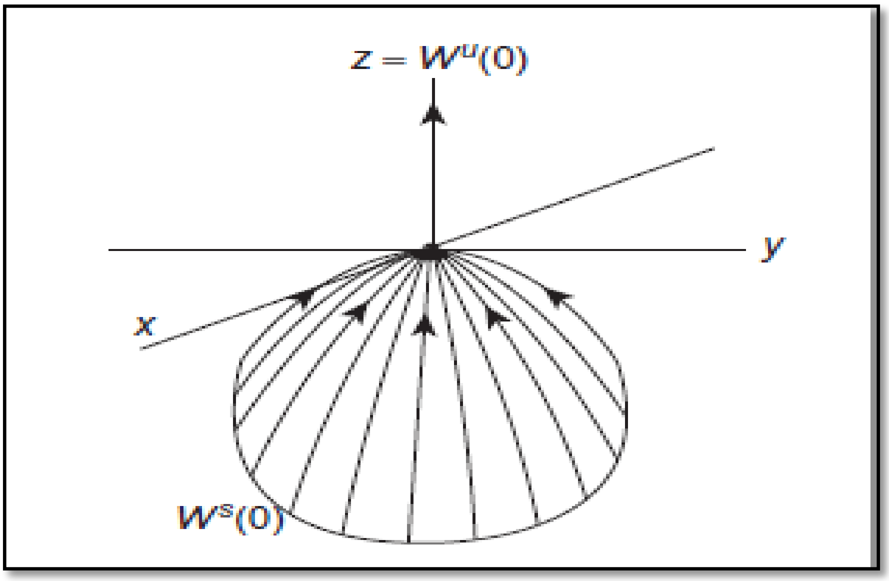


Figure (1.2)

“Phase portrait for$x' = -x, y' = -y, z' = z + x^2 + y^2$"

Which is a paraboloid passing through the origin in R^3 and opening downward. All solutions tend to the origin on this surface; we call this the stable surface for the nonlinear system. See Figure (1.2).

1.2.5. Stability

An equilibrium is said to be stable if nearby solutions stay nearby for all future time. In applications of dynamical systems one cannot usually pinpoint positions exactly, but only approximately, so an equilibrium must be stable to be physically meaningful

1.3. Basic Concepts [17]

$VS_1: \ x + y = y + x$

$x + 0 = x$,

$x + (-x) = 0,$

$(x + y) + z = x + (y + z),$

Here $x, y, z \in R^n, -x = (-1)\, x$, and $0 = (0, \ldots \ldots \ldots \ldots \ldots, 0) \in R^n$.

$VS_2: (\lambda + \mu)\, x = \lambda x + \mu\, x$

$\lambda(x + y) = \lambda x + \lambda y$

$1.x = x$

$0.x = 0 \quad (the\ first\ 0\ in\ R\ , the\ second\ in\ R^n).$

These operations satisfying VS_1 and VS_2, define the vector space structure on R^n . And for each $x= (x_1, \ldots\ldots, x_n)$ in R^n .let A be a map from:$R^n \rightarrow R^n$ It is easy to check that this map satisfies,

for $x, y \in R^n, \lambda \in R$;

$L_1: A(x + y) = Ax + Ay$

$L_2: A(\lambda x) = \lambda A\, x$

These are called linearity properties any map $A: R^n \rightarrow R^n$ satisfying L_1 and L_2 is called a linear map .

The set of all operator's on R^n is denoted by $L(R^n)$

1.3.1. Review of Matrices:

A matrix of size $m \times n$ is an array of elements a_{ij}

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}),\ \ 1 \le i \le m, 1 \le j \le n$$

We consider only square matrices, i.e. m=n

(I)- Addition : $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

(II)- Scalar multiple: α A = $(\alpha . a_{ij})$

(III)- Transpose: A^T is witch the a_{ij} with $a_{ji.}$. (A^T)= A

(IV)- Product: for A. B = C, it means c_{ij} is the inner product of (i^{th} row of A) and (j^{th} column of B)

Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} x & y \\ x & v \end{pmatrix} = \begin{pmatrix} ax + bx & ay + bv \\ cx + dx & cy + dv \end{pmatrix}$$

We can express system of linear equation using matrix product

Example:

$$x_1 - x_2 + 3x_3 = 4$$

$$2x_1 + 5x_3 = 0$$

$$x_2 - x_3 = 7$$

Can be expressed as:

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}$$

$$x_1{}' = a(t)\, x_1 + b(t)\, x_2 + g_1(t)$$

$$x_2{}' = c(t)\, x_2 + d(t)\, x_2 + g_2(t),$$

$$\rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Some properties:

(a)- Identity I: I= diag (1,1,…..,1)

$AI = IA =$ A.

(b)- Determinant det (t):

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$det\begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix} = avz + bwx + cuy - xvc - ywa - zub \; .$$

(c)- Inverse $inv(A) = A^{-1}, A^{-1}A = A\,A^{-1} = I$.

(d)- The following statements are all equivalent :

(I)- A is invertible;

(II)-A is non – singular;

(III)- $det\ (A) \neq 0$;

(IV)-Row vectors in A are linearly independent;

(V)-Column vectors in A are linearly independent.

(VI)-All eigenvalues of A are non – zero.

1.3.2. Autonomous:[13]

A differential equation is called autonomous if the right hand side does not explicitly depend upon the time variable:

$$\frac{du}{dt} = F(u).$$

1.3.3. The exponential of a matrix:[16]

We begin with the study of the autonomous linear first order system

$$x'(t) = Ax(t)\,,\ x(0) = x_0$$

Where A is an n by n matrix here as usual, we write Ax for the matrix product whose components are given by

$$(Ax)_i = \sum_{j=i}^{n} A_{ij} x_{ji}$$

Where $(A_{ij}), 1 \leq i, j \leq$n are the entries of A and (x_j) $1 \leq j \leq n$ the components of x.

We also recall the definition of the scalar product and norm

$x.y = \sum_{j=i}^{n} x_j\, y_i$

(1)- The inner product of vectors x and y in R^n is

$\langle \mathrm{x}, \mathrm{y} \rangle = \mathrm{x}_1\mathrm{y}_1 + \cdots + \mathrm{x}_n\mathrm{y}_n$

(2)- The Euclidean norm of x is

$|x| = \langle x, x \rangle^{1/2} = ({x_1}^2 + \cdots . {x_n}^2)^{1/2}$

Basic properties of the inner product are:-

(I)- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$;

(II)- Bilinearity: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$\langle \alpha\, x, y \rangle = \alpha \langle x, y \rangle$, $\alpha \in R$;

(III)- Positive definiteness: $\langle x, x, \rangle \geq 0$ *and* $\langle x, x, \rangle = 0$ If and only if $x = 0$

An important inequality is Cauchy's inequality: $\langle x, y \rangle \leq |x||y|$.

The basic properties of the norm are:

1. $|x| \geq 0$ *and* $|x| = 0$ if and only if $x = 0$;
2. $|x + y| \leq |x| + |y|$;
3. $|\acute{\alpha}\, x|$=1 $|\acute{\alpha}||x|$;

   Where $|\acute{\alpha}|$ is the ordinary absolute Value of the scalar α'
4. $|x - y| \geq 0$ and $|x - y| = 0$ if and only if $x = y$;
5. $|x - z| \leq |x - y| + |y - z|$.

1.3.4. Norm on E: [12]

Let E be a Vector space with addition and multiplication by scalar λ in R or C.

A norm on E is a map $\|.\|$: E→R which satisfies the following three properties

N_1: $\|x\| \geq 0$ and $\|x\|=0$ if and only if $x = 0$,

N_2: $\|\lambda x\|$: $|\lambda|\|x\|$,

N_3: $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

A vector space E equipped with a norm $\|.\|$ is called a normed vector space.

1.3.5. Euclidean norm:[15]

If x is a vector in R^n, then the Euclidean norm of x is defined as

$\|x\|_2=\sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2}$.

1.3.6. Homogeneous function of degree n: [10]

A function $f(x, y)$is called homogeneous of degree n if

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$

For a polynomial homogeneous says that all of the terms have same degree

Example :

The following are homogeneous function of various degrees:

$3x^6 + 5x^4y^2$ Homogeneous of degree 6.

$3x^6 + 5x^3y^2$ Nonhomogeneous.

1.3.7. n^{th} order liner differential equation: [10]

An n^{th} order liner differential equation is an equation of the form

$$\frac{d^n y}{dx^n}+p_1(x)\frac{d^{n-1} y}{dx^{n-1}}+p_2(x)\frac{d^{n-2} y}{dx^{n-2}}+\cdots+p_n(x)y=Q(x) \ldots\ldots\ldots (1)$$

An n^{th} order linear equation can be written as a linear system as follows let.

$$y_1(x)=y(x), y_2(x)=\frac{dy}{dx}, y_3(x)=\frac{d^2 y}{dx^2}, \ldots\ldots. y_n(x)=\frac{d^{n-1} y}{dx^{n-1}}$$

Then

$$\frac{dy_1}{dx}=y_2, \frac{dy_2}{dx}=y_3, \ldots\ldots. \frac{dy_{n-1}}{dx}=y_n$$

And we have

$$\frac{dy_n}{dx}=\frac{d^n y}{dx^n}=-p_1(x)\frac{d^{n-1} y}{dx^{n-1}}\cdots\cdots\cdots\cdots -p_n(x)y+Q(x)$$

$$=-p_1(x)y_n-\cdots-p_n(x)y_1+Q(x)$$

Therefore

$$\frac{dy}{dx}=A(x)Y+B(x)$$

Where

$$A(x)=\begin{bmatrix} 0 & 1 & 0 & \ldots\ldots\ldots\ldots & 0 \\ 0 & 0 & 1 & \ldots\ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots\ldots & 1 \\ -p_n(x) & -p_{n-1}(x) & -p_{n-2}(x) & \ldots\ldots & -p_1(x) \end{bmatrix}$$

And

$$B(x) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ Q(x) \end{bmatrix}$$

Homogeneous linear Equations:-

In (1) if Q(x) ≡0 called the homogenous case.

Where

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = 0$$

Linear system [9]

Linear system of ordinary differential equation:

$$X' = AX \ldots\ldots\ldots\ldots\ldots\ldots (1^*)$$

Where $X \in R^n$, A is an n×n matrix

And

$$x' = \frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

It is shown that the solution of linear system (1^*) together with the initial condition

$x(0) = x_0$is given by

$$x(t) = e^{AT}x_0$$

Where e^{AT} is an n ×n matrix function defined by its Taylor series.

1.3.8. Matrix exponentiale^A:

Suppose A is an n×n constant matrix .the matrix exponential e^A is defined in terms of the Maclaurin series expansion of the used exponential function.

That is,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Which is a sum involving power of the matrix A.

Then tA is matrix, so

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

If t=0 →e^0=1 and if k=0 then $A^0 = I$

1.3.9. Characteristic equation: [15]

Given a square matrix A, the equation $\det(A - \lambda \mathrm{I}) = 0$ is called the characteristic equation of A .the expression $\det(\mathrm{A} - \lambda \mathrm{I})$ is a polynomial in the variable λ , and is called the characteristic polynomial of A.

Example: let $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & a_{22} \ldots & a_{2n} \\ & \vdots & \vdots \\ 0 & 0 \quad \ldots & a_{nn} \end{bmatrix}$

Then the characteristic equation $\det(A - \mathrm{I}\lambda) = 0$ is simply

$$(\lambda - a_{11})(\lambda - a_{22}) \ldots\ldots (\lambda - a_{nn}) = 0$$

$$\rightarrow \lambda_1 = a_{11}, \ldots \ldots \ldots \ldots\ldots, \lambda_n = a_{nn}$$

And T=$\sum_{i=1}^{n} a_{ii}$ is trace (A).

Diagonal matrices:

D is an n×n diagonal matrix if it is writhen in the form (the no diagonal terms are all zeros).

$$D = \begin{bmatrix} d_1 & \cdots & . \\ \vdots & d_2 & \vdots \\ . & \cdots & d_n \end{bmatrix}$$

Which we will sometimes denote by $D = diag\ \{d_1, d_2, \dots d_n\}$

1.3.10. Eigenvector and eigenvalues:[11]

A nonzero vector V_0 is called an eigenvector of A if $AV_0 = \lambda V_0$ for some λ, λ is called an eigenvalue.

Theorem:

Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ.then the function

$x(t) = e^{\lambda t} v_0$ is a solution of the system $x' = Ax$.

1.3.11. Complex Eigenvalue:[12]

A class of operators that have no real eigenvalue are the planar operators.

$T_{a,b}: R^2 \to R^2$ Represented by matrix

$$A_{a,b} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad , \quad b \neq 0$$

The characteristic polynomial is

$$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0, \lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2}$$

Where roots are

$a + ib\,, a - ib\,; \quad i = \sqrt{-1}$

We interpret $T_{a,b}$ geometrically as follows .introduce the numbers r, θ by

$r = (a^2 + b^2)^{1/2}$

$\theta = cos^{-1}\left(\frac{a}{r}\right), \cos\theta = \frac{a}{r}$,

$a = rcos\ \theta$

$b = rsin\ \theta$

In the standard basis, the matrix of R_θ is

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

1.3.12. ODEs (linear, nonlinear): [15]

Consider the system$x' = f(x)$, where $f: R^n \to R^n$. the system of ODEs is linear if the function f satisfies $f(\alpha x + y) = \alpha f(x) + f(y)$ for all vectors $x, y \in R^n$ and all scalars $\alpha \in R$.

Otherwise, the system of ODEs is called nonlinear.

