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Post-Optimality in Linear Programming Problem With Bounded Variables

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Submitted :

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بسم الله الرحمن الرحيم

﴿ نرفع درجات من نشاء وفوق كل ذي علم عليم ﴾

صدق الله العظيم

سورة يوسف الآية (76)

إهـداء

أهدي هذا العمل المتواضع

إلى

- الوالدين الكريمين حفظهما الله
- و إلى كل أفراد أسرتي
- و إلى صديقتي أسماء الطيب
- و إلى كل من لم يدخر جهداً في مساعدتي
- و إلى كل من ساهم في تلقيني و لو بحرف في حياتي الدراسية

شكر و تقدير

الحمد و الشكر لله أولاً و أخيراً . .

أقدم شكري و امتناني إلى جميع من أعانوني و ساعدوني في إخراج هذا البحث بفضلهم و جهدهم على الآراء القيمة التي أبدوها لي و خصوصاً مشرف البحث الدكتور الفاضل علي محمد إبراهيم و إلى الهيئة التدريسية في القسم عموماً, و راجيتاً من الله أن أكون قد أصبت أكثر مما أخطأت و أن يستفاد مما بذلت من جهد, آمله أن أكون قد أعطيت الموضوع بعض حقه, و أسأل الله أن يعلمنا ما ينفعنا, و ينفعنا بما علمنا

و الله ولي التوفيق.

ملخص البحث

عند حصول تغيرات على صيغة مسألة البرمجة الخطية المحدودة BLPP فإن الإجابة على السؤال المتعلق بدراسة إيجاد الحل الأمثل من جديد لمسألة البرمجة الخطية تسمى ما بعد الأمثليّة (أو تحليل الحساسية).

بديهي عندما يتغير تركيب المسألة الأصلية BLPP بإجراء بعض التعديلات عليها, فإن المسألة الجديدة BLLP يمكن دراستها من البداية إذا كانت التغيرات ليست رئيسية, و يجب أن لا نتجاهل الفوائد القيمة الناتجة عن حل المسألة الأصلية.

تضمن دراستنا على الطريقة البيانية لحل مسألة البرمجة الخطية في متغيرين, حيت تناولنا عدة حالات التي يظهر بها حل تلك المسائل.

كما أن الدراسة تبين الطرق و الخوارزميات لإيجاد الحلول المثلى للمسألة الجديدة BLPP مستخدمين الطريقة (الخوارزمية) المبسطة القرينة و جدول الحل الأمثل للمسألة الأصلية.

ABSTRACT

When the optimal solution to a bounded linear programming problem (BLPP) is reached, we want to answer questions concerning changes in its formulation, the study is called post-optimality analysis (or sensitivity analysis).

Obviously when the original bounded linear programming problem BLPP is modified (making some changes in its formulation) the new problem could be solved from scratch, if the change are minor, however, this means we ignore the valuable information gained in solving the original problem.

Also, Our study included a graphical method to solve the linear programming problem in the two variables, where we dealt with several cases shows that the solution to those problem.

And the study shows haw the new optimal solution for the new BLPP can be found using the primal dual simplex algorithm and the solution of the original problem.

CONTENTS

CHAPTER 3. BOUNDED-VARIABLE SIMPLEX METHOD

CHAPTER4. DUAL SIMPLEX AND SENSITIVITY ANALYSIS FOR BOUNDED LINEAR PROGRAMMING PROBLEMS

ITRODUCTION

The linear programming with bounded variables have been studied by many authors (Dantzing, 1963), (Turnves,1968), (Duguay, 1972), (Hussain, 2000). In 1954, Dantzig, developed the method for solving linear programming with upper bound restrictions on the variables. In 1972, Duguay et al. studied linear programming with relative bounded variables (BLPP). Later on, various method like revised simplex algorithm, modified decomposition algorithm have been developed by various authors (Murty, 1976), (Ho, 1991). This study concerned with sensitivity analysis for linear programming with bounded variables. Sensitivity analysis (also called post optimality analysis) is the study of the behavior of the optimal solution with respect to changes in input parameters of the original optimization problem.

Sometimes, when we use sensitivity analysis to resolving modified problem, we get a solution, which is optimal but not feasible. The dual simplex algorithm is the method of choice when linear programs have to be reoptimized when data in problem is perturbed.

Our study contains four chapters. In chapter 1, as an introduction (Primarily), we defined the linear programming problem and how we can get the solution by graphical method (for two variables case), we define convex sets and extreme points, and we study some special cases in graphical method. In chapter 2, we explain algebra of the simplex method and the simplex Method in Tableau form . And we explain duality and sensitivity analysis in linear programming problem. In chapter 3, we defined the linear programming with bounded variables and discuss the simplex method for bounded variables. In chapter 4, we defined duality in linear programming problem with bounded variables and discuss many cases in sensitivity analysis, numerical illustration is given.

Chapter (1)

Foundations of The Simplex Method

Chapter1

1.1. Introduction:

In this chapter we defined the linear programming problem and how we can get the solution by graphical method (for two variables case), we define convex sets and extreme points, and we study some special cases in graphical method.

1.2. Linear programming problem LPP [4],[7]

A linear programming problem LPP is an optimization (maximum or minimum) problem in which the objective function is linear in the unknowns nonnegative variables (1.1) and the constraints consist of linear equalities or linear inequalities (1.2 – 1.4). The exact form of these constraints may differ from one problem to another, that a linear program may be written in the general form :

$$\textbf{Optimize} \begin{pmatrix} maximum \\ or \\ minimum \end{pmatrix} \boldsymbol{Z} = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad (1.1)$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \; \{\leq, =, or \; \geq\} \; b_1 \qquad (1.2)$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \; \{\leq, =, or \; \geq\} \; b_2 \qquad (1.3)$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \; \{\leq, =, or \; \geq\} \; b_m \qquad (1.4)$$

$$x_1, x_2, \ldots, x_n \geq 0, \; b_1, b_2, \ldots, b_m \geq 0 \qquad (1.5)$$

Or written in the form,

$$\textbf{Optimize} \begin{pmatrix} maximum \\ or \\ minimum \end{pmatrix} \boldsymbol{Z} = \sum_{j=1}^{n} c_jx_j \qquad j = 1, \ldots, n \qquad (1.6)$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j \; \{\leq, =, or \geq\} \; b_i \quad ; \quad i = 1, \ldots, m \qquad (1.7)$$

$$x_j \geq 0 \quad ; \quad j = 1, \ldots, n \qquad (1.8)$$

$$b_i \geq 0 \quad ; \quad i = 1, \ldots, m \qquad (1.9)$$

Where the b_i's, c_j's and a_{ij}'s are fixed real constants, and the x_j's are real numbers to be determined.

Standard linear programming problem [7],[11]

A linear programing is said to be in standard form if all constraints are equalities and all variables are nonnegative.

An inequality can be easily transformed into an equation. To illustrate, consider the constraint given by $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$. This constraint can be written in an equation form by subtracting the nonnegative surplus or slack variable x_{n+i} (sometimes denoted by s_i) leading to

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \text{ and } x_{n+i} \geq 0. \tag{1.10}$$

Similarly, the constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ is equivalent to

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \text{ and } x_{n+i} \geq 0. \tag{1.11}$$

We shall consider the standard form of the objective function to be maximization. This in no way eliminates the consideration of minimization-type objective because if a function $\boldsymbol{Z}$ is to be minimized, we can use the simple equivalence:

$$\text{Minimize } \boldsymbol{Z} \equiv -\text{Maximize } (-\boldsymbol{Z})$$

Example 1.1:

LPP1: minimize $\boldsymbol{Z} = 5x_1 - 3x_2$

subject to

$$x_1 + 2x_2 \geq 4$$
$$x_1 + 3x_2 \leq 6$$
$$x_1,\ x_2 \geq 0$$

We can convert this problem into standard form as follows :

Chapter1

LPṔ1: maximize $\acute{\boldsymbol{Z}} = -5x_1 + 3x_2 + 0x_3 + 0x_4$
subject to

$$x_1 + 2x_2 - x_3 = 4$$
$$x_1 + 3x_2 + x_4 = 6$$
$$x_1,\ x_2,\ x_3, x_4 \geq 0$$

Notation and Definitions

We may write the standard linear programming problem in the matrix form :

LPP: maximize $\boldsymbol{Z} = \boldsymbol{c}^T\mathbf{x}$ (1.12)
subject to

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b} \quad (1.13)$$
$$\mathbf{x} \geq 0 \quad (1.14)$$

Where $\boldsymbol{A} = m \times n$ matrix of the coefficients of the constraints, that is,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n}) \ ,$$

where $\boldsymbol{a_j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is the column j in the matrix $\boldsymbol{A}$; $j = 1, 2, \dots, n$

$\boldsymbol{b} = m$-vector of right-hand sides, that is, $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$\boldsymbol{c}^T = n$-vector of objective coefficients, that is, $\boldsymbol{c}^T = (c_1, c_2, \dots, c_n)$ and the variables are given by the n-vector :

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n \quad \text{(Euclidean } n\text{-space)}$$

Definition 1.1 (Feasible and Infeasible Solution)[7]: A solution is feasible if it satisfies all the constraints (1.13) and the nonnegativity conditions (1.14) of the linear programming problem, otherwise it is called ***Infeasible Solution***. The set of all feasible solution is called feasible region.