Example The right hand side of the ODE

$\frac{dx}{dt} = x^2$ is $f(x) = x^2$,

And$f(x + y) = (x + y)^2 = x^2 + 2xy + y^2 \neq x^2 + y^2 = f(x) + f(y)$

Therefore the ODE, is nonlinear.

1.3.13. Derivative of (f at x_0) [9]

The function $f: R^n \to R^n$ is differentiable at $x_0 \in R^n$ if there is a liner transformation $Df(x_0) \in L(R^n)$that satisfies

$$\lim_{|h|\to 0} \frac{|f(x_0+h)-f(x_0)-Df(x_0)h|}{|h|} = 0$$

The linear transformation $Df(x)$ is called the derivative of f at x_0.

Theorem:

If $f: R^n \to R^n$ is differentiable at x_0 ,then the partial derivatives

$\frac{\partial f_i}{\partial xi}$, i,j= 1,…………,n ,all exist at x_0 and All $x \in R^n$,

$$\mathrm{D}f(x_0)x = \sum_{j=1}^{n} \frac{\partial f}{\partial xj}(x_0)x_j$$

Thus, if f is a differentiable function the derivative Df is given by the n×n Jacobian matrix.

$$\mathrm{Df} = \left[\frac{\partial f_i}{\partial x_i}\right].$$

1.3.14. Differentiable on E:

Suppose that $f: E \to R^n$ is differentiable on E, then $f \in C^1(E)$if the derivative

D$f: E \to L(R^n)$ is continuous on E.

1.3.15. Flow$\varphi_t(x)$: [14]

$\varphi_t(x_0)$ is the solution at time t of $x' = f(x)$ starting at x_0

When $t = 0$ is called the flow through x_0 at $t = 0$

Thun $\varphi_0 (x_0) = x_0$, $\varphi_s(\varphi_t(x_0)) = \varphi_{s+t}(x_0)$. etc.

(Continuos semi- group).

We sometime with $\varphi_t\, f(x_0)$ to identify the particular dynamic system leading to this flow.

1.3.16. Review of Topology in R^n:[12]

An operator T on R^n matrix $[a_{ij}]$ by the rule

$T_{ej} = \sum_i a_{ij}\text{-}e_i$; $i = 1 \dots \dots n$

Where $\{e_1 , \dots \dots , e_n\}$ is the standard basis of R^n is defined by

$e_i = (0, \dots ,0,1,0,,, \dots 0)$; $i = 1 \dots \dots n$

Equivalently, the i^{th} coordinate of T_x , $x = (x_1, \dots \dots, x_n)$ is $\sum_i a_{ij}x_j$.

If $\epsilon > 0$ the $\epsilon -$ neighborhood of $x \in R^n$

is $B_\epsilon(x) = \{y \in R^n : |y - x| < \epsilon\}$

A neighborhood of x is any subset of R^n containing a $\epsilon -$neighborhood of X.A set $X \subseteq R^n$ is open if it is a neighborhood of every $x \in$ X.

Explicitly, X is open if and only if for every $x \in$ X there exists $\epsilon > 0$ depending on X, such that $B_\epsilon(X) \subseteq x$

A sequence$\{x_k\} = x_1, x_2, \dots \dots \dots in$ R^n

Converges to the limit $y \in R^n$ if $\lim_{k\to\infty} |x_k - y| = 0$

A sequence $\{x_k\}$ in R^n is Cauchy sequence if for every $\epsilon > 0$ there exists an integer k_0 such that

$|x_j - x_k| < \epsilon$ If $k \geq k_0$ and $j \geq k_0$

The following basic property of R^n is called metric completeness:

A sequence converges to a limit if and only if it is a Cauchy sequence

A subset $Y \subseteq R^n$ is closed if every sequence of point in Y that is convergent has its limit in Y it is easy to see that this is equivalent to:

Y is closed if the complement R^n–Y is open

Let $X \subseteq R^n$ be any subset, A map $f: x \to R^m$ is continuous if it takes convergent ,sequence to convergent sequences .this means: for every sequence$\{x_k\}$ in X with

$\lim_{k\to\infty} x_k = y , \quad y \epsilon X$

It is true that

$\lim_{k\to\infty} f(x_k) = f(y)$

A subset $X \subseteq R^n$ is bounded if there exists

$a > 0$ Such that $X \subseteq B_a(0)$.

A subset X is compact if every sequence in X has a subsequence converging to a point in X.

The basic theorem of Bolzano –Weierstrass says:

A subset of R^n is compact if and only if it is both closed and bounded let $k \subseteq R^n$ be compact and $f: K \to R^m$ be a continuous map . Then f (k) is compact. A nonempty compact subset of R has maximal element and a minimal element.

Chapter two

2.1.Linear Systems

2.1.1. Equilibrium point:[7]

An important notion when considering system dynamics is that of equilibrium point.

Equilibrium points are considered for autonomous systems (no explicit control input).

A point x_0 in the state space is an equilibrium point of the autonomous system $x' = Ax$ if when the state x reaches x_0 , it stays at x_0 for all future time.

2.1.2. Stability:

The system $x' = Ax$ is stable if Re $[\lambda_i] < 0$ for i=1,…,n., where λ_i is eigenvalues.

2.1.3. Theorem (Unique solution of the linear systems):[12]

Let A be an operator on R^n having n distinct, real eigenvalues .then

$$x' = Ax \ldots\ldots\ldots\ldots\ldots (1) \quad ; x(0) = x_0$$

Has unique solution

Proof:-Such that matrix QAQ^{-1} is diagonal

$$QAQ^{-1} = diag\{\lambda_1, \ldots\ldots\ldots\ldots\ldots., \lambda_n\} = B$$

Where $\lambda_1, \ldots\ldots\ldots\ldots\ldots\ldots, \lambda_n$ are the eigenvalues of A.

Introducing the new coordinates

$$y = Qx \ \ in \ \ R^n,$$

$$\rightarrow \ \ x = \ Q^{-1}y$$

$$\rightarrow y' = Qx' = QAx = QAQ^{-1}y$$

Then

$y' = By$ ………(2)

Since B is diagonal,

$\therefore y_i{}' = \lambda_i y_i \,; i = 1, \ldots\ldots, n$ ……………..(2′)

We know that (2′) has unique solution for every initial condition $y_i\,(0)$:

$y_i\,(t) = y_i(0)\exp(t\lambda_i)$

To solve (1) ,put $y(0) = Qx_0$.

If $y(t)$ is the corresponding solution of (2) , then the solution of (1) is

$x(t) = Q^{-1}y(t)$

$= Q^{-1}\,(y_1\,(0)e^{\lambda_1 t}, \ldots\ldots\ldots.. y_n\,(0)e^{\lambda_n t})$.

$\therefore x' = Q^{-1}y'$

$= Q^{-1}By$

$= Q^{-1}(QAQ^{-1})y$

$= AQ^{-1}y$

$= Ax$

And $x(0) = Q^{-1}y(0) = Q^{-1}Qx_0 = x_0$

Thus x (t) really dose solve (1).

To prove that there are no other solutions to (1), we note that x (t) is

A solution to (1) if and only if is a solution to

$y' = By$ ……………(3), $y(0) = Qx_0$

Hence two different solutions to (1) would lead to two different solutions to (3), which is impossible since B is diagonal. This proves the theorem.

2.1.4. Theorem (The solution of the linear system):

Let A be an operator on R^n . The solution of the initial value problem.

$x' = Ax \dots\dots\dots\dots\dots (1) , x(0) = k \epsilon R^n$

is

$x(t) = e^{tA}\ k \dots\dots\dots\dots\dots\dots\dots (2)$

And there are no other solutions

Proof.

$\frac{d}{dt}(e^{tA}k) = \left(\frac{d}{dt}e^{tA}\right)k$

$= Ae^{tA}\ k$;

Since $e^{0A}\ k = k$; if follows that (2) is a solution of (1).

Let x(t) be any solution of (1) and put

$y(t) = e^{-tA}\, x(t)$

Then

$y'(t) = \left(\frac{d}{dt}e^{-tA}\right)x(t) + e^{-tA}x'(t)$

$= -Ae^{-tA}\, x(t) + e^{-tA}\, x(t)$

=0

→ $y(t)$ is a constant, setting t=0

Shows $y(t) = k$

This completes the proof of the theorem.

2.1.5. Theorem (The solution of DE is special form):

Let the n×n matrix A have n distinct real eigenvalues $\lambda_1 , \dots \dots , \lambda_n$

Then every solution to the Differential equation.

$x' = Ax \,, \quad x(0) = u$

Is of the form

$x_i(t) = C_{i1}e^{t\lambda_1} + \cdots C_{in}e^{t\lambda_n}; \quad i = 1 \dots . . n$

For unique constants $C_{i1}, \dots \dots . , C_{in}$ depending on u

2.2. Linear system in R^2:[9]

$x' = Ax \dots \dots \dots \dots (1)$ Where $x \epsilon R^2$ and A is a 2× 2matrix.

We begin by describing the phase portraits for the linear system

$x' = Bx \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (2)$

Where the matrix $B = Q^{-1}AQ$

If B=$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ $, B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $, or\ B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

With $X(0) = x_0$ is given by

$x(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} x_0 \,, \quad x(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0 \,,$

$or\ \ x(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} x_0$

2.2.1.*Case I*. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ With $\lambda < 0 < \mu$,

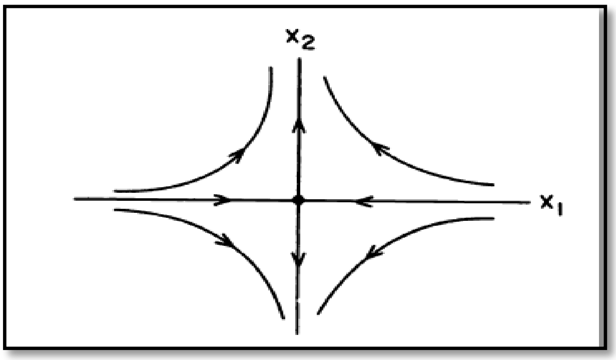


Figure (2.1)
"A saddle at origin"

The system (2) is said to have a saddle at the origin in the is case.

If $\mu < 0 < \lambda$, the arrows in Figure(2. 1) are reversed. whenever A has two real eigenvalues of opposite sign , $\lambda < 0 < \mu$, the phase portrait for the linear system (1) is linearly equivalent to the phase portrait shown in figure(2.1).; i.e, it is obtained from Figure(2.1) by a linear transformation of coordinates; and the stable and unstable subspaces of (1) are determined by the eigenvectors of A.

2.2.2. Case II. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \leq \mu < 0$

$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with $\lambda < 0$

The phase portraits for the linear system (2) in these case are given in Figure(2. 2).

The origin is referred to as a stable node in each of these cases.

It is called a proper node in the first case with $\lambda = \mu$ and an improper node in the other two cases.

If $\lambda \geq \mu > 0$ or if $\lambda > 0$ in case II, the arrows in Figure(2.2) are reversed and the origin is referred to as an unstable node.

Whenever A has two negative eigenvalues $\lambda \leq \mu < 0$, the phase portrait of linear system (1) is linearly equivalent to one of the phase portraits shown in Figure(2.2) The stability of the node is determined by the sign of the eigenvalues : stable if $\lambda \leq \mu < 0$ and unstable if

$\lambda \geq \mu > 0$. Note that each trajectory in Figure (2.2) approaches the equilibrium point at the origin along a well –defined tangent line $\theta = \theta_0$, determined by an eigenvector of A, as $t \to \infty$

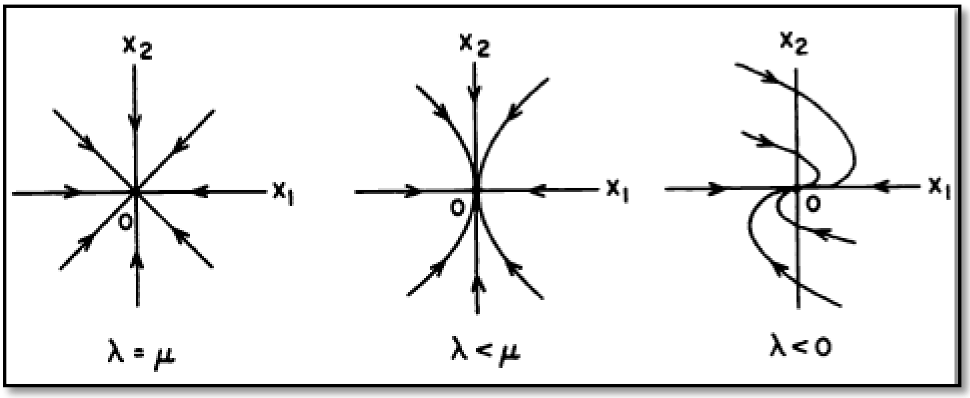


Figure (2.2)
"A stable node at the origin"

2.2.3. Case III.$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ With $a < 0$.

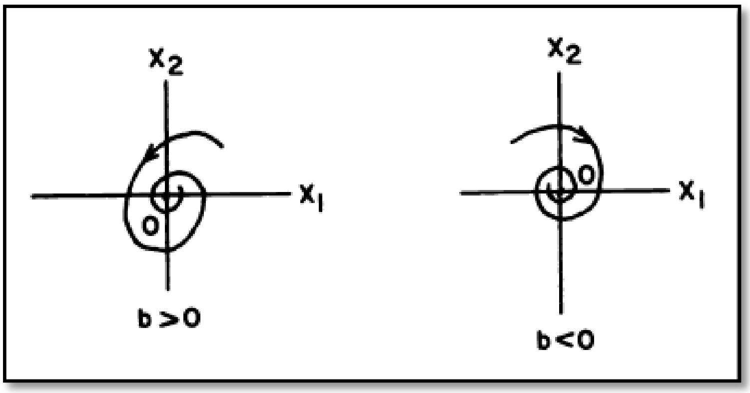


Figure (2. 3)
"A stable focus at the origin"

The phase portrait for the linear system (2) in this case is given in Figure (2.3) the origin is referred to as a stable focus in these cases. If $a > 0$, the trajectories spiral away from the origin with increasing t and the origin is called an unstable focus.

Whenever A has a pair of complex conjugate eigenvalues with nonzero real part,$a \pm ib$,with $a < 0$, the phase portraits for the system (1) is linearly equivalent to one of the phase portraits shown in figure (2.3). Note that the trajectories in figure (2.3) do not approach the origin along well-defined tangent lines; i.e. ,the angle $\theta(t)$ that the vector $x(t)$ makes with the x_1-axis does not approach a constant θ_0 as $t \to \infty$, but rather $|\theta(t)| \to \infty$ as $t \to \infty$ and $|x(t)| \to 0$ as $t \to \infty$ in this case .

2.2.4. Case IV. $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$

The phase portrait for the linear systems (2) in this case is given in

Figure (2.4) the system (2) is said to have a center at the origin in this case.

Whenever A has a pair of pure imaginary complex conjugate eigenvalues,$\pm ib$, The phase portrait of the linear system (1) is linearly equivalent to one of the phase portraits shown in Figure (2. 4)

Note that the trajectories or solution curves in Figure (2. 4) lie on circles

$|x(t)| = constant$.

In general, the trajectories of the system (1) will lie on ellipses and the solution $x(t)$ of (1) will satisfy

$m \leq |x(t)| \leq M \;\; for\ all\ \ t \in R;$

The angle $\theta(t)$ also satisfies $|\theta(t)| \to \infty$ as $t \to \infty$ in this case.

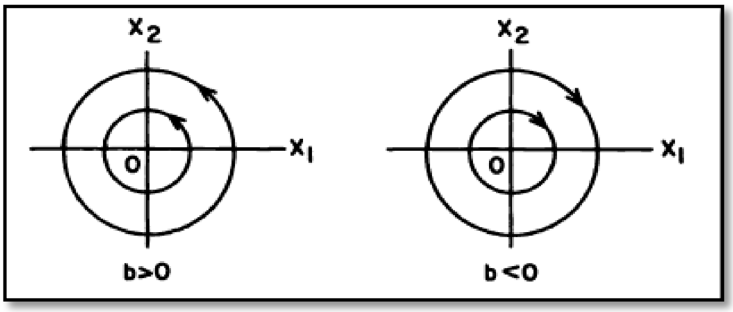


Figure (2. 4)
"A center at the origin"

If one (or both) of the eigenvalues of A is zero,

i.e., if det A=0, the origin is called an equilibrium point of (1).

2.2.5. A saddle, a node and a focus or center at the origin:

The linear system (1) is said to have saddle, a node, a focus or a center at the origin if the matrix A is similar to one of the matrices B in cases I,II,III or IV respectively,

i.e., if its phase portrait is linearly equivalent to one of the phase portraits in figures (2.1),(2.2),(2.3),or(2.4) respectively .

Theorem:

Let $\delta = \det A$ and T = trace A and consider the linear system

$x' = Ax \ldots\ldots\ldots\ldots\ldots\ldots(1)$

(a)- If $\delta < 0$ then (1) has a saddle at the origin.

(b)- If $\delta > 0$ and $T^2 - 4\delta \geq 0$ then (1) has a node at the origin;

It is stable if $T < 0$ and unstable if $T > 0$

(c)-If$\delta > 0$,$T^2 - 4\delta < 0$, and $T \neq 0$ then (1) has a focus at the origin ;

It is stable $T < 0$ and unstable if $T > 0$.

(d) If $\delta > 0$ and T=0 then (1) has a center at the origin.

Note that in case (b),$T^2 \geq 4|\delta| > 0$

i.e.,$T \neq 0$.

Proof .The eigenvalues of the matrix

A are given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4\delta}}{2}$$

Thus (a) if $\delta < 0$ there are two real eigenvalues of opposite sign.

(b) If $\delta > 0$ and $T^2 - 4\delta \geq 0$ then there are two real eigenvalues of the same sign as T;

(c) If $\delta > 0$; $T^2 - 4\delta < 0$ and $T \neq 0$ then there are two complex conjugate eigenvalues $\lambda = a \pm ib$ and,

A is similar to the matrix B in case III above with $a = T/2$;

(d) If $\delta > 0$ and T=0 then there are two pure imaginary complex conjugate eigenvalues. Thus case a, b, c, II, III and IV discussed above and we have a saddle, node, focus or center respectively.

2.2.6. Sink and source of the linear system:

A stable node or focus of (1) is called a sink of the linear system and an unstable node or focus of (1) is called a source of the linear system.

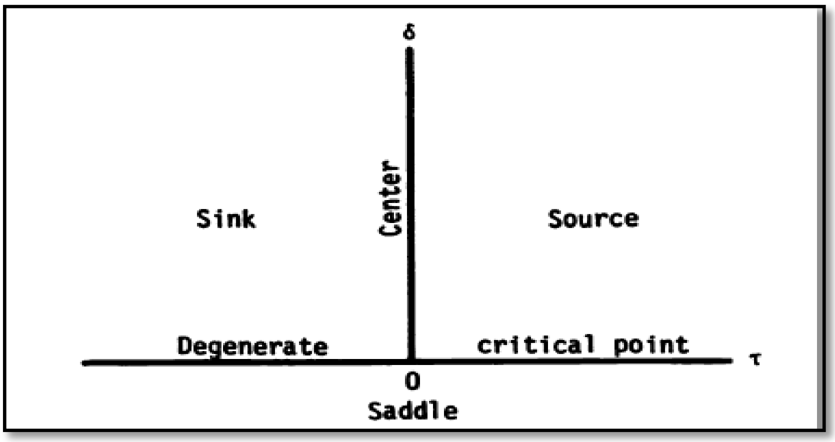


Figure (2. 6)

"A bifurcation diagram for the linear system (1)"

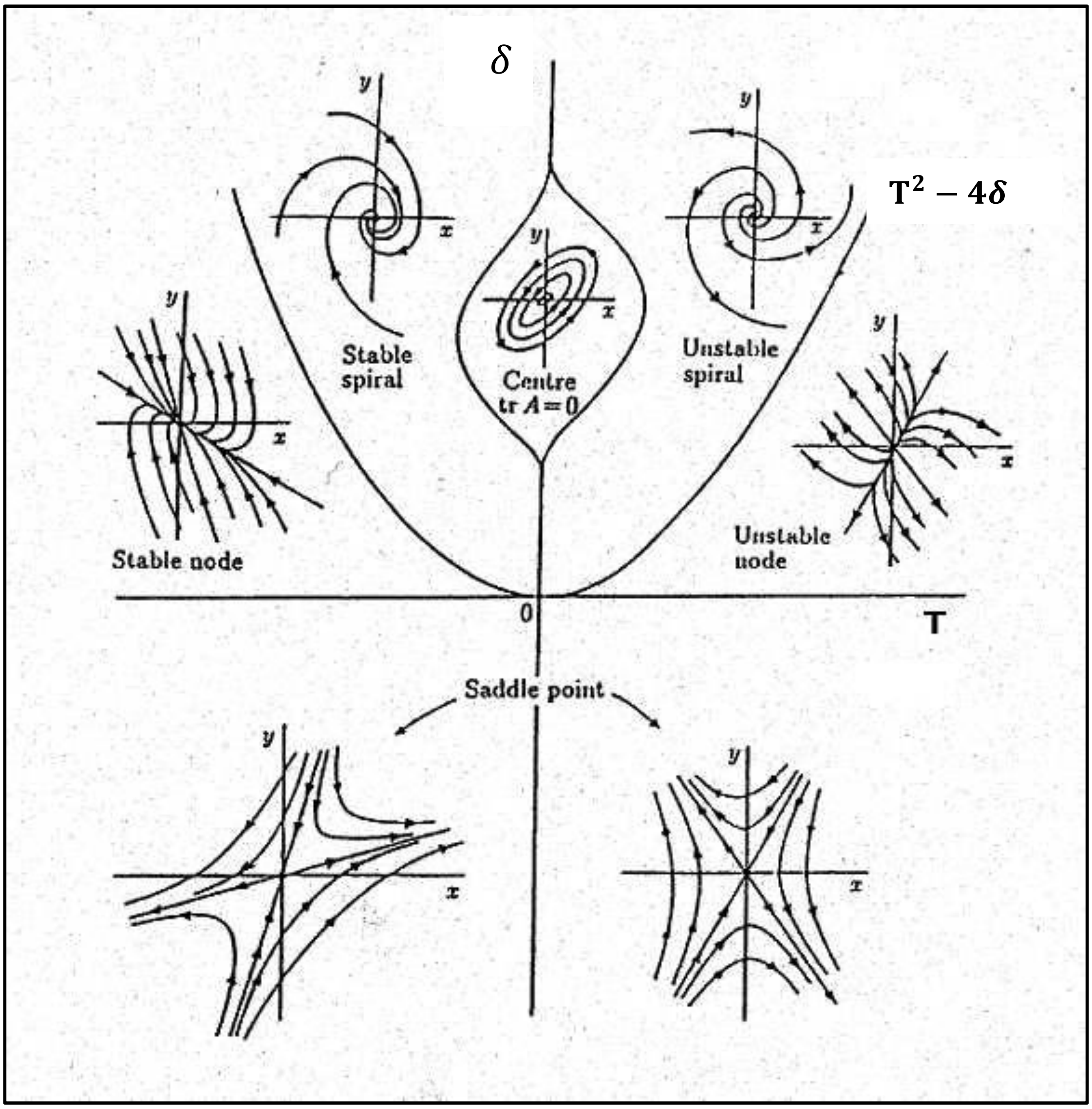


Figure (2. 7) "Depending on the sign of the trace T and of the determinant δ of the Jacobian matrix and on $T^2 - 4\ \delta$, several approaches to the steady state can be observed."

Chapter three

3.1.Nonlinear System:[9]

Let $f: E \rightarrow R^n$ and E is an open subset of R^n .

Then $x' = f(x) \ldots \ldots (1)$ is called nonlinear system of differential equations.

So, in this case, unique solution through each point $x_0 \in E$, defined on a maximal interval of existence $(\alpha, \beta) \subseteq R$.

3.1.1. Equilibrium point :[7],[9]

We suppose f is C^1 a point $\tilde{x} \in E$ is called an equilibrium point of (1) if $f(\tilde{x}) = 0$.

A point $x_0 \in R^n$ is called an equilibrium point or critical point of (1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.

The linear system $x' = Ax$ with the matrix $A = Df(x_0)$ is called the linearization of (1) at x_0. If $x_0 = 0$ is an equilibrium point of (1), then $f(0) = 0$ and, by Taylors' Theorem.

$$f(x) = Df(0)x + \frac{1}{2}D^2 f(0)(x, x) + \cdots$$

Clearly, the constant function $x(t) = \tilde{x}$ is a solution of (1). By uniqueness of solution, no other solution curve can pass through $\tilde{x}$.

If E is the state space of some physical or biological, economic, (or the like) system described by (1) then $\tilde{x}$ is an "equilibrium state "; if the system is at $\tilde{x}$ it always will be (and always was) at $\tilde{x}$.

If $F(t, c) = 0$ for all t ,then $c \in R^n$ is said to be an equilibrium (or critical) state.

Let $\emptyset : \Omega \to E$ be the flow associated with (1); $\Omega \subseteq R \times E$ is an open set, and for each $x \in E$ the map $t \to \emptyset\,(t, x) = \emptyset_t\,(x)$ is the solution passing through x when $t = o$; it is defined for t in some open interval.

If $\tilde{x}$ is an equilibrium, then $\emptyset_t(\tilde{x}) = \tilde{x}$ for all $t \in R$. for this reason, $\tilde{x}$ is also called a stationary point, or fixed point, of the flow. Another name for $\tilde{x}$ is a zero or singular point of the vector field f .

Suppose f is linear: $E = R^n$ and $f(x) = Ax$ where A is a linear operator on R^n. Then the origin $0 \in R^n$ is an equilibrium of (1).

Then solutions $\emptyset_t(t)$ approach 0 exponentially:

$$|\emptyset_t\,(x)| \le ce^{\lambda t}$$

For some $c > 0$.

Now suppose f is C^1 vector field (not necessarily linear) with equilibrium point $0 \in R^n$. we think of the derivative $\mathrm{D}f(0) = A$ of f at 0 as a linear vector field which approximates f near 0.

3.1.2.Sink, source and saddle: [9]

An equilibrium point x_0 of (1) is called a sink if all of the eigenvalues of the matrix $\mathrm{D}f(x_0)$ have negative real part, it is called a source if all of the eigenvalues of $\mathrm{D}f(x_0)$ have positive real parts and it is called a saddle if it is a hyperbolic equilibrium point and $\mathrm{D}f(x_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part .

In general, linear systems have one equilibrium point at the origin. Nonlinear systems may have many equilibrium points.[5]

3.2. Stability of Equilibria: [12]

An equilibrium $\breve{\tilde{x}}$ is stable if all nearby solutions stay nearby .it is asymptotically stable if all nearby solutions not only stay nearby, but also tend to $\tilde{x}$.