Definition 1.2 (Optimal Solution) [7]: A point $\mathbf{x}^*$ is an optimal solution to a maximization linear program if $\mathbf{x}^*$ is a feasible solution and $\boldsymbol{c}\mathbf{x}^* \geq \boldsymbol{c}\mathbf{x}$ for all feasible solutions $\mathbf{x}$.

1.3. Graphical Solution of Two-Dimensional (Two variables) linear programs [7]

Prior to presenting the geometrical concepts that form the foundation of the simplex method, we present a graphical method for solving simple problems involving only two variables. We now use the following example to illustrate how we can graphically solve a linear program with two decision variables.

Example 1.2:

LPP2 **:** maximize $\boldsymbol{Z} = x_1 + 2x_2$

subject to

$$-x_1 + x_2 \leq 1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (\boldsymbol{I})$$
$$x_1 - x_2 \leq 3 \ldots\ldots\ldots\ldots\ldots\ldots \quad (\boldsymbol{II})$$
$$x_1 \qquad \leq 3 \ldots\ldots\ldots\ldots\ldots\ldots \quad (\boldsymbol{III})$$
$$x_1,\ x_2 \geq 0$$

First, we must identify the feasible region of the problem. Labeling one axis x_1 and the other x_2 . Note that the nonnegativity restrictions, $x_1, x_2 \geq 0$, require that we only consider points, (x_1, x_2), in the first quadrant. Next, the region identified by each constraint is plotted. Considering the first constraint $(-x_1 + x_2 \leq 1)$ initially, we graph the corresponding linear equation **I**: $(-x_1 + x_2 = 1)$ and, identify the region defined by this constraint in the first quadrant. We repeat this process with the second and third constraints, as showing in Figure 1.1. The set of all feasible points or solutions called the Feasible Region (F.R.) is the quadrilateral **ABCD** (including its interior), that is, the set of points that satisfy all of the constraints.

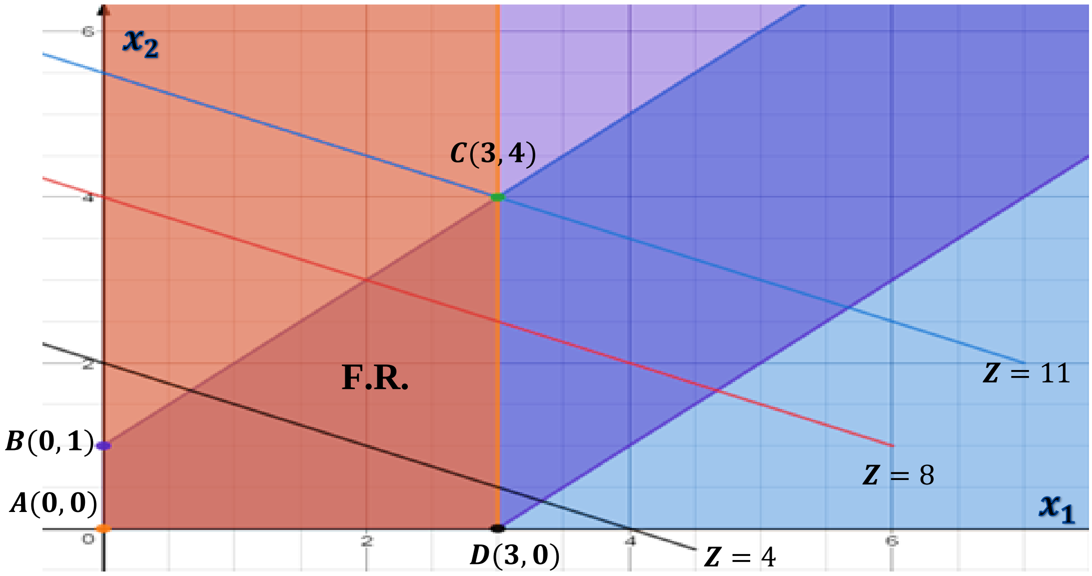

Figure 1.1

The final step is to determine the point which yield the maximum value of the objective function $\boldsymbol{Z} = x_1 + 2x_2$ in the feasible region. Let us begin by examining the level curves (isoprofit lines, isocost lines) of the objective function. For example, $\boldsymbol{Z} = 4$ defines the line $x_1 + 2x_2 = 4$. That is, any point on this line gives an objective function value of $\boldsymbol{Z} = 4$. Similarly, $\boldsymbol{Z} = 8$ defines the line $x_1 + 2x_2 = 8$. These represent parallel lines because they have the same slope. Thus, the level curves of the objective function are family of parallel lines. We simply need to identify the level curve that contacts the feasible region (that is, contains at least one feasible point) and corresponds to the greatest objective value. Thus, once we have defined the slope of the parallel lines, we only need to slide this line of fixed slope through the set of feasible points in the direction of improving $\boldsymbol{Z}$. the direction of improving $\boldsymbol{Z}$ can be quite easily identified by examining the gradient of the objective function. Recall that the gradient of the objective function $\boldsymbol{Z} = f(x_1, x_2) = c_1x_1 + c_2x_2$ is

$$\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial \boldsymbol{Z}}{\partial x_1} \\ \frac{\partial \boldsymbol{Z}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

and for our example,

$$\nabla f(x_1, x_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Recall also that the gradient of function at point is normal to the level curve of the function and always points in the direction of steepest a scent, that is, the direction of greatest increase of a two-variable linear program, we only need to sketch the vector corresponding to the gradient of the objective function. This is illustrated graphically in Figure 1.1. The level curves of the objective are then normal to this vector. For a maximization problem, we would slide the level curves in the direction of the gradient (direction of increasing $\boldsymbol{Z}$)until they reach the boundary of the solution space. Similarly, for a minimization problem, we would slide the level curves in the direction opposite the gradient (direction of decreasing $\boldsymbol{Z}$) until they reach the boundary of the solution space.

By using the foregoing technique, the optimal solution to Example 1.2 is determined be $\mathbf{C}(x_1^*, x_2^*) = (3{,}4)$, as illustrated in Figure 1.1. The corresponding optimal objective value is computed as

$$\boldsymbol{Z}^* = 1(3) + 2(4) = 11.$$

1.4. Convex and Polyhedral Sets [3]

In this section we discusses the geometry of the problem by presenting several definitions that form the foundation of the development that is to follow:

Definition 1.3 (Hyperplane): A hyperplane (line in two dimensions, plane in three dimensions) is the set of points $\mathbf{x} = (x_1, \dots, x_n)^T \in R^n$, that satisfy $\boldsymbol{a}\mathbf{x} = \boldsymbol{b}$, where $\boldsymbol{a} = (a_1, \dots, a_n) \in R^n$, $\boldsymbol{a} \neq 0$, and $\boldsymbol{b} \in R^1$ (i, e., $\boldsymbol{b}$ is a scalar).

Definition 1.4 (Halfspace): A closed halfspace corresponding to the hyperplan $\boldsymbol{a}\mathbf{x} = \boldsymbol{b}$ is either of the sets $H^+ = \{\mathbf{x}: \boldsymbol{a}\mathbf{x} \geq \boldsymbol{b}\}$ or $H^- = \{\mathbf{x}: \boldsymbol{a}\mathbf{x} \leq \boldsymbol{b}\}$. When these halfspaces are defined as $\{\mathbf{x}: \boldsymbol{a}\mathbf{x} > \boldsymbol{b}\}$ or $\{\mathbf{x}: \boldsymbol{a}\mathbf{x} < \boldsymbol{b}\}$, they are called open halfspaces.

Note the vector $\boldsymbol{a}$ is the gradient of linear function $f(\mathbf{x}) = \boldsymbol{a}\mathbf{x}$, and thus is normal to the hyperplane and points in the direction of increasing $\boldsymbol{a}\mathbf{x}$ as depicted in Figure 1.2.