Theorem:

Let $\tilde{x} \in w$ be a sink of equation (1) suppose every eigenvalue of $Df(\tilde{x})$ has real part less than $-c, c > 0$. Then there is a neighborhood $U \subseteq w$ of $\tilde{x}$ such that

a) $\emptyset_t(x)$ is defined and in U for all $x \in U, t > o$.
b) There is a Euclidean norm on R^n such that
$|\emptyset_t(x) - \tilde{x}| \leq e^{-tc}|x - \tilde{x}|$
For all $x \in U, t \geq 0$.
c) For any norm on R^n, there is constant $B > 0$ such that
$|\emptyset_t\,(x) - \tilde{x}| \leq Be^{-tc}|x - \tilde{x}|$
For all $x \in U, t \geq 0$.

In particular, $\emptyset_t\,(x) \to \tilde{x} \quad as \quad t \to \infty$ for all $x \in U$.

Proof:

For convenience we assume $\tilde{x} = 0$ (If not, give R^n new coordinates

$y = x - \tilde{x}$;In y-coordinates f has an equilibrium at 0 etc.)

Put $A = Df(0)$. choose $b > 0$ so that the real parts of eigenvalues of A are less than $-b < -c$,

$\langle Ax, x\rangle \leq -b|x|^2$ For all $x \in R^n$

Since $A = Df(0)$ and $f(0) = 0$, by the definition of derivative.

$$\lim_{x\to 0} \frac{|f(x)-Ax|}{|x|} = 0$$

Therefore by Cauchy's inequality.

$$\lim_{x\to 0} \frac{\langle f(x)-Ax, x\rangle}{|x|^2} = 0$$

It follows that there exists $\delta > 0$ so small that if $|x| \leq \delta,$ then $x \in w$

And $\langle f(x), x\rangle \leq -c|x|^2$.

Put $U = \{x \in R^n ||x| \leq \delta\}$. let $x(t), 0 \leq t \leq t_0$

Be a solution curve in U, $x(t) \neq 0$ $Then$

$$\frac{d}{dt}|x| = \frac{1}{|x|}\langle x', x\rangle .$$

Hence, since $x' = f(x)$:

$$\frac{d}{dt}|x| \leq -c|x|.$$

This shows, first, that $|x(t)|$ is decreasing;

Hence $|x(t)| \in U$ for all t $\in [0, t_0]$.

Since U is compact.

$$|x(t)| \leq e^{-tc}|x(0)|$$

For all $t \geq 0$, Thus (a) and (b) are proved and (c) follows from equivalence of norms.

The phase portrait at a nonlinear sink $\tilde{x}$ looks like that of the linear part of the vector field: in a suitable norm the trajectories point inside all sufficiently small spheres about $\tilde{x}$ Fig.A. (3.1)

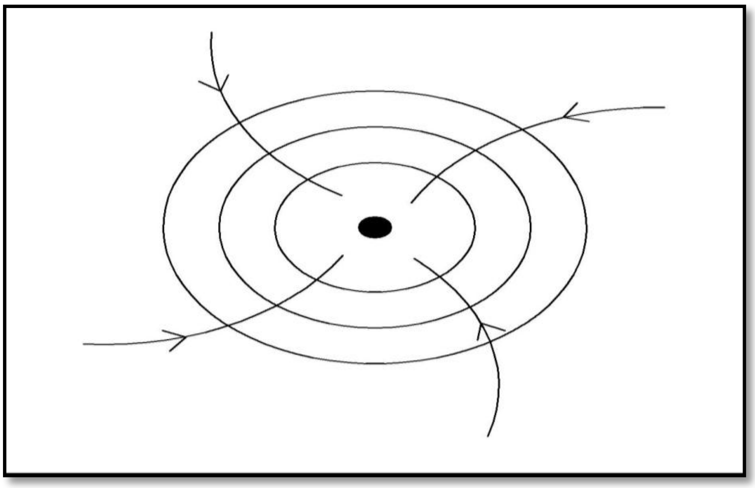


FIG .A. (3.1)
"Nonlinear sink"

3.3. Stability [12]

3.3.1.Stable equilibrium: suppose $\tilde{x} \in w$ is an equilibrium of the differential equation $x' = f(x)$ Where $f: w \to E$ is a C^1 map from an open set W of the vector space E into E. Then $\tilde{x}$ is a stable equilibrium if for every neighborhood V of $\tilde{x}$ in w there is a neighborhood V_1 of $\tilde{x}$ in Vsuch that every solution $x(t)$ with $x(0)$ in V_1 is defined and in V for all $t > 0$. (see Fig .A.(3.2))

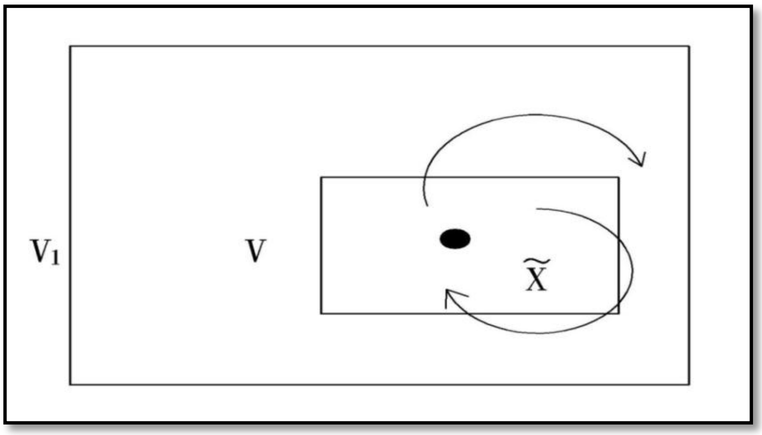


FIG .A. (3.2)
“Stability"

3.3.2. Asymptotically stable: if V_1 can be chosen so that in addition to properties described in definition 3.3.1.,$\lim_{t\to\infty} x(t) = \tilde{x}$, then $\tilde{x}$ is asymptotically stable .(see Fig B.(3.3))

An asymptotically stable equilibrium is also called an attracting equilibrium.[6]

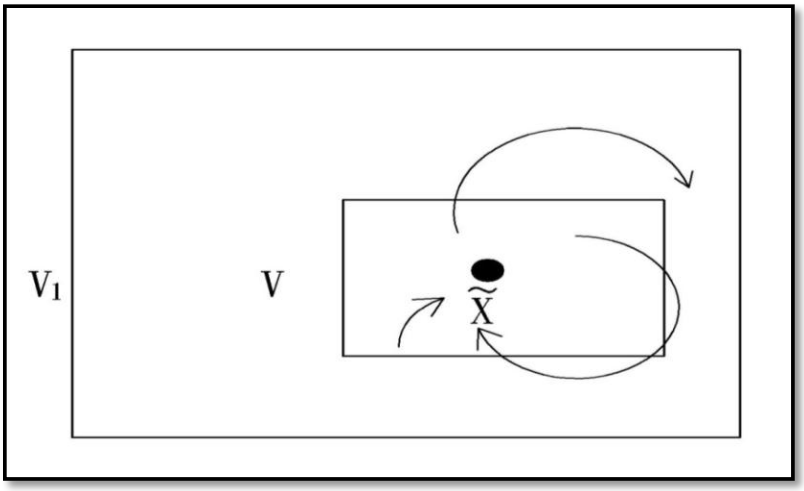


FIG.B. (3.3)
"Asymptotic stability"

3.3.3. Equilibrium unstable: An equilibrium $\tilde{x}$ that is not stable is called unstable. This means there is a neighborhood V of $\tilde{x}$ such that for every neighborhood V_1 of $\tilde{x}$ in V There is at least one solution $x(t)$ starting at $x(0) \in V_1$, which does not lie entirely in V.(See Fig .C. (3.4))

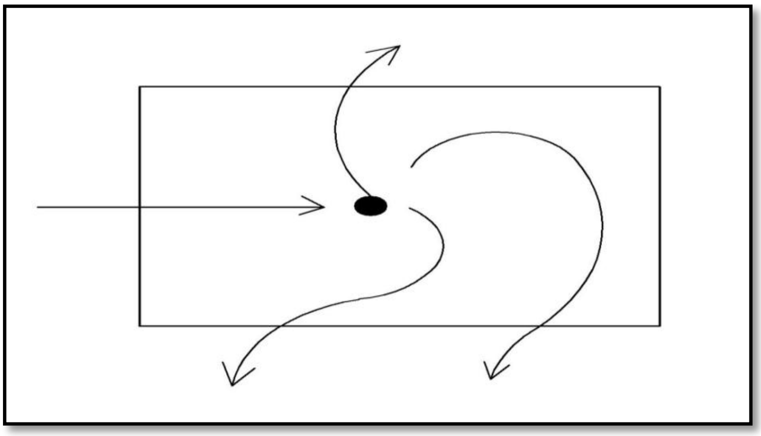


FIG.C. (3.4)
“Instability”

A sink is asymptotically stable and therefore stable. An example of an Equilibrium that is stable but not asymptotically stable is the origin in

R^2 for a linear equation

$x' = Ax$ ……………….. (2)

Where A has pure imaginary eigenvalues, The orbits are all ellipses (Fig .D. (3.5))

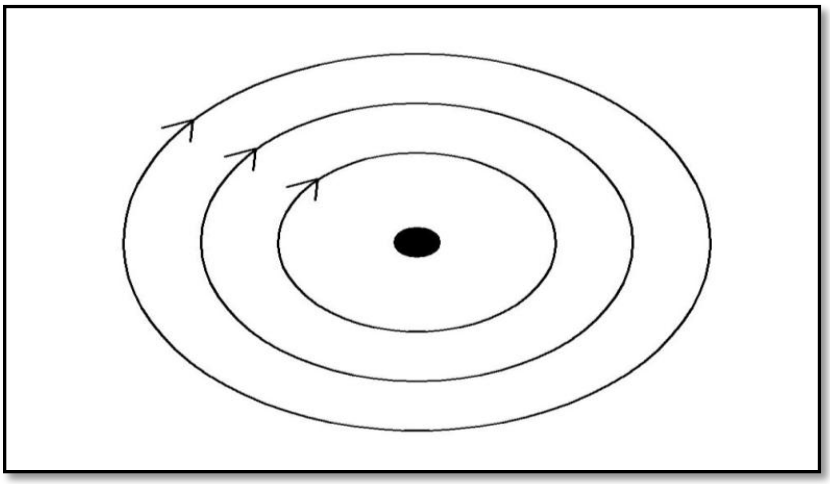


FIG.D. (3.5)
"Stable, but not asymptotically stable"

Theorem: [12]

Let $w \subseteq E$ be open and $f: w \to E$ continuously differentiable suppose $f(\tilde{x}) = 0$ and $\tilde{x}$ is a stable equilibrium point of the equation

$x' = f(x)$ Then no eigenvalue of $Df(\tilde{x})$ has positive real port.

We say that an equilibrium $\tilde{x}$ is hyperbolic if the derivative $Df(\tilde{x})$ has no eigenvalue with real part zero.

Corollary: A hyperbolic equilibrium point is either unstable or asymptotically stable.

Theorem:[3]

Let x_e be a critical point of the autonomous system $x' = f(x)$.

(a)- The critical point x_e is stable iff $f'(x_e) < 0$.

(b)- The critical point x_e is unstable iff $f'(x_e) > 0$.

Example:[7]

Find all the equilibrium points of the nonlinear system

$x' = x(3 - x - 2y),$

$y' = y(2 - x - y),$

And determine their stability.

The equilibrium points are determined by solving

$f(x, y) = x(3 - x - 2y) = 0,$

$g(x, y) = y(2 - x - y) = 0.$

There are four equilibrium points $(\tilde{x}, \tilde{y})$:$(0,0), (0,2), (3,0)$ and $(1,1)$. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Stability of the equilibrium points may be considered in turn. With J_* the Jacobian matrix evaluated at the equilibrium point, We have at $(\tilde{x}, \tilde{y}) = (0,0)$: $J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$

The eigenvalues J_* are $\lambda_{1,2} = 3,2$ So that the equilibrium point $(0,0)$ is unstable node.

At $(\tilde{x}, \tilde{y}) = (0,2)$: $J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}.$

The eigenvalues of J_* are $\lambda_{1,2} = -1, -2$ so that the equilibrium point

(0, 2) is stable node.

At $(\tilde{x}, \tilde{y}) = (3,0)$: $J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$.

The eigenvalues of J_* are $\lambda_{1,2} = -3, -1$ so that the equilibrium point (3,0) is also a stable node. Finally,

$(\tilde{x}, \tilde{y}) = (1,1)$: $J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$.

The eigenvalues of J_* are $\lambda_{1,2} = -1 + \sqrt{2}, -1 - \sqrt{2}$.

Since one eigenvalue is negative and the other positive the equilibrium point (1,1) is an unstable saddle point.

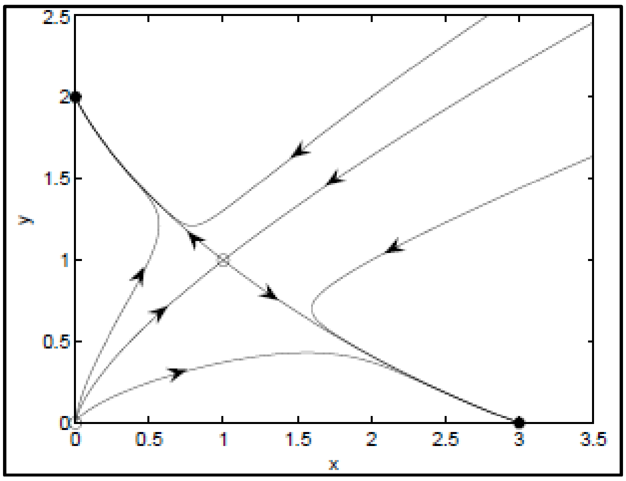


FIG (3.6):
"Phase space plot for two-dimensional nonlinear system"

3.4. Theorem of Lyapunov's second method: [5]

Very important note: This theorem provides a sufficient condition, not a necessary condition

Consider the system: $\dot{x} = f(x,t)$, $f(0,t) = 0 \ \forall t$

If scalar function is defined such that:

i. $V(0,t) = 0$

ii. $V(x,t)$Is positive definite. i.e. There exists a continuous

Non-decreasing scalar function $\alpha(x)$ such that $\alpha(0) = 0$ and

$\forall\, x \neq 0, \quad 0 < \alpha(\|x\|) < V(x,t)$

iii. $\dot{V}(x,t)$ is negative definite , that is $\dot{V}(x,t) \leq -\gamma(\|x\|) < 0$ where γ is a continuous non- decreasing scalar function such that $\gamma(0) = 0$

iv. $V \leq \beta(\|x\|)$ where β is a continuous non-decreasing function and $\beta(0) = 0$ i.e. V is decrescent ,i.e. the Lyapunov function is upper bounded

v. V is radically unbounded, that is $\alpha(\|x\|) \to \infty$ as$\|x\| \to \infty$.

Then the equilibrium point is uniformly asymptotically stable in the large and $V(x,t)$ is called a Lyapunov function.

3.5. Stability in the sense of Lyapunov:[5]

Assume $x_e = 0$

3.5.1. Stable:

The equilibrium $x = 0$ is stable iff $\forall\ \epsilon > 0, \forall\, t_0 \geq 0,$

$\exists\, \delta > 0\ such\ that\ \ \|x(t_0)\|_2 < \delta \to \|x(t)\|_2 < \epsilon\ , \forall\, t \geq t_0\,.$

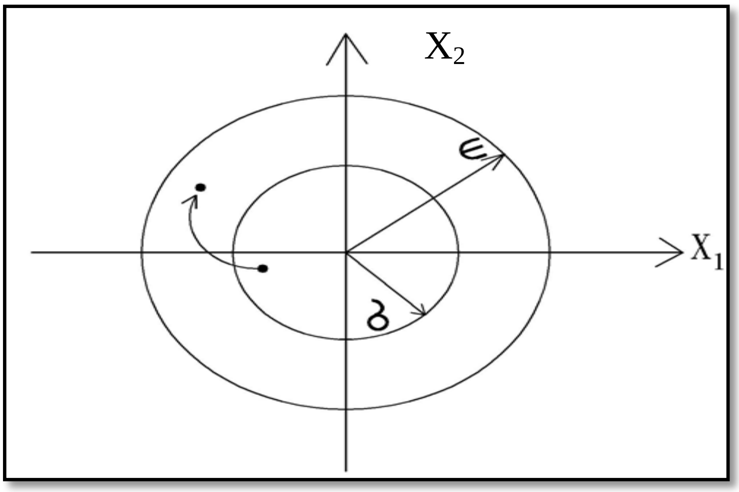


FIG.E. (3.7)

" that is, if I start within δ, I Stay within ϵ . In general, I give you an ϵ ,you give me the corresponding δ, Things remain bounded"

3.5.2. Asymptotically stable: [5]

The equilibrium $x = 0$ is asymptotically stable if:

i. $x = 0$ is a stable equilibrium

ii. $\forall\, t_0 \geq 0 , \exists\, \delta(t_0) \|x(t_0)\|_2 < \delta \rightarrow \lim_{t\rightarrow+\infty}|x(t)| = 0$

3.5.3. Uniformly stable: [5]

The equilibrium $x = 0$ is uniformly stable if:

i. $x = 0$ is a stable equilibrium

ii. $\delta(\epsilon, t_0) = \delta(\epsilon)$

These conditions refer to stability in the sense of lyapunov.

3.6. Method of Lyapunov:

3.6.1. The second method of Lyapunov. [5]

(a)- Originally proposed by lyapunov(around1890) to investigate stability in the small (local stability)

(b) - Later extended to cover globlal stability

(C) - Stability can be determined without explicitly solving the system solution.

(d)- Generalization of an "energy" argument for non –energetic

Systems(e.g.forecasting the stock market,etc….)

(e)- Difficulty is finding the Lyapunov function.

3.6.2. Positive Definite functions: [5]

A function v is positive definite if $v(x) > 0 \ \forall x \neq 0$ and $v(0) = 0$

Example: For example, suppose $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. note that x is a vector, while $V(x)$ is a Scalar function.

Suppose $V_1(x) = {x_1}^2 + {x_2}^2$ .isV_1 positive definite?

(i)- Does $V_1 = 0 \ \rightarrow x_1 = 0, x_2 = 0$?

(ii)-Do we have: $V_1(x) > 0$, $\forall$ $(x_1, x_2) \in R^+$.

(iii)- Suppose $V_2(x) \equiv {x_1}^2$. Is V_2 positive definite?

V_2 Is positive but not definite for example.$x_1 = 0$ And

$x_2 = 10 \ \rightarrow V_2 = 0$This property is called positive semi definiteness.

3.6.3. Intuition for Lyapunov's theorem consider a second order system, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:[5]

Let $V(x_1, x_2)$ be a positive definite function

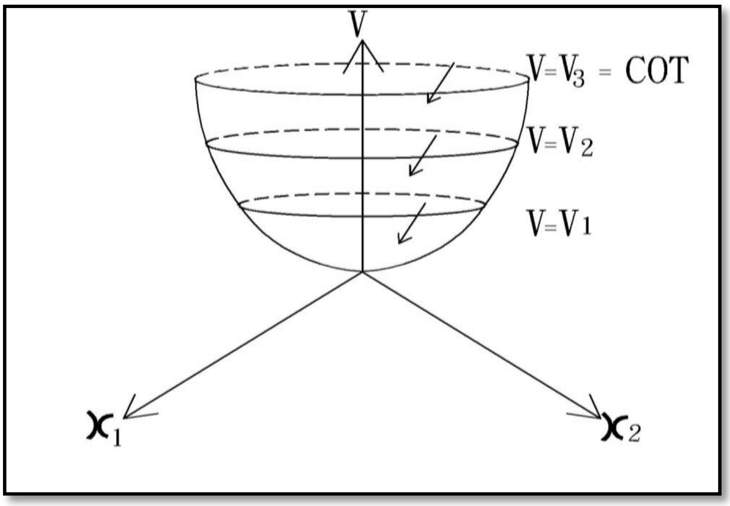


FIG.F. (3.8)

If $V(x_1, x_2)$ always decreases, then it must reach zero eventually. That is. For a stable system all trajectories must move so that the values of V are decreasing. this is similar to the energy argument for stability of mechanical systems.

To relate V to the system dynamics we compute $\dot{V}$.

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x_i} \dot{x}_i.$$

$$= \frac{\partial V}{\partial t} + \nabla V^T . f$$

The second term of this expression relates V to the vehicle dynamics.

In our notation we have assumed:

$$f(x) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad and \quad \nabla V^T = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]$$

We need $\dot{V}$ to be negative definite for our intuitive condition to be true.

$\dot{V}$ is the rate of change of the scalar field V along the flow of the vector Field f.

$\dot{V} = \frac{\partial V}{\partial X} . \frac{\partial X}{\partial t} = l_f\, V.$ Lie derivative of V (if V does not depend on t).

Example:[1] consider a unit mass suspended from a fixed support by a spring, Z being the displacement from the equilibrium. If first the spring is assumed to obey Hooke's law, then the equation of motion is

$$z'' + kz = 0 \ldots\ldots\ldots\ldots\ldots.(3.1)$$

Where k is the spring constant.

Taking $x_1 = z$, $x_2 = z'$,(3.1) becomes

$$x_1' = x_2$$

$$x_2' = -kx_1,$$

Since the system is conservative, the total energy

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}x_2^2$$

Is a lyapunov function and it is easy to see that

$$V' = kx_1x_2 - kx_2x_1 = 0$$

So by lyapunov's second Theorem the origin is stable.

3.7. Poincare-Bendixson Theorem:[14]

This remarkable result, which only holds in R^2, is very useful for proving the existence of periodic orbits.

3.7.1. Limit sets:[8]

A point $\breve{x}$ is called a positive limit point of the orbit $\gamma(x_0)$ if there exists an increasing sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that

$\breve{x} = \lim_{n\to\infty} x(t_n)$. A negative limit point of $\gamma(x_0)$ is defined similarly.

We denote by $\omega(\gamma)$ the set of all positive limit points for the orbit γ. It is called the ω-limit set of γ. Similarly we denote $\alpha(\gamma)$ the set of all negative limit points for the orbit γ (the α-limit set of γ).

3.7.2. Theorem (Poincare'-Bendixson*).[14]

Consider the system $x' = f(x)$, $x \in R^2$, and suppose that f is continuously differentiable. If the forward orbit $O^+(x)$ remains in a compact (closed and bounded) set containing no fixed points then $\omega(x)$ contains a periodic orbit.

3.7.3Theorem (Bendixson-Dulac)[8]**

Suppose f and g are defined in a simply connected region R. If the expression $\frac{\partial \varphi f}{\partial x} + \frac{\partial \varphi g}{\partial y}$ is either always positive or always negative on R (except perhaps a small set such as on isolated points or curves) then the system $x' = f(x,y)$, $y' = g(x,y)$, Where the functions f and g have continuous derivatives. Has no closed trajectory inside R.

3.7.4. Dulac's Criteria[15]

Consider the autonomous planar system $x' = f(x,y)$,$y' = g(x,y)$ and a continuously differentiable function φ defind on an annular region R contained in some open set. If $\frac{\partial \varphi f}{\partial x} + \frac{\partial \varphi g}{\partial y}$ does not change sign in R, then there is at most one limit cycle contained entirely in R.

*(Ivav Otto Bendixson(1861-1935) was a Swedish mathematician).

**(Henri Dulac(1870-1955)was a French mathematician).

3.7.5. Direction field and nullclines [13]

The direction field of the differential equation is the function $(P, Q): R^2 \to R^2$ that associates a two trajectory to know if the vectors of the direction field point up or down, and to the left or to the right.

In order to see this the nullclines

$N_1 = \{p \in R^2: P(p) = 0\}$, $N_2 = \{p \in R^2: Q(p) = 0\}$ can help.

The nullcline N_1 divides the phase plane into two parts (these are not necessarily connected sets).In one of these, in which $p > 0$, trajectories move to the right (since $x' > 0$ there), in the other part, in which $p < 0$, trajectories move to the left (since $x' < 0$ there). Similarly, the nullcline N_2 divides the phase plane into two parts (these are not necessarily connected sets). In one of these, in which $Q > 0$, trajectories move up (since $y' > 0$ there), in the other part, in which $Q < 0$, trajectories move down (since $y' < 0$ there).

Thus the nullclines N_1 and N_2 divide the phase plane into four parts, in each of them it can be decided if the trajectories move up or down, and to the left or to the right. (We use the terminology that "the trajectory moves" , in fact the point $\varphi(t, p)$ moves along the trajectory as t is varied.) The intersection points of the nullclines are the equilibria where both P and Q are equal to zero.

Example: - we have a system of linear differential equation

$x' = y + x^3 - x$

$y' = -x - y + y^3$,

To find the curves of this system let $x' = y' = 0$ such that

$y = -x^3 + x$

$x = y^3 - y,$

By substitutions by the value of $y = (-x^3 + x)$ in the second Differential equation we have.

$$-x - (-x^3 + x) + (-x^3 + x)^3 = 0 = x^9 - 3x^7 + 3x^5 - 2x^3 + 2x$$

$x(x^8 - 3x^6 + 3x^4 - 2x^2 + 2) = 0$

$x(poly \ \ of \ \ degree \ \ eight\,) = 0$

at$x = 0 \rightarrow 0[poly \ \ of \ \ degree \ \ eight] = 0$

$\rightarrow y = o \ \ then \ \ (0,0) \ is \ an \ equilibrium \ \ point.$

The eight degree polynomial has no real roots and the origin is the only equilibrium point of the system

Now to find the Jacobian[The matrix of partial dirvitri]

$$J = \begin{bmatrix} -1 + 3x^2 & 1 \\ -1 & -1 + 3y^2 \end{bmatrix} \rightarrow at\ (0,0) \rightarrow\rightarrow J(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

Now we have to find the eigenvalue of $J(0,0)$

det$[A - I\lambda] = 0 \rightarrow \begin{bmatrix} -1 - \lambda & 1 \\ -1 & -1 - \lambda \end{bmatrix} \rightarrow (-1 - \lambda)^2$+1=0

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

All the eigenvalues have negative real parts i.e. $Re\{\lambda_1\} = -1$, $Re\{\lambda_2\} = -1 \rightarrow \lambda_1, \lambda_2 < 0$

This case is a sink.

To find graphs of $y = -x^3 + x$, $x = y^3 - y$

Let choose same points of x and y

Hence.

x	$y = -x^3 + x$	y	$x = y^3 - y$
1	0	1	0
2	-6	2	6
3	-24	3	24
4	-60	4	60
5	-120	5	120
0	0	0	0
-1	0	-1	0
-2	+6	-2	-6
-3	24	-3	-24
-4	60	-4	-60
-5	120	-5	-120

At the graph "next page”$(\lambda + 1)^2 + 1 = 0$ the origin is a stable spiral point for the system that any solution(closed enough) in the phase plane will spiral toward the origin ,since the critical point is an unique point .