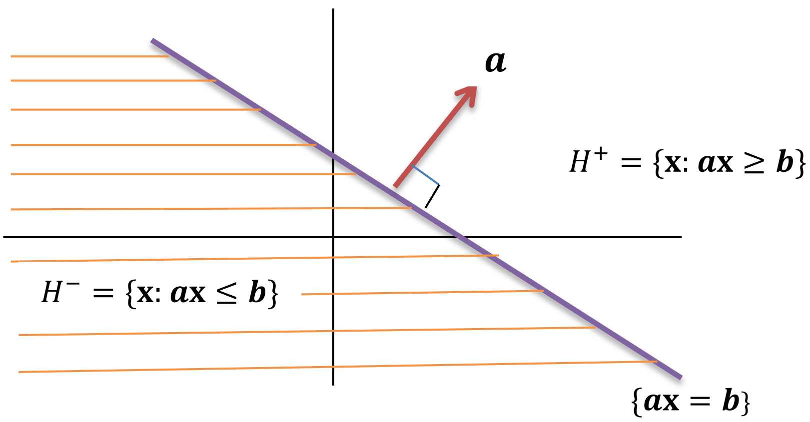

Figure 1.2

Definition 1.5 (Polyhedral Set): A polyhedral set is the intersection of a finite number of halfspaces. Thus, the constraint set $\boldsymbol{S} = \{\mathbf{x}: \boldsymbol{A}\mathbf{x} \leq \boldsymbol{b}, \mathbf{x} \geq 0\}$ is a polyhedral set because it is the intersection of m halfspace corresponding to $\boldsymbol{A}\mathbf{x} \leq \boldsymbol{b}$ and n halfspaces corresponding to $\mathbf{x} \geq 0$.

Definition 1.6 (convex set): A set $\boldsymbol{S}$ is convex if, for any two points, say $\mathbf{x}_1, \mathbf{x}_2 \in \boldsymbol{S}$, then the line segment joining these two points lies entirely within $\boldsymbol{S}$. Mathematically, this means that if $\mathbf{x}_1, \mathbf{x}_2 \in \boldsymbol{S}$, then $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \boldsymbol{S}$ for all $\alpha \in [0,1]$.

The expression $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \boldsymbol{S}$, $\alpha \in [0,1]$ defines the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ and is called the convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Figure1.3 depicts some examples of convex and nonconvex sets.

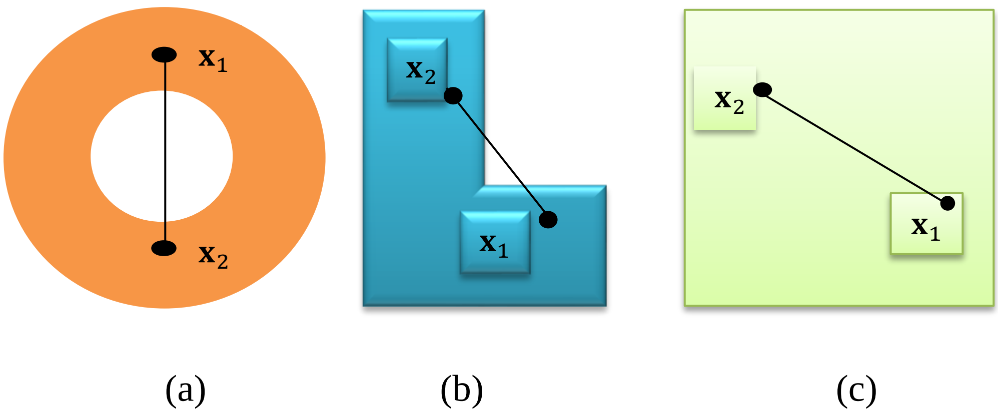

Figure 1.3: (a) and (b) nonconvex sets, (c) convex set.

Theorem 1.1: [7]

The set $\boldsymbol{S} = \{\mathbf{x}: \boldsymbol{A}\mathbf{x} = \boldsymbol{b}, \mathbf{x} \geq 0\}$ is a convex set.

Proof

Let $\mathbf{x}_1$, $\mathbf{x}_2 \in \boldsymbol{S}$ and let $\alpha \in [0,1]$. To complete the proof, it is sufficient to show that $\bar{\mathbf{x}} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \boldsymbol{S}$.

Because $\mathbf{x}_1 \in \boldsymbol{S}$, then $\boldsymbol{A}\mathbf{x}_1 = \boldsymbol{b}$ and $\mathbf{x}_1 \geq 0$ (from the definition). Similarly, $\boldsymbol{A}\mathbf{x}_2 = \boldsymbol{b}$ and $\mathbf{x}_2 \geq 0$. Also $\alpha \in [0,1]$ implies that $\alpha \geq 0$ and $(1-\alpha) \geq 0$.

Now, combining these results yields

$$\alpha\, \mathbf{A}\mathbf{x}_1 = \alpha \boldsymbol{b} \tag{1.15}$$

$$\alpha \mathbf{x}_1 \geq 0 \tag{1.16}$$

$$(1-\alpha)\, \boldsymbol{A}\mathbf{x}_2 = (1-\alpha)\boldsymbol{b} \tag{1.17}$$

$$(1-\alpha)\mathbf{x}_2 \geq 0 \tag{1.18}$$

Summing the expressions in (1.15) and (1.17) yields

$$\alpha\, \mathbf{A}\mathbf{x}_1 + (1-\alpha)\, \boldsymbol{A}\mathbf{x}_2 = \alpha\boldsymbol{b} + (1-\alpha)\boldsymbol{b} \tag{1.19}$$

Similarly, summing (1.16) and (1.18), we obtain

$$\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \;\geq 0 \tag{1.20}$$

Now, rearranging, (1.19) and (1.20) yield, respectively,

$$\boldsymbol{A}[\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2] = [\alpha + (1-\alpha)]\boldsymbol{b} = \boldsymbol{b} \tag{1.21}$$

and

$$\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \;\geq 0 \tag{1.22}$$

From (1.21) and (1.22), it is clear that $\boldsymbol{A}\bar{\mathbf{x}} = \boldsymbol{b}$ and $\bar{\mathbf{x}} \geq 0$, and thus $\bar{\mathbf{x}} \in \boldsymbol{S}$. □

Definition 1.7 (Extreme point): A point $\mathbf{x}$ is an extreme point of a given convex set $\boldsymbol{S}$ if it can't be written as a strict convex combination of two other distinct point of $\boldsymbol{S}$. Geometrically, this means that $\mathbf{x}$ is an extreme point of $\boldsymbol{S}$ if it does not lie on the interior of the line segment joining two other distinct points of $\boldsymbol{S}$. Mathematically there does not exist $\mathbf{x}_1, \mathbf{x}_2 \in \boldsymbol{S}$, $\mathbf{x}_1 \neq \mathbf{x}_2$, and $\alpha \in (0,1)$ such that $\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$.

Note that in polyhedral sets, these extreme points occur only at the intersection of the hyperplanes that form the boundaries of the polyhedral set.

Definition 1.8 (Adjacent Extreme points): Two distinct extreme points, say, $\mathbf{x}_1$ and $\mathbf{x}_2$, are adjacent if the line segment joining them is an edge of the convex set.

1.5. Some Special Cases

1.5.1. Alternative optimal solution [3]

A linear programming problem may have more than one optimal solution. In this case it will actually have an infinite number of optimal solutions.

Example 1.3:

Consider the linear programming problem:

LLP3: maximize $\boldsymbol{Z} = \frac{1}{2}x_1 + \frac{3}{2}x_2$

subject to

$$x_1 + 3x_2 \leq 6 \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \quad \textbf{(I)}$$
$$x_1 + \ x_2 \geq 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \quad \textbf{(II)}$$
$$x_1,\ x_2 \geq 0$$

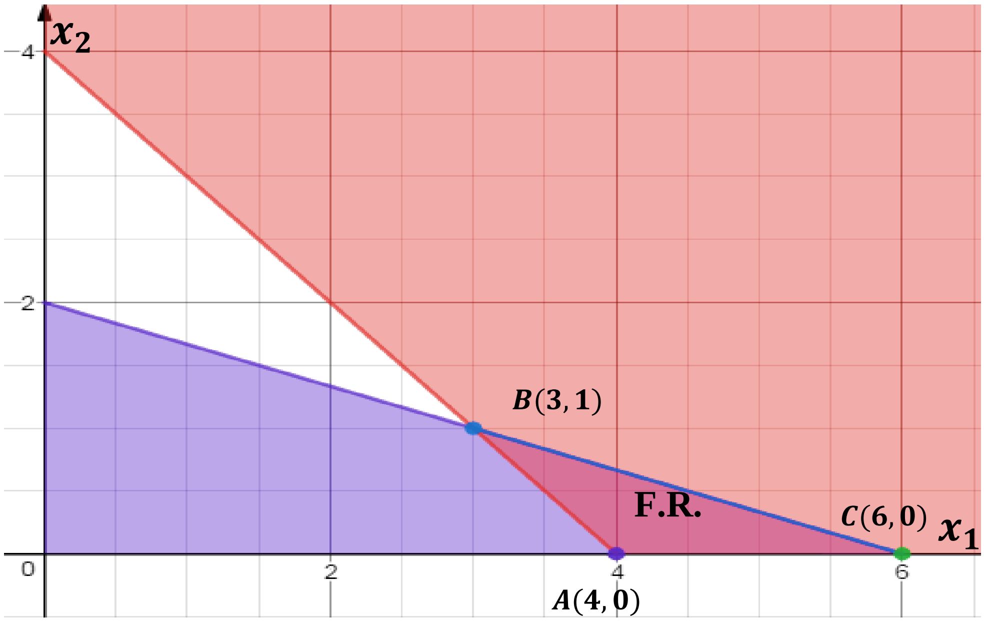

Figure 1.4

The convex set of all feasible solution (*feasible region*) is **ABC** as show in Figure 1.4. The extreme points and corresponding values of the objective function are given in Tableau 1.1, we see that both (3,1) and (6,0) are optimal solutions to the problem. The line segment joining these points is

$$\begin{aligned}(x_1, x_2) &= \alpha(3,1) + (1-\alpha)(6,0)\\ &= (3\alpha, \alpha) + (6-6\alpha, 0)\\ &= (6-3\alpha, \alpha) \qquad \text{for } \alpha \in [0,1]\end{aligned}$$

For any point (x_1, x_2) on this line segment we have

$$\begin{aligned}\boldsymbol{Z} = \tfrac{1}{2}x_1 + \tfrac{3}{2}x_2 &= \tfrac{1}{2}(6-3\alpha) + \tfrac{3}{2}(\alpha)\\ &= \tfrac{6}{2} - \tfrac{3}{2}\alpha + \tfrac{3}{2}\alpha = 3.\end{aligned}$$

Any point on this segment is an optimal solution.