Now to find out if all the solution of the system have trajectories with spiral towards the origin?

Now we consider the autonomous system.

$$\frac{dx}{dt} = F(x,y) \quad , \quad \frac{dy}{dt} = G(x,y)$$

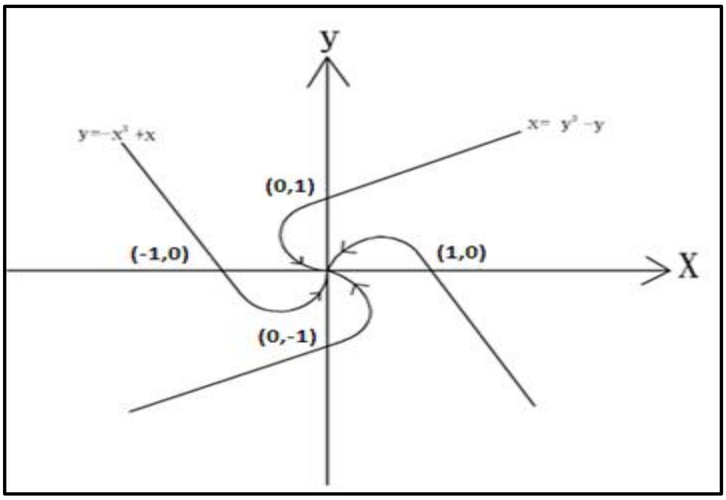


FIG.G. (3.9)

By the theorem : let F,G have continuous is first partial derivative in a simply connected domain D in xy plane .If $F_x + G_y$ has the same sign through D → ∃ no particles solutions in D lying entirely in D.

Of $F_x + G_x$changes sign in D no conclusion is possible.

$F_x = -1 + 3x^2, G_y = -1 + 3y^2$

$\rightarrow F_x + G_y = -1 + 3x^2 - 1 + 3y^2$

$= -2 + 3(x^2 + y)^2 \rightarrow \quad ((no \;\; particles \;\; solution\,))$

Let $\quad r^2 = x^2 + y^{\,2}, \;\; x = rcos\theta \,, y = rsin\theta$

$2rr' = 2xx' + 2yy'$

$\rightarrow rr' = xx' + yy'$

$=x(-x + y + x^3) + y(-x - y + y^3)$

$= -x^2 + xy + x^4 + (-xy) - y^2 + y^4$

$= -(x^2 + y^2) + (x^4 + y^4)$

$= -r^2 + (r^4cos^4\theta + \; r^4sin^4\theta)$

$= -r^2 + r^4(cos^4\theta + sin^4\theta)$

Let $w = r^2 = (x^2 + y^2)$

$$\frac{dw}{dt} = -2w + 2w^2 - 4x^2y^2$$

$\rightarrow -2w + 2w^2 \leq \frac{dw}{dt}$

$$\rightarrow \frac{dw}{dt} \geq -2w + 2w^2 = -2w(1-w)$$

By applying Poincare –Bendixson theorem:-since the system has the origin as a critical point. It must be excluded

Now for $\frac{dw}{dt} \geq 0 \rightarrow -2w(1-w) > 0 \rightarrow +2w(w-1) > 0$

$\therefore w(w-1) > 0 \quad \rightarrow w > 0, w > 1$

$\rightarrow r^2 > 1, \ r < -1, \ r > 1$

For $\frac{dw}{dt} < 0 \rightarrow -2w + w^2 < 0 \rightarrow w(w-2) < 0$

$\rightarrow \quad 0 < w < 2 \quad \rightarrow \quad 0 < r^2 < 2 \quad \rightarrow \quad r < \sqrt{2}$.

For $r > 1$ in this case $\frac{dw}{dt} > 0 \rightarrow$ r increasing and the solution spiral out, in case $r < \sqrt{2}$ $\rightarrow r$ decreasing by Poincare –Bendixson theorem any solution of this system starting in the origin $r < \sqrt{2}$ must stay in the origin $\rightarrow$ by the same theorem there is a periodic solution of the system whose trajectory is closed curve in the region $r < \sqrt{2}$.

Now we have the following equation.

$-r^2 + r^4(cos^4\theta + sin^4\theta)$=0

To show that the fulol pointed in an $r = 1$ and point out on $r = \sqrt{2}$

Since $\quad cos^4\theta + sin^4\theta \leq 1$

At $\quad r = 1 \ \rightarrow \ -1 + 1(k) \leq 1 \quad , \quad k \leq 1$

$\rightarrow on \quad r = 1 \qquad the \ \ fulol \ \ point\,.$

In at $\ r = \sqrt{2} \rightarrow \ rr' = -2 + 4(cos^4\theta + sin^4\theta)$

$$= 2(-1 + 2(cos^4\theta + sin^4\theta))$$

$$= -2(1 - 2(cos^4\theta + sin^4\theta))$$

To show $\quad cos^4\theta + sin^4\theta \geq \frac{1}{2}$

At the term $[1 - 2(cos^4\theta + sin^4\theta)]$

$1 - 2cos^2\theta sin^2\theta = 1 - 2(cos\theta sin\theta)^2$

$$= 1 - \frac{1}{2}sin^2 2\theta \geq \frac{1}{2}$$

$tan\theta = \frac{y}{x} \quad \rightarrow \qquad \theta = tan^{-1}\frac{y}{x}$

$\theta' = \frac{xy' - yx'}{x^2 + y^2}$ Substitute by $x', y', r^2 = x^2 + y^2$ we have

$$\theta' = \frac{1}{r^2}[x(-x - y + y^3) - y(-x + x^3 + y)]$$

$$= \frac{1}{r^2}[-x^2 - xy + xy^3 + xy - x^3y - y^2]$$

$$= \frac{1}{r^2}\left[-(x^2 + y^2) + xy\,[x^2 - y^2]\right]$$

$$= \frac{1}{r^2}[-r^2 + r^4 cos\theta sin\theta(cos^2\theta - sin^2\theta)]$$

$$= \left[-1 + r^2\frac{1}{4}\sin 4\theta\right]$$

Suppose $\quad -1 + \frac{r^2}{4}\sin 4\theta = 0 \rightarrow \ \frac{r^2}{4}\sin 4\theta = 1$

$\rightarrow\ \ 1 \leq \frac{1}{2}|\sin 4\theta|\ \leq \frac{1}{2}$

Contradiction $\rightarrow \theta' \neq 0$

Then at $r = \sqrt{2}$ the fulod pointed out

“Because $-2(1-2(cos^4\theta - sin^4\theta)) \geq 0$ "

By Poinear –Bendexson→ ∃ at least one, limit cycle in the annulus

Now we will show that the limit cycle in unique

$d\,iv(\underline{X}) = 3x^2 - 1 + 3y^2 - 1 = -2 + 3(x^2 + y^2)$

At

$d\,iv(\underline{X}) = 0 \rightarrow\ -2 + 3(x^2 + y^2) = 0$

$x^2 + y^2 = \frac{2}{3}\ \ \rightarrow \frac{2}{3} < 1$. An the *annulus* $\rightarrow\ \ d\,iv(\underline{X}) < 0$

∴ At must one limit cycle.

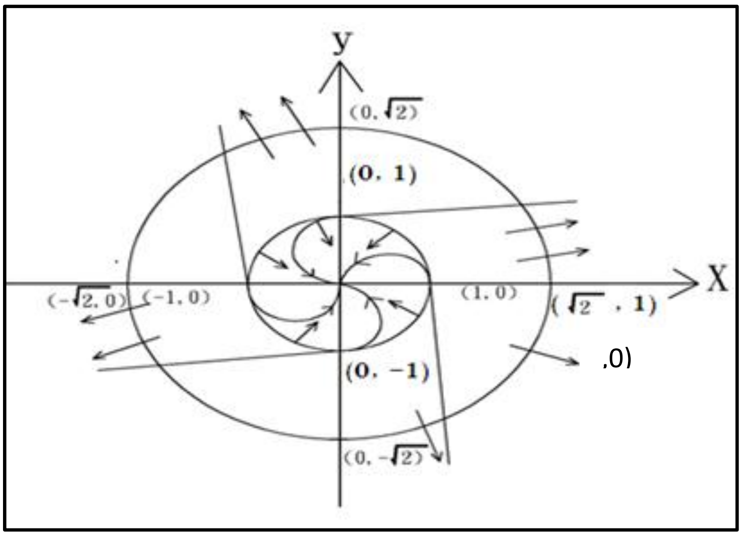


FIG.H. (3.10)

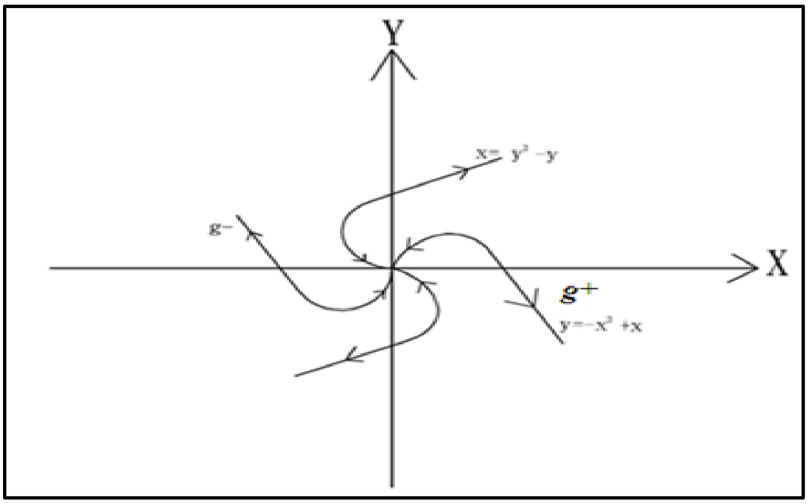


FIG.I. (3.11)

Chapter four

4.1. Application: [4]

One variable system: syntheses and degradation of one compound consider one compound, X, synthetized with a rate constant k_s and degraded at rate k_d :

$$\xrightarrow{k_s} X \xrightarrow{k_d}$$

Evolution equation: the evolution equation of the concentration X is:

$$\frac{dX}{dt} = F(X) = k_s - k_d X \ldots\ldots\ldots\ldots\ldots\ldots.(1')$$

Note that this is a linear system steady state:

$$\frac{dX}{dt} = 0 \rightarrow X_s = {}^{k_s}/_{k_d} \ldots\ldots\ldots\ldots\ldots.(2')$$

Linear stability analysis: the evolution of the perturbation x is given by:

$$\frac{dX}{dt} = (\frac{dF}{dX})_{X_s} X = -k_d X \ldots\ldots\ldots\ldots\ldots(3')$$

The solution of this differential equation is:

$$X = e^{-kdt} \ldots\ldots\ldots\ldots\ldots.(4')$$

This means that the perturbation will be damped, in an exponential way, with respect to time. The steady state is therefore stable.

Analytical solution: note that this system is relatively simple and the solution of equation (1′) can be found analytically (2):

$$X = X_0 e^{-k_d t} + \frac{k_s}{k_d}(1 - e^{-k_d t}) \ldots\ldots\ldots\ldots\ldots\ldots.(5')$$

Where $X_0 = X(0)$,

We can check that

$$\lim_{t\to\infty} X(t) = \frac{k_s}{k_d} = X_s$$

(2)Recall that: $\int \frac{1}{p+qx}\, dx = \frac{1}{q} ln(p+qx) + c$

4.2. one-variable system: logistic synthesis and linear degradation:

Consider now one compound, X, synthetised with a logistic synthesis rate and linearly

Degraded: $\frac{dX}{dt} = F(X) = k_s \text{x}\left(1 - \frac{X}{N}\right) - k_d X \ldots\ldots\ldots\ldots\ldots\ldots (7')$

Steady states:

$X_{s_1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8')$

$X_{s_2} = N\frac{k_s - k_d}{k_s} (exists\ if\ k_s > k_d) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9')$

Linear stability analysis:

$F = k_s X - \frac{k_s}{N} X^2 - k_d X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10')$

$F' = k_s - 2\frac{k_s}{N} X - k_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11')$

$F'(X_{s_1}) = k_s - k_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12')$

$F'(X_{s_2}) = k_d - k_s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13')$

The following table summarizes the stability conditions for each steady state:

	$k_s < k_d$	$k_s > k_d$
x_{s1}	Stable	Unstable
x_{s2}	$\nexists$	Stable

Graphical analysis:

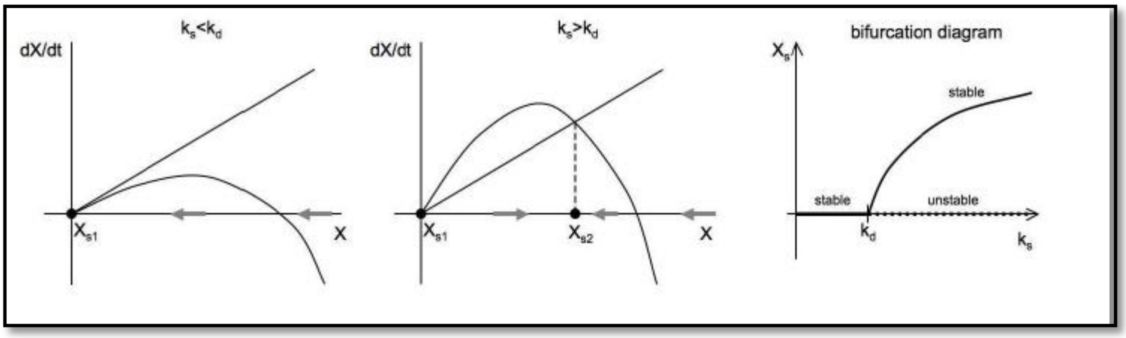


Figure (4.1)

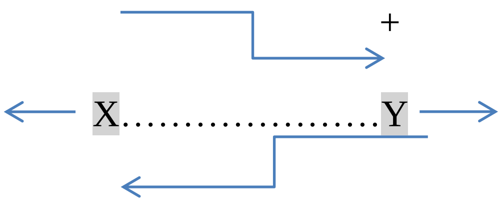
"Graphical analysis and bifurcation diagram (transcrtical bifurcation)"

Analytical solution: note that Eq. $(7')$ can be rewritten as a standard logistic equation and can be solved analytically.