Extreme point	Value of $\boldsymbol{Z} = \frac{1}{2}x_1 + \frac{3}{2}x_2$
A(4,0)	2
B(3,1)	3
C(6,0)	3

Tableau 1.1

1.5.2. Unbounded objective value [3]

In some linear programming problems, the values of some of the variables may be increased indefinitely without violating any of the constraints, meaning that, the feasible solutions is unbounded in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) infinitely. In this case, both the feasible region and the optimum objective value are unbounded.

Example 1.4:

Consider the linear programming problem.

LPP4: maximize $\boldsymbol{Z} = 2x_1 + x_2$
subject to

$$x_1 - 2x_2 \leq 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(I)}$$
$$x_1 + x_2 \geq 6 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(II)}$$
$$x_1,\ x_2 \geq 0$$

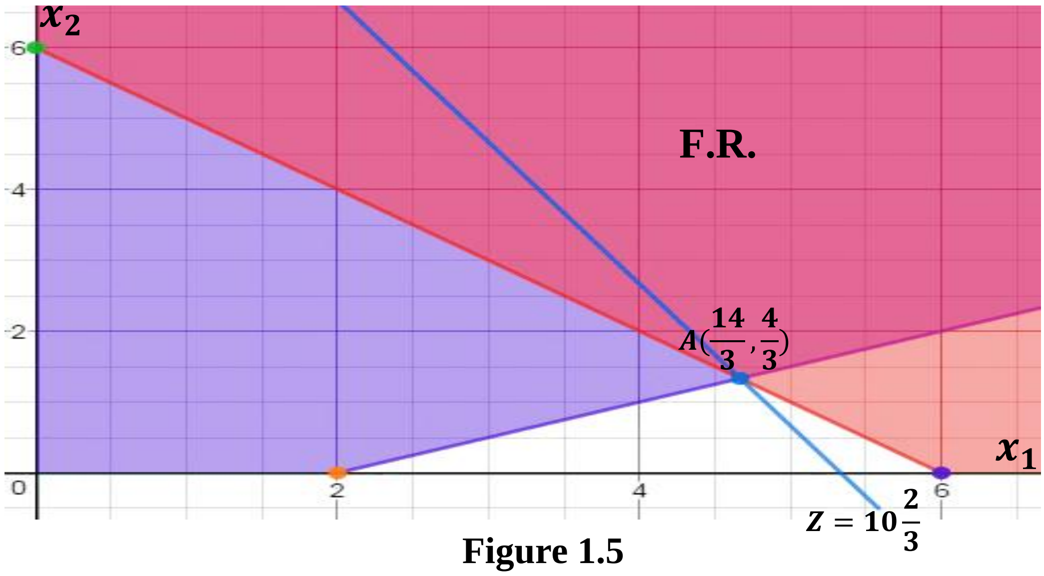

Figure 1.5

The convex set $\boldsymbol{S}$ of all feasible solutions is shown in Figure1.5. Note that it is unbounded, and that the value of objective function $\boldsymbol{Z}$ be increases as x_1 and as x_2 increases. On the other hand, a linear programming problem with an unbounded convex set of feasible solutions may have an optimal solution.

Example 1.5:

Consider the same set of constraints as in Example 1.4. Suppose that the problem was instead to

LPP5: minimize $\boldsymbol{Z} = 2x_1 + x_2$
subject to

$$x_1 - 2x_2 \leq 2$$
$$x_1 + x_2 \geq 6$$
$$x_1,\ x_2 \geq 0$$

Note that the minimum value of $\boldsymbol{Z}$ is in the point **B**$(0{,}6)$, thus it is optimal point.

1.5.3. Infeasible Solution [3]

It is possible that the feasible region of linear problem (LP) to be empty, resulting an infeasible LP. Because the optimal to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.

Example 1.6:

Consider the linear programming problem

LPP6: maximize $\boldsymbol{Z} = 3x_1 + 2x_2$
subject to

$$x_1 + x_2 \geq 6$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Which is solved in Figure 1.6 below.

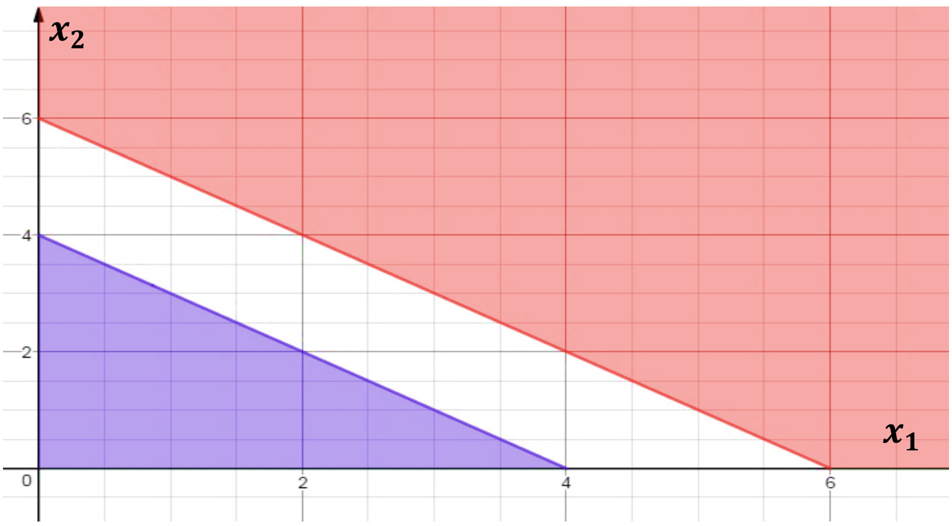

Figure 1.6

Note that the set of feasible solutions is empty. This situation will arise when conflicting constraints are put on a problem. The assumptions for the model must be changed to yield a nonempty set of feasible solutions.

1.6. Basic Feasible Solutions and Extreme Points [7],[11]

In this section we present a method for characterizing extreme points algebraically, and this will enable us to algebraically describe the simplex method.

Consider a linear system of equations given by

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b} \tag{1.23}$$
$$\mathbf{x} \geq 0 \tag{1.24}$$

Where $\boldsymbol{A}$ is a given $m \times n$ matrix,

$\boldsymbol{b}$ is a given m-vector, i,e. , $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$

Assume that the rank$(\boldsymbol{A}) = m \leq n$. That is, assume that $\boldsymbol{A}$ has full row rank, or, equivalently, the rows of $\boldsymbol{A}$ are linearly independent. Also assume that the columns of $\boldsymbol{A}$ can be reordered so that $\boldsymbol{A}$ can be written in partitioned form as

$$\boldsymbol{A} = [\boldsymbol{B}\ \ \boldsymbol{N}] \tag{1.25}$$

Where

$\boldsymbol{B} = m \times m$ nonsingular matrix. Designated the basis matrix

$\boldsymbol{N} = m \times (n - m)$ matrix (the matrix of nonbasic columns)

Based on this partitioning of matrix $\boldsymbol{A}$, the linear system given in (1.26) can be recast in the form

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \boldsymbol{b} \tag{1.27}$$

Where vector $\mathbf{x}$ has been partitioned as $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$ to correspond precisely to the partitioning of matrix $\mathbf{A}$. Because $\boldsymbol{B}$ is nonsingular, the inverse of

$\boldsymbol{B}$ exists (using any method to finding the inverse of a matrix), and we may premultiply both sides of (1.26) by $\boldsymbol{B}^{-1}$ to obtain

$$\boldsymbol{B}^{-1}\boldsymbol{B}\mathbf{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} \qquad (1.28)$$

This simplifies to

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N \qquad (1.29)$$

Now setting $\mathbf{x}_N = 0$, we see that (1.28) results in $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$. The solution

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix}$$

is called a basic solution, with vector $\mathbf{x}_B$ called the vector of basic variables, and $\mathbf{x}_N$ is called the vector of nonbasic variables.

If, in addition, $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, then $\mathbf{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix}$ is called a basic feasible solution of the system (1.23) and (1.24).

Finally, if $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} > 0$, then $\mathbf{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix}$ is called a nondegenerate basic feasible solution. Otherwise, if at least one element of $\mathbf{x}_B$ is zero, then $\mathbf{x}$ is called a degenerate basic feasible solution.

Example 1.7:

Consider the polyhedral set defined by the following inequalities (as illustrated in Figure 1.7).

$$x_1 + x_2 \leq 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \quad \textbf{(I)}$$
$$x_2 \leq 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots.. \quad \textbf{(II)}$$
$$x_1, x_2 \geq 0.$$

By introducing the slack variables x_3 and x_4 , we obtain

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_2 + x_4 &= 2 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0 \end{aligned}$$

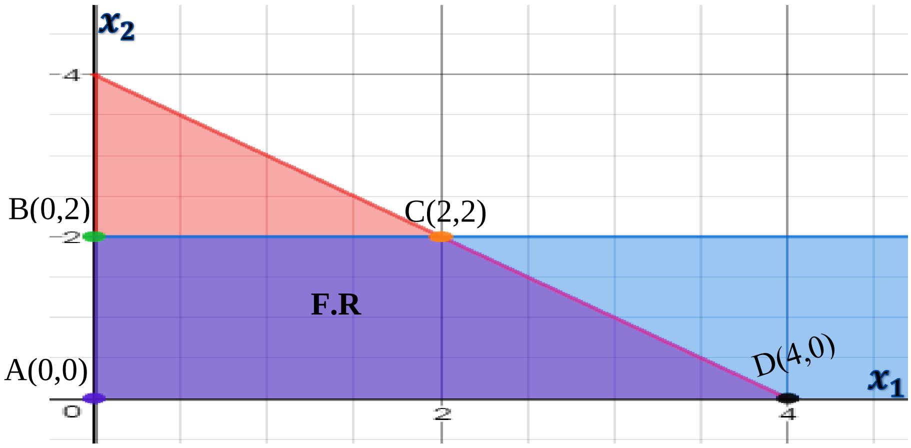

Figure 1.7

Note that, the constraint matrix $\boldsymbol{A} = [\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4}] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Because $\boldsymbol{A}$ is a 2×4 matrix, each basic solution will have $m = 2$, basic variables and $n - m = 2$ nonbasic variables. From the foregoing definition, basic feasible solutions correspond to finding 2×2 basis $\boldsymbol{B}$ with nonnegative $\boldsymbol{B}^{-1}\boldsymbol{b}$. The following are the possible ways of extracting $\boldsymbol{B}$ out of $\boldsymbol{A}$.

1. $\boldsymbol{B} = [a_1, a_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2. $\boldsymbol{B} = [a_1, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

3. $\boldsymbol{B} = [a_2, a_3] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. $\boldsymbol{B} = [a_2, a_4] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5. $\boldsymbol{B} = [a_3, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \qquad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that the points corresponding to 1, 2, 3 and 5 are nondegenerate basic feasible solutions because $\mathbf{x}_B > 0$. The point obtained in 4 is a basic solution, but it not feasible because it violates the nonnegativity restrictions. In other words, we have four basic feasible solutions, namely:

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}; \ \mathbf{x}_j = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

These points (basic feasible solutions) belong to R^4, projected in R^2 that is, in the (x_1, x_2) space give rise to the four points:

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ \text{and} \ \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

These four points are illustrated in Figure (1.7). Note that these points are precisely the extreme points of the feasible region.

In this example, the possible number of basic feasible solutions is bounded by number of ways of extracting two columns out of four columns to form the basis. Therefore the number of basic feasible solutions is less than or equal to

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Out of these six possibilities, one point violates the nonnegativity of $\boldsymbol{B}^{-1}\boldsymbol{b}$. Furthermore, a_1 and a_3 could not have been used to form a basis, since $a_1 = a_3 = \begin{bmatrix}1\\0\end{bmatrix}$ are linearly dependent, and hence the matrix $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ does not qualify as a basis. This leaves four basic feasible solution. In general, the number of basic feasible solutions is less than or equal to

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}\ .$$

Theorem 1.2: [3]

The problem determined by

$$\text{LPP:}\quad \text{maximize}\quad \boldsymbol{Z} = \boldsymbol{c}\mathbf{x}$$
$$\text{subject to}$$
$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b}$$
$$\mathbf{x} \geq 0$$

Where $\boldsymbol{A}$ is an $n \times m$ matrix, $\boldsymbol{c} \in \boldsymbol{R}^n$, $\mathbf{x} \in \boldsymbol{R}^n$, and $\boldsymbol{b} \in \boldsymbol{R}^m$ has a finite number of basic feasible solutions.

Poof

The number of basic solutions to the problem is not more than

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \binom{n}{n-m}$$

Because there are $n - m$ choice for which of the n variables will be set to zero. The number of basic feasible solutions may be smaller than the number of basic solutions, since not all basic solutions need to be feasible.

Chapter (2)

The Simplex Method, Duality And Sensitivity Analysis

Chapter2

2.1. Introduction

In this chapter, we present a systematic method for iteratively moving from one extreme point to an adjacent extreme point in the search for an optimal solution. The method is first discussed algebraically, the tabular formats method is next, and we will discuss the problem of finding an initial basic feasible solution to a linear programming problem, also we study the duality of LPP and we develop a variant of the simplex method known as the dual simplex method and discuss how to deal with changes that are made to a linear program after it has been solved.

2.2. Algebra of The Simplex Method [7],[11],[12]

Consider the standard linear programming problem:

LPP: maximize $\boldsymbol{Z} = \boldsymbol{c}^T\mathbf{x}$
subject to

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b}$$
$$\mathbf{x} \geq 0$$

Where $\boldsymbol{A}$ is an $m \times n$ matrix with rank m. Recall that a basic feasible solution to this problem corresponds to an extreme point of the feasible region and is characterized mathematically by partitioning matrix $\boldsymbol{A}$ into a nonsingular basis matrix $\boldsymbol{B}$ and the matrix of nonbasic columns $\boldsymbol{N}$. That is

$$\boldsymbol{A} = [\boldsymbol{B}\ \ \boldsymbol{N}] \quad ; \quad \mathbf{x} = \begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix} \qquad (2.1)$$

The linear system $\boldsymbol{A}\mathbf{x} = \boldsymbol{b}$ can be rewritten to yield

$$\boldsymbol{B}\mathbf{x}_B + \boldsymbol{N}\mathbf{x}_N = \boldsymbol{b} \qquad (2.2)$$

This simplifies to

$$\mathbf{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} \qquad (2.3)$$

and solving for $\mathbf{x}_B$ yields

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N \qquad (2.4)$$

Now setting $\mathbf{x}_N = 0$, we see that (2.4) results in $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$. The solution

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix}$$

is called a basic solution, with vector $\mathbf{x}_B$ called the vector of basic variables, and $\mathbf{x}_N$ is called the vector of nonbasic variables. If, in addition, $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq 0$, then

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix}$$

is called a basic feasible solution. Otherwise it is called basic solution but not feasible (infeasible).

Now, consider the objective function $\boldsymbol{Z} = \boldsymbol{c}^T\mathbf{x}$. partitioning the cost vector $\boldsymbol{c}^T$ into basic and nonbasic components (i, e. , $\boldsymbol{c}^T = [\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\ \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}}]$, the objective function can be recast as

$$\boldsymbol{Z} = [\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\ \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}}] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$$

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\mathbf{x}_B + \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}}\mathbf{x}_N \qquad (2.5)$$

Now, substituting the expression for $\mathbf{x}_B$ defined in (2.4) into (2.5) yields

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N) + \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}}\mathbf{x}_N \qquad (2.6)$$

Which can be rewritten as

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - (\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}})\mathbf{x}_N \qquad (2.7)$$

Now setting $\mathbf{x}_N = 0$, we see that (2.7) results in $\overline{\boldsymbol{Z}} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b}$, which is the objective value corresponding to the current basic feasible solution.

Therefore, the current extreme point solution can be represented in canonical form as shown in (2.4) and (2.7) respectively.

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$$

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - (\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}})\mathbf{x}_N$$

with the current basic feasible solution given as

$$\overline{\boldsymbol{Z}} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} \tag{2.8}$$

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix} \geq 0 \tag{2.9}$$

Letting J denote the index set of the nonbasic variables, observe that

(2.7) and (2.4) can be rewritten as follows:

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - (\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_{\boldsymbol{N}}^{\boldsymbol{T}})\mathbf{x}_N$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - (\textstyle\sum_{j\in J} \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\, \boldsymbol{a}_{\boldsymbol{j}} - \sum_{j\in J} c_j)x_j$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J}(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\, \boldsymbol{a}_{\boldsymbol{j}} - c_j)x_j \tag{2.10}$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}\prime}\boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J}(z_j - c_j)\, x_j$$

$$= \overline{\boldsymbol{Z}} - \textstyle\sum_{j\in J}(z_j - c_j)\, x_j \tag{2.11}$$

where $z_j = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{a}_{\boldsymbol{j}}$ for each nonbasic variable.