4.3. Two- variable system:

two mutually activated compound (bestiality)

Consider a system involving two compounds which activates each other according a Hilltypekinetics :



$$\begin{cases} \frac{dX}{dt} = v_1 \frac{Y^n}{\theta^n + Y^n} - k_1 X \\ \frac{dY}{dt} = v_2 \frac{X^n}{\theta^n + X^n} - k_2 Y \end{cases} \ldots\ldots\ldots (14')$$

Evolution equation

To simplify, we assume that $v_1 = v_2 = 1$and$k_1 = k_2 = 1$.

The equation become:

$$\begin{cases} \frac{dX}{dt} = \frac{Y^n}{\theta^n + Y^n} - X \\ \frac{dY}{dt} = \frac{X^n}{\theta^n + X^n} - Y \end{cases} \ldots\ldots\ldots\ldots (15')$$

Steady state:

Let's $\theta = 1/2$. We can check that $X_s = Y_s = \frac{1}{2}$is a steady state .we also see that $X_s = Y_s = 0$ is also a steady state.

Note that those two state may not be the only solutions.

Representation in the phase plane:

We can represent the nullclines in the phase space .the unllclines are defined by:

$$\frac{dX}{dt} = 0 \to \frac{Y^n}{\theta^n + Y^n} = X \ldots\ldots\ldots\ldots\ldots\ldots (16')$$

$$\frac{dY}{dt} = 0 \to \frac{X^n}{\theta^n + X^n} = Y \ldots\ldots\ldots\ldots\ldots\ldots.. (17')$$

Linear stability analysis:

The jacobian matrix for the steady state $\left(\frac{1}{2}, \frac{1}{2}\right)$ is:

$$J = \begin{pmatrix} \frac{\partial X'}{\partial X} & \frac{\partial X'}{\partial Y} \\ \frac{\partial Y'}{\partial X} & \frac{\partial Y'}{\partial Y} \end{pmatrix} \dots \dots \dots \dots \dots \dots \dots (18')$$

Recall that:

$$\frac{\partial\left(\frac{X^n}{\theta^n+X^n}\right)}{\partial X} = \frac{(X^n)'(\theta^n+X^n)-(X^n)(\theta^n+X^n)'}{(\theta^n+X^n)^2}$$

$$= \frac{n\theta^n X^{n-1}}{(\theta^n+X^n)^2} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (19')$$

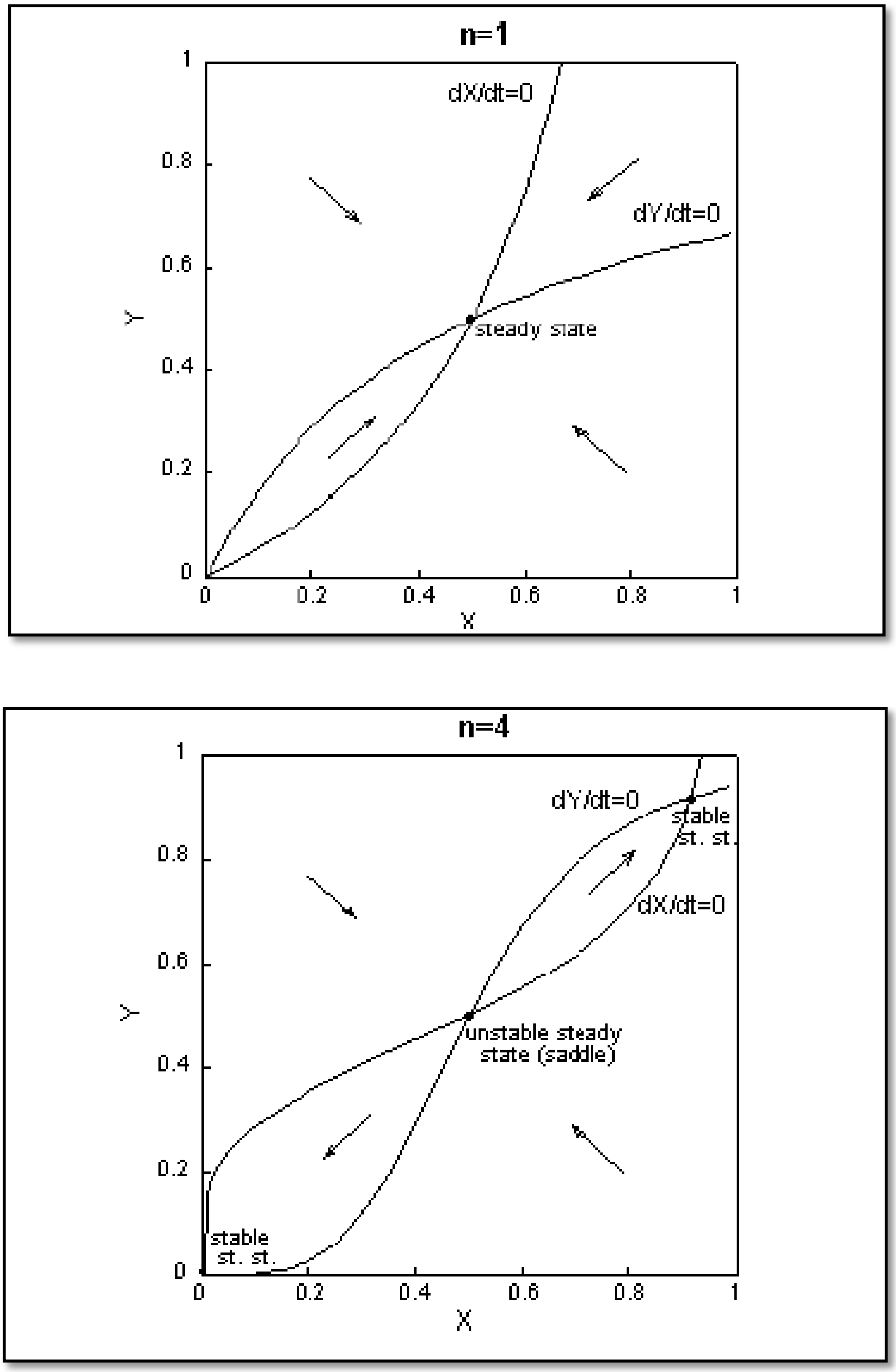


Figure (4.2)

"Phase space and nullclines corresponding to the system (15′) for n=1 (upper panel) and n=4 (bottom panel)."

If $\theta = 1/2$:

$$\frac{\partial\left(\frac{X^n}{\theta^n+X^n}\right)}{\partial X} = \frac{n\frac{1}{2^n}X^{n-1}}{\left(\frac{1}{2^n}+X^n\right)^2} \ldots\ldots\ldots\ldots\ldots\ldots..(20')$$

At the steady state, $X_s = 1/2$:

$$\frac{\partial\left(\frac{X^n}{\theta^n+X^n}\right)}{\partial X} = \frac{n\frac{1}{2^n}\frac{1}{2^{n-1}}}{\left(\frac{1}{2^n}+\frac{1}{2^n}\right)^2} = \frac{n}{2} \ldots\ldots\ldots\ldots\ldots..(21')$$

The trace T and the determinant Δ of the Jacobian matrix are:

$T = -2$ And $\Delta = 1 - {n^2}/{4}$ and $T^2 - 4\Delta = 4 - 4\left(1 - {n^2}/{4}\right) = n^2$

We can see that trace T is always negative and that $T^2 - 4\Delta$ is always positive .the sign of Δ depends on n:

(I)- If $n < 2$, we have $\Delta > 0$ → the steady state (1/2 ,1/2) is thus a stable node .

(II)- If $n > 2$, we have $\Delta < 0$ → the steady state (1/2, 1/2) is thus an (unstable) saddle point.

4.4Two–variable system: Brusselator (limit–cycle oscillations)

Consider the following system of 4 chemical reactions:

$A \xrightarrow{k_1} X$

$B + X \xrightarrow{k_2} Y + C$

$2X + Y \xrightarrow{k_3} 3X$

$X \xrightarrow{k_4} D$

Evolution equations

The concentration of *A* and *B* are supposed to be constant and noted a and *b*, respectively.

The evolution equations of the concentrations X and Y are:

$$\begin{cases} \dfrac{dX}{dt} = k_1 a - k_2 bX + k_3 X^2 Y - X \\ \dfrac{dY}{dt} = \qquad k_2 bX - k_3 X^2 Y \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(22')$$

To simplify, we will consider that

$k_1 = k_2 = k_3 = k_4 = 1$ The system then reduced to:

$$\begin{cases} \dfrac{dX}{dt} = a - bX + X^2 Y - X \\ \dfrac{dY}{dt} = bX - X^2 Y \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(23')$$

Steady state:

The steady state of the system is:

$$\begin{cases} X_s = a \\ Y_s = b/a \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots..(24')$$

Linear stability analysis:

The Jacobian matrix is:

$$J = \begin{pmatrix} \dfrac{\partial X'}{\partial X} & \dfrac{\partial X'}{\partial Y} \\ \dfrac{\partial Y'}{\partial X} & \dfrac{\partial Y'}{\partial Y} \end{pmatrix} = \begin{pmatrix} -b + 2XY - 1 & X^2 \\ b - 2XY & -X^2 \end{pmatrix} \ldots\ldots..(25')$$

At the steady state $X = X_s = a$ and $Y = Y_S$ = b/a and thus:

$$J = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (26')$$

The trace T and the determinant Δ of the Jacobian matrix are:

$T = b - 1 - a^2$ And $\Delta = a^2$

The characteristic equation is:

$$\omega^2 - T\omega + \Delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27')$$

We have to study the sign of $T^2 - 4\Delta$

$$T^2 - 4\Delta = (b - 1 - a^2)^2 - 4a^2$$

$$= (b - 1 - a^2 - 2a)(b - 1 - a^2 + 2a)$$

$$= (b - (a+1)^2)(b - (a-1)^2) \ldots\ldots (56)$$

The determinant Δ is always positive.

The trace T is positive if $b > a^2 + 1$ and negative otherwise.

$T^2 - 4\Delta$ is negative if $(a-1)^2 < b < (a+1)^2$ and positive otherwise.

The following table summarizes the different possible behaviors as a

Function of the parameters a and b:

B		$(a-1)^2$		a^2+1		$(a+1)^2$	
T	-	-	-	0	+	+	+
Δ	+	+	+	+	**+**	+	+
$T^2 - 4\Delta$	+	0	-	-	-	0	+
Type of steady state	Stable node		Stable focus		Unstable focus		Unstable node

When parameter b increases, the steady state turns from a stable node to a stable focus, then, it lost its stability (the system then evolves towards a limit cycle) and the steady state turns from an n unstable focus to an unstable node).

These be haviours are shown in figure (4.3) (results obtained by numerical integration of the differential equations):

(I)- For a=2 and b=0.5: the system evolves towards a steady state (stable node)

(II)- For a=2 and b=4: the system evolves towards a steady state (stable focus)

(III)- For a=2 and b=6: the system leaves its steady state (unstable focus) to reach a limit cycle (sustained oscillations)

(IV)- For a=2 and b=12: the system leaves its steady state (unstable node) to reach a limit cycle (sustained oscillations)

The stability diagram showing the stable and unstable regions as a function of a and b is give in figure (4.4). The solid curve, satisfying $b = a^2 + 1$ delimits the stability region.