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$$

$$= \boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J}(\boldsymbol{B}^{-1}\, \boldsymbol{a}_{\boldsymbol{j}})x_j$$

$$= \overline{\boldsymbol{b}} - \textstyle\sum_{j\in J} \alpha_j\, x_j \tag{2.12}$$

Where $\overline{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$, and $\boldsymbol{\alpha}_{\boldsymbol{j}} = \boldsymbol{B}^{-1}\boldsymbol{a}_{\boldsymbol{j}}$.

2.2.1. Checking for Optimality [7]

The first question that should be answered is: when will such an exchange improve the objective function? This can be answered in a rather straightforward manner by examining the canonical representation of $\boldsymbol{Z}$ in (2.10). In the current basic feasible solution, $\mathbf{x}_N = 0$, that is, the nonbasic variables are at their lower bound and can only be increased from their current value of zero. Observe that the coefficient $-(\boldsymbol{c}_B\boldsymbol{B}^{-1}\boldsymbol{a_j} - c_j)$ of x_j represents the *rate of change* of $\boldsymbol{Z}$ with respect to the non-basic variable x_j . That is,

$$\frac{\partial \boldsymbol{Z}}{\partial x_j} = -(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{a_j} - c_j)$$

Thus, if $\partial \boldsymbol{Z}/\partial x_j > 0$, then increaseing the nonbasic variable x_j will increase $\boldsymbol{Z}$. The quantity $(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{a_j} - c_j)$ is sometimes referred to as *reduced cost* and for convenience is usually denoted by $\left(z_j - c_j\right)$. We can thus state the optimality conditions for a maximization linear programming problem.

Optimality conditions (maximization problem). The basic feasible solution represented by (2.9) will be optimal to (LPP) if

$$\frac{\partial \boldsymbol{Z}}{\partial x_j} = -\left(z_j - c_j\right) = -(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{a_j} - c_j) \leq 0, \quad \text{for all } j \in J$$

Or equivalently, if

$$z_j - c_j = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{a_j} - c_j \geq 0, \quad \text{for all } j \in J$$

Note that, if $z_j - c_j > 0$, for all $j \in J$, then the current basic solution will be the unique optimal solution. However, if some nonbasic variable x_k has $z_j - c_j = 0$, then there exist an ***alternative optimal solutions***. For more detail (see reference [9]).

2.2.2. Determining the Entering and Departing Variables

Suppose there exist some nonbasic variable x_k with $z_k - c_k < 0$, then the objective function can be improved (increased) by increasing x_k from its current value of zero, we choose to increase the nonbasic variable with the most negative $z_j - c_j$. The selected variable x_k is called the entering variable. That is x_k enter the basis vector.

Now if there such that nonbasic variable x_k which increasing from its current value of zero, while holding all other nonbasic variables at zero, then the basic variables will change according to the relationship

$$\mathbf{x}_B = \bar{\boldsymbol{b}} - \alpha_k x_k \tag{2.13}$$

we can write as

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_r} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} \alpha_{1k} \\ \alpha_{2k} \\ \vdots \\ \alpha_{rk} \\ \vdots \\ \alpha_{mk} \end{bmatrix} x_k \tag{2.14}$$

If $\alpha_{ik} \leq 0$, then x_{B_i}, increases as x_k increase, and so x_{B_i} continues to be nonnegative. If $\alpha_{ik} > 0$, then x_{B_i} will decreases as x_k increases. In order to satisfy nonnegative, x_k is increased until the first point at which some basic variable x_{B_r} drops to zero. Examining equation (2.14), we obtain.

$$x_k = \underset{1 \leq i \leq m}{minimum} \{ \frac{\bar{b}_i}{\alpha_{ik}} : \alpha_{ik} > 0 \}$$

This process is termed the *minimum ratio test*, the basic variable x_{B_r} which drops to zero as the nonbasic variable x_k increasing is called the *departing variable*.

<u>Note that</u>:

1. If there exist more than one index r implies to the same value of the *minimum ratio test*, then the new solution is degenerate (see reference [11]).
2. If there is no any positive component α_{ik}, that is $\alpha_{ik} \leq 0$ for all i. In this case, the optimal objective value is unbounded (Explaining in Example 2.2).

Example 2.1:

LPP7: maximize $\boldsymbol{Z} = 4x_1 + 3x_2$
subject to

$$\begin{aligned} -x_1 + x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 20 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The problem is illustrated Graphically in Figure 2.1. After introducing the slack variable x_3, x_4 and x_5, we get the following system of constraints:

$$\begin{aligned} -x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + x_4 &= 20 \\ x_1 + x_2 + x_5 &= 12 \\ x_1, x_2, x_3, x_4,\ x_5 &\geq 0. \end{aligned}$$

The data for this problem can be summarize as follows.

$$\boldsymbol{A} = [\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}\,] = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix}, \quad \text{and}$$

$\boldsymbol{c^T} = (4,3,0,0,0)$.

Iteration 1

Since $\boldsymbol{b} \geq 0$, then we can choose an initial basis as

$$\boldsymbol{B} = [\,\boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \boldsymbol{I} = \boldsymbol{B^{-1}},$$

and $\boldsymbol{N} = [\boldsymbol{a_1}, \boldsymbol{a_2}] = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$**.**

Solving the system $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N$ leads to

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

put $\mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$**,** we obtain the basic feasible solution

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} > 0 \ , \ \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the objective value of this solution is zero, where

$$\boldsymbol{Z} = \boldsymbol{c_B^T}\mathbf{x}_B = (0,0,0) \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} = 0.$$

To see if we can improve the solution, calculate $z_1 - c_1$ and $z_2 - c_2$ as follows

$$z_1 - c_1 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_1} - c_1 = (0,0,0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - 4 = -4$$

$$z_2 - c_2 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_2} - c_2 = (0,0,0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3 = -3$$

since the most negative $z_j - c_j$ is $z_1 - c_1 = -4$, thus x_1 is the entering variable. The current solution is not optimal. Determining the departing variable x_{B_r} by *minimum ratio test* as follows

$$minimum \left\{ \frac{\bar{b}_2}{\alpha_{21}} = \frac{20}{2} = 10, \frac{\bar{b}_3}{\alpha_{31}} = \frac{12}{1} = 12 \right\} = 10 \ = \frac{\bar{b}_2}{\alpha_{21}}$$

Therefore, the index $r = 2$, that is, $x_{B_2} = x_4$ leaves the basis, obviously noting that

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} x_1$$

and x_4 drops to zero when $x_1 = 10$.

Iteration 2

The variable x_1 enters the basis and x_4 leaves the basis

$$\boldsymbol{B} = [\, \boldsymbol{a_3}, \boldsymbol{a_1}, \boldsymbol{a_5} \,] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix},$$

and $$\boldsymbol{N} = [\boldsymbol{a_4}, \boldsymbol{a_2}] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now, $\mathbf{x}_B$ can be determined by solving $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The objective value $\boldsymbol{Z} = \boldsymbol{c_B^T}\mathbf{x}_B = \boldsymbol{c_B^T}\overline{\boldsymbol{b}} = (0,4,0)\begin{bmatrix} 16 \\ 10 \\ 2 \end{bmatrix} = 40.$

Now, calculate $z_2 - c_2$and $z_4 - c_4$ as follows

$$z_2 - c_2 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_2} - c_2 = (0,4,0)\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - 3 = -1$$

$$z_4 - c_4 = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \boldsymbol{B}^{-1} \boldsymbol{a}_{\boldsymbol{4}} - c_4 = (0,4,0) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} - 0 = 2 .$$

Because $z_2 - c_2 < 0$, thus x_2 entering the basis. The current solution is not optimal.

Determining the departing variable x_{B_r}.

$$minimum \left\{ \frac{\bar{b}_1}{\alpha_{12}} = \frac{16}{3/2}, \ \frac{\bar{b}_2}{\alpha_{22}} = \frac{10}{1/2}, \ \frac{\bar{b}_3}{\alpha_{32}} = \frac{2}{1/2} \right\} = 4 = \frac{\bar{b}_3}{\alpha_{32}} .$$

Therefore, the index $r = 3$, that is, $x_{B_r} = x_5$ leaves the basis.