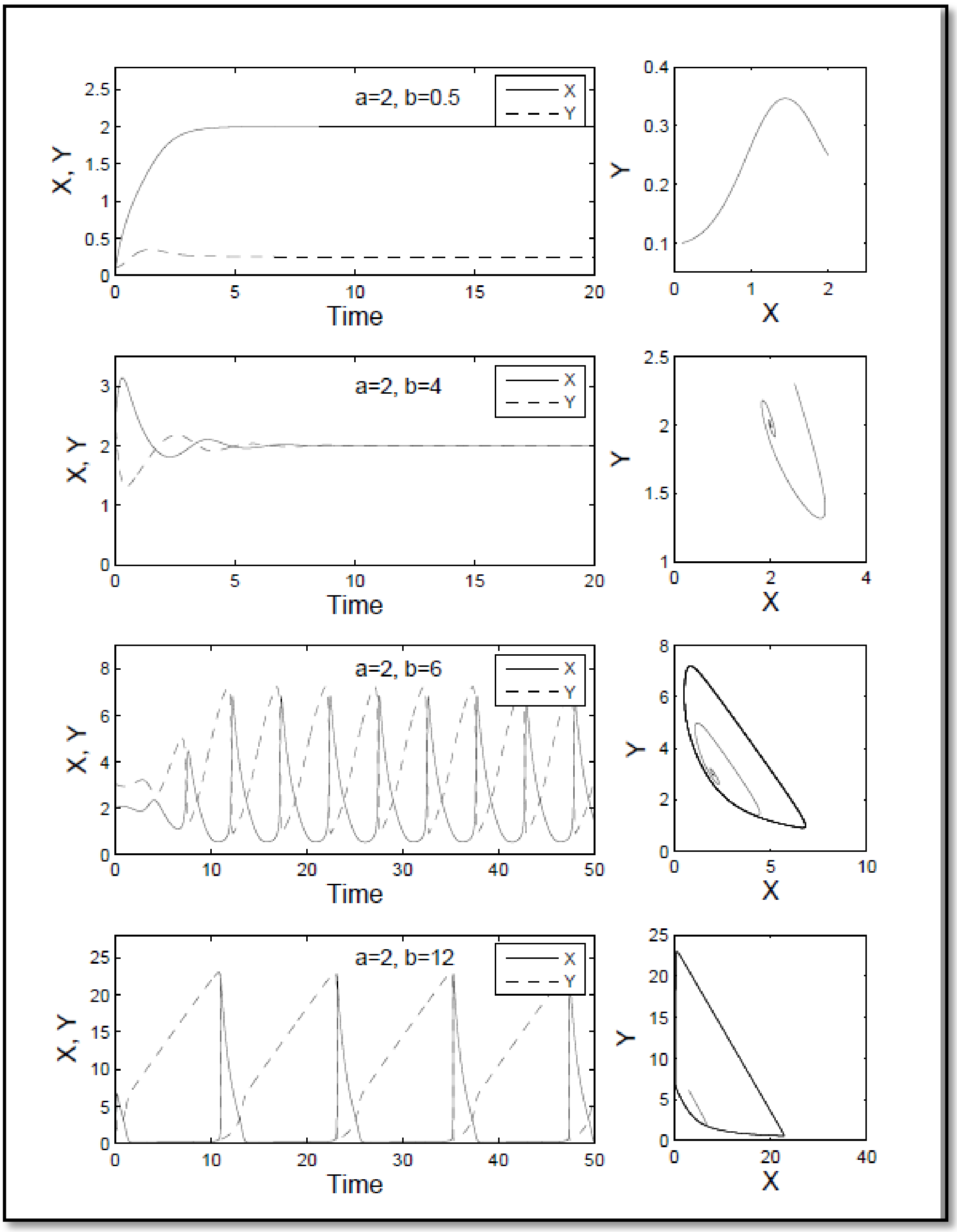
The dotted curve corresponding to $b = (a+1)^2 \; and \; b = (a-1)^2$

Separate the node from the focus in the stable and unstable regions respectively.

Figure (4.5) is bifurcation diagram:

It shows how the steady state of X changes as a function of the parameter b(for a fixed to 2).

Figure (4.6) shows how the period varies in the oscillatory domain .These diagrams have been obtained numerically.



(Figure 4.3)

"Different kind of behavior obtained for the brusselator model with different parameter (a and b) values. Left panels: time evolution."

Right panels: phase space**.**

Note that the period at the bifurcation can be calculated. We know that the frequency is given by the imaginary part of the eigenvalues .at the Bifurcation point, T=0 and the eigenvalues are $\omega_1 = \omega_2 = \sqrt{\Delta} = a.$

The period is therefore equal to $\frac{2\pi}{a}$ for a=2, as illustrated here, the period is thus egal to $\pi \approx 3.14$.

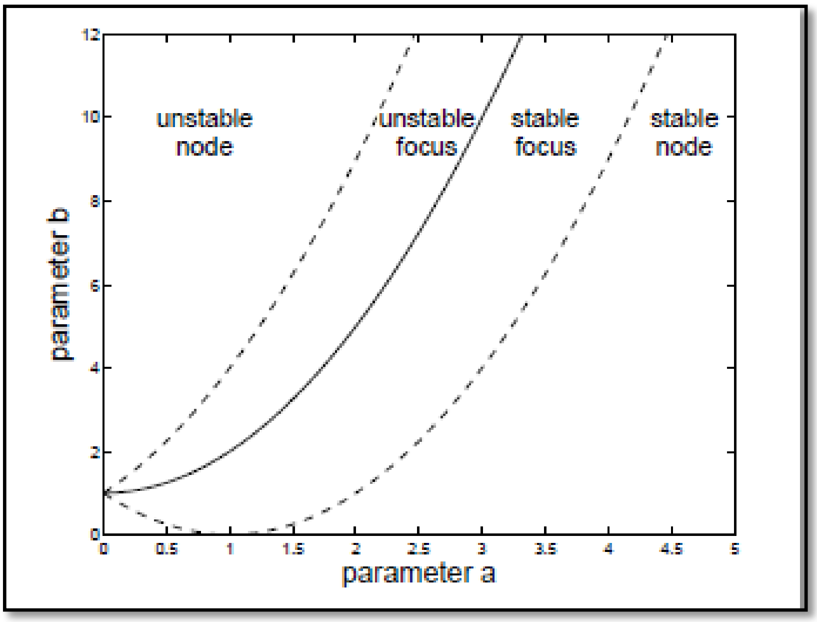


Figure (4.4)
"Stability diagram for the Brusselator."

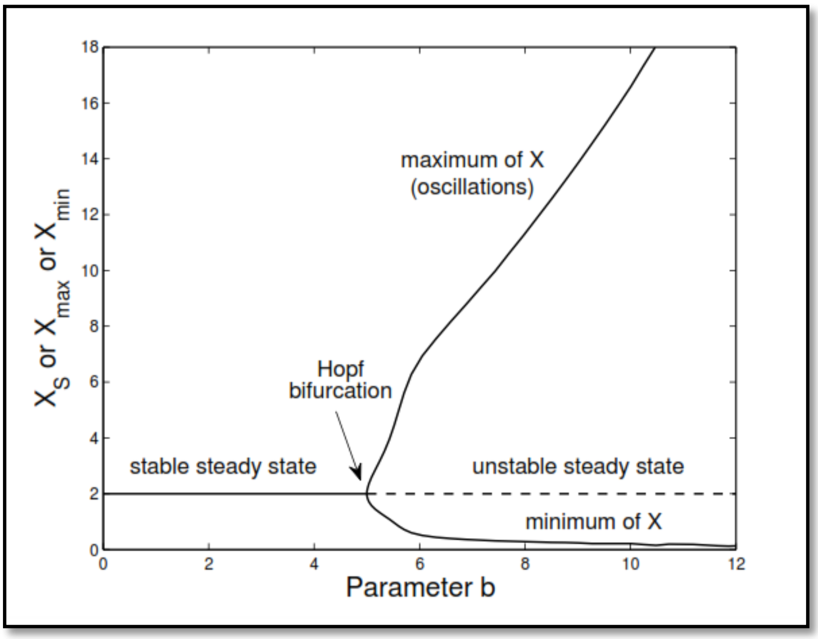


Figure (4.5)
“Bifurcation diagram for the Brussdator as a function of parameter b (with a=2)”

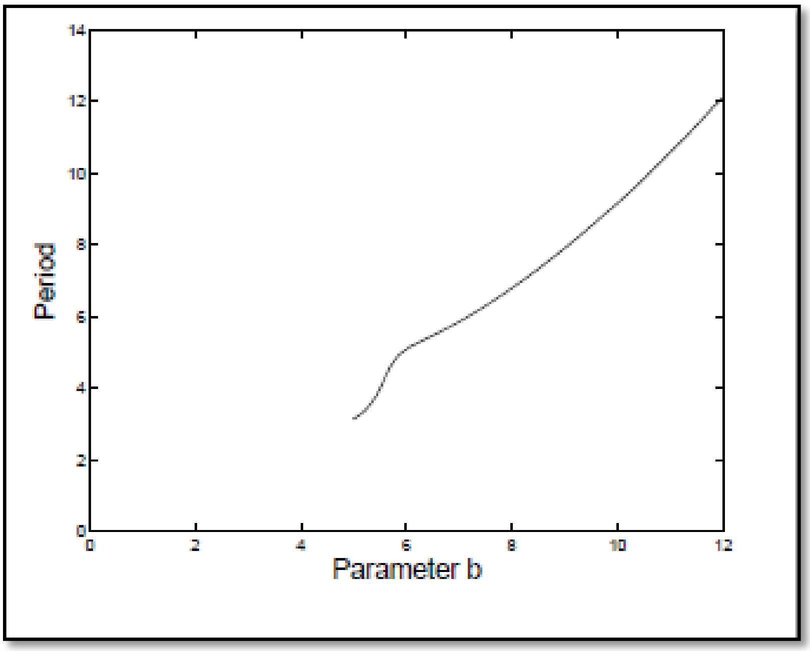


Figure (4.6)

"Period for the Brusseator as function of parameter b (with a=2)."

Nullclines and direction field:

The nullclines, defined by:

$$\frac{dX}{dt} = 0 \rightarrow Y = \frac{(1+b)}{X} - \frac{a}{X^2} (\, X - nullcline)$$

$$\frac{dY}{dt} = 0 \rightarrow Y = {}^{b}/_{X} \,(\, Y - unllcline \,) \qquad \ldots \ldots \ldots . (29')$$

Are shown in (figure (4.7)). They delimit regions in the phase space where the vector field has a particular direction (figure (4.8)).

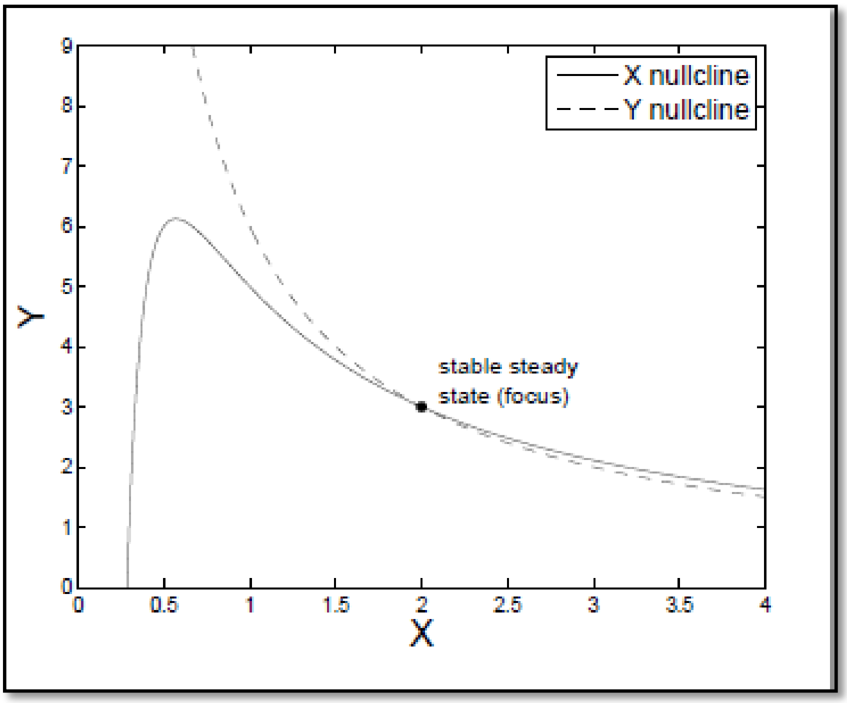


Figure (4.7)

“Brusselator: unllclines (for a=2 and b=6)"

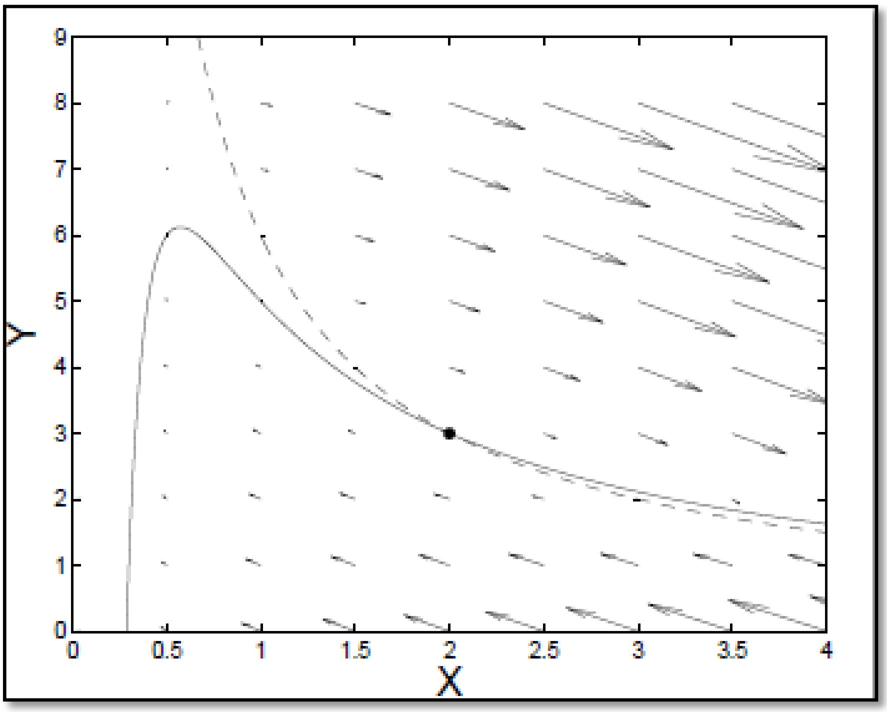


Figure (4.8)

"Brusselator: direction field (for a=2 and b=6)"

Conclusion

It is often impossible to find explicit solutions of nonlinear systems of differential equations. The one exception to this occurs when we have equilibrium solutions. Provided we can solve the algebraic equations, we can get the equilibria explicitly. Often, these are the most important solutions of a particular nonlinear system. More important, given our extended work on linear systems, we can usually use the technique of linearization to determine the behavior of solutions near equilibrium points.