Iteration 3

The variable x_2 enters the basis and x_5 leaves the basis

$$\boldsymbol{B} = [\ \boldsymbol{a}_{\boldsymbol{3}}, \boldsymbol{a}_{\boldsymbol{1}}, \boldsymbol{a}_{\boldsymbol{2}}\ ] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

and $$\boldsymbol{N} = [\boldsymbol{a}_{\boldsymbol{4}}, \boldsymbol{a}_{\boldsymbol{5}}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we find $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ x_{B_3} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 20 \\ 12 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 4 \end{bmatrix}, \text{ and}$$

$$\mathbf{x}_N = \begin{bmatrix} x_{N_1} \\ x_{N_2} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The objective value $\boldsymbol{Z} = \boldsymbol{c}_B^T\overline{\boldsymbol{b}} = (0,4,3)\begin{bmatrix}10\\8\\4\end{bmatrix} = 44.$

Now, calculate $z_4 - c_4$ and $z_5 - c_5$ as follows

$$z_4 - c_4 = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{a}_4 - c_4 = (0,4,3)\begin{bmatrix}2\\1\\-1\end{bmatrix} - 0 = 1$$

$$z_5 - c_5 = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{a}_5 - c_5 = (0,4,3)\begin{bmatrix}-3\\-1\\2\end{bmatrix} - 0 = 2\ .$$

Since $z_j - c_j \geq 0$, for all nonbasic variable. Therefore the current basic feasible solution is optimal. The optimal solution given by $(x_1^*,x_2^*,x_3^*,x_4^*,x_5^*)$ = (8,4,10,0,0), with the objective value 44.

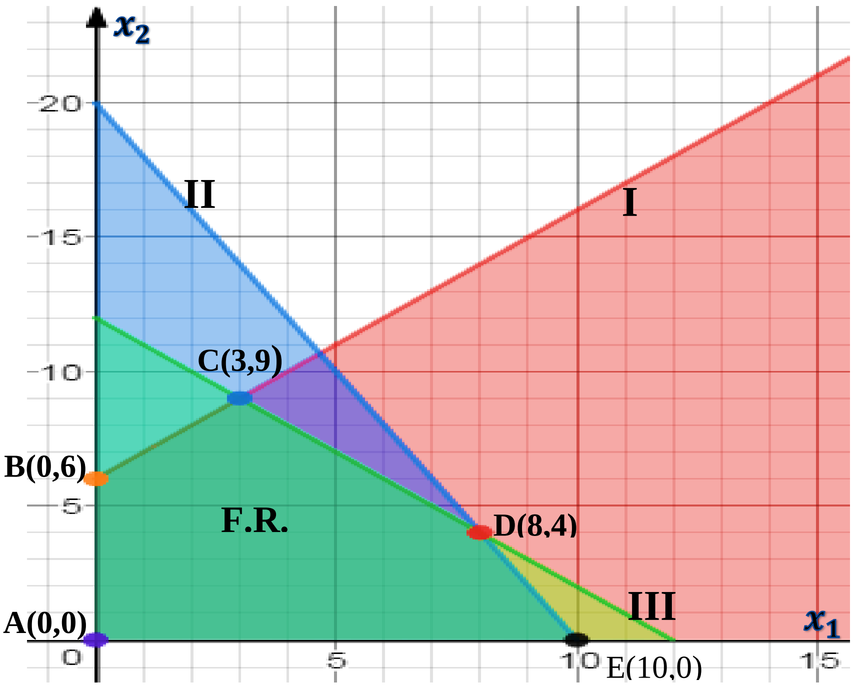

Figure 2.1

Chapter2

Example 2.2:

(*unboundedness*)

LPP8: maximize $\boldsymbol{Z} = 4x_1 + x_2$
subject to

$$2x_1 - 3x_2 \leq 12$$
$$-4x_1 + \ x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

The problem illustrated Graphically in Figure 2.2, has an unbounded optimal objective value. After introducing the slack variable x_3 and x_4, we get the constraint matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -3 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{bmatrix}$. Now, consider the basic feasible solution whose basis $\boldsymbol{B} = [\boldsymbol{a_3}, \boldsymbol{a_4}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{N} = [\boldsymbol{a_1}, \boldsymbol{a_2}] = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$, $\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{c_B^T} = (0,0)$, $\boldsymbol{c_N^T} = (4,1)$

$$\bar{\mathbf{x}}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}, \ \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{Z} = \boldsymbol{c_B^T}\bar{\mathbf{x}}_B = (0,0)\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 0$$

Calculate $z_1 - c_1$ and $z_2 - c_2$ as follows, noting $\boldsymbol{c_B^T}\boldsymbol{B}^{-1} = (0,0)$;

$$z_1 - c_1 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_1} - c_1 = -c_1 = -4,$$

$$z_2 - c_2 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_2} - c_2 = -c_2 = -1.$$

So, we increase x_1, which has the most negative $z_j - c_j$, thus x_1 enter the basis. Note that $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - [\boldsymbol{B}^{-1}\boldsymbol{a_1}]x_1$, and hence,

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ -4 \end{bmatrix} x_1.$$

The maximum value of x_1 is 6, at which instant x_3 drops to zero. The new basis is $\boldsymbol{B}$=$[\boldsymbol{a_1}, \boldsymbol{a_4}] = \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix}$, with the inverse $\boldsymbol{B}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & 1 \end{bmatrix}$,

and $\boldsymbol{N} = [\boldsymbol{a_3}, \boldsymbol{a_2}] = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{c_B^T} = (4{,}0)$, and $\boldsymbol{c_N^T} = (0{,}1)$ so we have

$$\bar{\mathbf{x}}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 32 \end{bmatrix},\ \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{Z} = 32$$

Now, we calculate $z_2 - c_2$ and $z_3 - c_3$ as follows:

$$z_2 - c_2 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_2} - c_2 = (2,0)\begin{bmatrix} -3 \\ 1 \end{bmatrix} - 1 = -7,$$

$$z_3 - c_3 = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{a_3} - c_3 = (2,0)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 2.$$

Note that $z_2 - c_2 < 0$ and $\boldsymbol{\alpha_2} = \boldsymbol{B}^{-1}\boldsymbol{a_2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} \\ -5 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, the optimal objective value is unbounded. In this case, if x_2 is increased and x_3 is kept zero, we get the following solution:

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - [\boldsymbol{B}^{-1}\boldsymbol{a_2}]x_2$$

i, e.,

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 32 \end{bmatrix} - \begin{bmatrix} \frac{-3}{2} \\ -5 \end{bmatrix} x_2 = \begin{bmatrix} 6 + \frac{3}{2} x_2 \\ 32 + 5x_2 \end{bmatrix},$$

With $x_2 \geq 0$, and $x_3 = 0$. Note that this solution is feasible for all $x_2 \geq 0$. In particular,

$$2x_1 - 3x_2 + x_3 = 2\left(6 + \frac{3}{2}x_2\right) - 3x_2 + x_3 = 12,$$

and

$$-4x_1 + x_2 + x_4 = -4\left(6 + \frac{3}{2}x_2\right) + x_2 + (32 + 5x_2) = 8$$

Furthermore, $\boldsymbol{Z} = 24 + 7x_2$, which approaches ∞ as x_2 approaches ∞. Thais means the optimal objective value is unbounded.

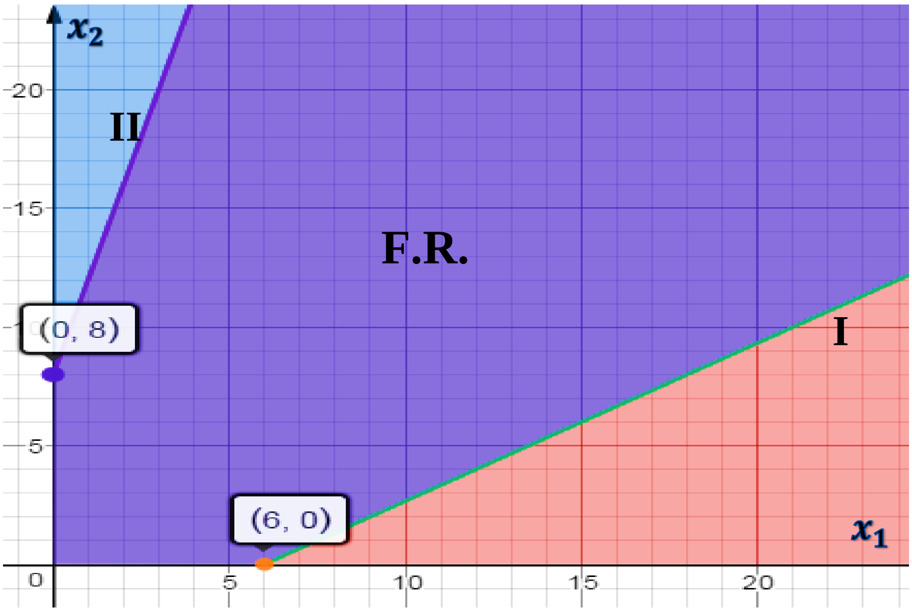

Figure 2.2

2.3. The Simplex Method in Tableau Form [7],[11]

Consider the canonical form represented in (2.7), and (2.4):

$$\boldsymbol{Z} = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} - (\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_N^T)\mathbf{x}_N \tag{2.15}$$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N \tag{2.16}$$

Now, rearrange terms as follows:

$$\boldsymbol{Z} + (\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_N^T)\mathbf{x}_N = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \tag{2.17}$$

$$\mathbf{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\mathbf{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} \tag{2.18}$$

The simplex tableau is simply a table used to store the coefficients of the algebraic representation in (2.17) and (2.18). the last row of the tableau consists of the coefficients in the objective equations (2.17), it is called *cost row*, and the body of the tableau (rows 1 to m) records the coefficient of the constraint equations (2.18). the general form is as show in Tableau 2.1.

	$\boldsymbol{Z}$	$\mathbf{x}_B$	$\mathbf{x}_N$	**RHS**	
$\mathbf{x}_B$	0	**I**	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	(rows 1 to m)
$\boldsymbol{Z}$	1	0	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N}-\boldsymbol{c}_N^T$	$\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$	(cost row)

Tableau 2.1

We now summarize the steps of the simplex algorithm as applied to the simplex tableau. The algorithmic steps follow directly from the preceding algebraic analysis.

Algorithm 1: (The simplex method)

STEP1 Check for possible improvement. Examine the $z_j - c_j$ values in the cost row of the simplex tableau. If $z_j - c_j \geq 0$, then the current basic feasible solution is optimal; stop. If , however, any $z_j - c_j < 0$, go to Step 2.

STEP2 Check for unboundedness. If, for any $z_j - c_j < 0$, there is no positive element in the associated $\boldsymbol{\alpha_j}$ vector (i.e., $\boldsymbol{\alpha_j} \leq 0$), then the problem has an unbounded objective value. Otherwise finite improvement in the objective is possible and we go to Step 3.

STEP3 Determine the entering variable. Select as the entering variable, the nonbasic variable with the most negative. Designate this variable as x_k. Ties in the selection of x_k may be broken arbitrary. The column associated with x_k is called the pivot column. Go to Step 4.

STEP4 Determine the departing variable. Use the *minimum ratio test* to determine the departing basic variable. That is, let

$$\frac{\bar{b}_r}{\alpha_{rk}} = \underset{1\leq i\leq m}{minimum}\left\{\frac{\bar{b}_i}{\alpha_{ik}} : \alpha_{ik} > 0\right\}.$$

Row r is called the pivot row, α_{rk} is called the pivot element, and the basic variable, x_{B_r} ; associated with row r is the departing variable. Go to Step 5.

	$\boldsymbol{Z}$	x_{B_1}	$\cdots$	x_{B_r}	$\cdots$	x_{B_m}	$\cdots$	x_j	$\cdots$	x_k	$\cdots$	**RHS**
x_{B_1}	0	1	$\cdots$	0	$\cdots$	0	$\cdots$	α_{1j}	$\cdots$	α_{1k}	$\cdots$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_r}	0	0	...	1	...	0	$\cdots$	α_{rj}	$\cdots$	α_{rk}	$\cdots$	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	0	...	0	...	1	$\cdots$	α_{mj}	$\cdots$	α_{mk}	$\cdots$	$\bar{b}_m$
$\boldsymbol{Z}$	1	0	...	0	...	0	$\cdots$	$z_j - c_j$	$\cdots$	$z_k - c_k$	$\cdots$	$\boldsymbol{c_B \bar{b}}$

Tableau 2.2: Before Pivoting

STEP5 Pivot and establish a new tableau .

a) The entering variable x_k is the new basic variable in row r.
b) Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consist of all zero elements except for a 1 at the pivot position α_{rk} (see Tables 2.2 and 2.3.)
c) Return to Step 1.

	x_{B_1}		x_{B_r}		x_{B_m}		x_j		x_k		**RHS**
x_{B_1}	1	$\cdots$	$\frac{-\alpha_{rk}}{\alpha_{rk}}$	$\cdots$	0	$\cdots$	$\alpha_{1k} - \frac{\alpha_{rj}}{\alpha_{rk}}\alpha_{1k}$	$\cdots$	0	$\cdots$	$\bar{b}_1 - \frac{\alpha_{1k}}{\alpha_{rk}}\bar{b}_r$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_k}	0	$\cdots$	$\frac{1}{\alpha_{rk}}$	$\cdots$	0	$\cdots$	$\frac{\alpha_{rj}}{\alpha_{rk}}$	$\cdots$	1	$\cdots$	$\frac{\bar{b}_r}{\alpha_{rk}}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{B_m}	0	$\cdots$	$\frac{-\alpha_{mk}}{\alpha_{rk}}$	$\cdots$	1	$\cdots$	$\alpha_{mj} - \frac{\alpha_{rj}}{\alpha_{rk}}\alpha_{mk}$	$\cdots$	0	$\cdots$	$\bar{b}_m - \frac{\alpha_{mk}}{\alpha_{rk}}\bar{b}_r$
$\boldsymbol{Z}$	0	$\cdots$	$\frac{z_k - c_k}{\alpha_{rk}}$	$\cdots$	0	$\cdots$	$(z_j - c_j) - \frac{\alpha_{rj}}{\alpha_{rk}}(z_k - c_k)$	$\cdots$	0	$\cdots$	$\boldsymbol{C_B \bar{b}} - (z_j - c_j)\frac{\bar{b}_r}{\alpha_{rk}}$

Tableau 2.3: After Pivoting

To illustrate this algorithm, we solve the LPP7 in example 2.1.

Example 2.3:

LPP7: maximize $\boldsymbol{Z} = 4x_1 + 3x_2$
subject to

$$-x_1 + x_2 \le 6$$
$$2x_1 + x_2 \le 20$$
$$x_1 + x_2 \le 12$$
$$x_1, x_2 \ge 0$$

Introduce the slack variables x_3, x_4 and x_5. The problem because the following

maximize $\boldsymbol{Z} = 4x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$
subject to

$$-x_1 + x_2 + x_3 = 6$$
$$2x_1 + x_2 + x_4 = 20$$
$$x_1 + x_2 + x_5 = 12$$
$$x_j \ge 0, j = 1, 2, \dots, 5$$

We can choose the initial basis as $\boldsymbol{B} = [\, \boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5} \,] = \boldsymbol{I}_3$, and $\boldsymbol{N} = [\, \boldsymbol{a_1}, \boldsymbol{a_2} \,]$,we indeed have $\overline{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} \ge 0$. This gives the following initial tableau:

Iteration 1

TABLEAU 2.4

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	-1	1	1	0	0	6
x_4	2	1	0	1	0	20
x_5	1	1	0	0	1	12
$\boldsymbol{Z}$	-4	-3	0	0	0	0

STEP1 The initial tableau appears in Tableau 2.4. Because there are $z_j - c_j < 0$ (both $z_1 - c_1 < 0$ and $z_2 - c_2 < 0$), we go to step 2.

STEP2 There are positive elements in $\boldsymbol{\alpha_j}$ associated with $z_j - c_j < 0$. Thus, finite improvement the objective $\boldsymbol{Z}$ is possible and we go to Step 3.

STEP3 The most negative $z_1 - c_1 = -4$. Thus $k = 1$ and x_1 is the entering variable. Go to Step 4.

STEP4 We now examine the ratios $\bar{b}_i/\alpha_{i1}$, where $\alpha_{i1} > 0$

$$minimum\left\{\frac{\bar{b}_2}{\alpha_{21}} = \frac{20}{2} = 10, \frac{\bar{b}_3}{\alpha_{31}} = \frac{12}{1} = 12\right\} = 10$$

Thus, $r = 2$, and the departing variable is $x_{B_2} = x_4$.

STEP5

a) Because x_1 is the entering variable and x_4 is the departing variable, x_1 is replaces x_4 in $\mathbf{x}_B$ as the basic variable in row 2.

b) Row $r = 2$ of the new tableau is obtained by dividing row r of the preceding tableau by $\alpha_{rk} = 2$ (the pivot element at the intersection of entering variable column and departing variable row). That is, the new objective row is obtained by multiplying new pivot row 2 by 4 and adding it to the old objective row. The new row 1 is obtained by adding the new row 2 to old row 1. Finally, the new row 3 is obtained by multiplying the pivot row by -1 and adding to old row 3. The completed second tableau is shown in Tableau 2.5, this corresponding to

$$\boldsymbol{B} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

TABLEAU 2.5

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	3/2	1	1/2	0	16
x_1	1	1/2	0	1/2	0	10
x_5	0	1/2	0	-1/2	1	2
$\boldsymbol{Z}$	0	- 1	0	2	0	40

c) Return to Step 1.

Iteration 2

STEP1 Because $z_2 - c_2 < 0$, we go to Step 2.

STEP2 There are positive elements in $\boldsymbol{\alpha_2}$. Thus the finite improvement in the objective is possible and go to Step 3.

STEP3 The most negative (and only negative) $z_j - c_j$ is $z_2 - c_2 = -1$. Thus, $k = 2$ and x_2 is the entering variable. Go to Step 4.

STEP4 The ratios $\bar{b}_i/\alpha_{i2}$, where $\alpha_{i2} > 0$, are

$$minimum\left\{\frac{\bar{b}_1}{\alpha_{12}} = \frac{16}{3/2}, \frac{\bar{b}_2}{\alpha_{22}} = \frac{10}{1/2}, \frac{\bar{b}_3}{\alpha_{32}} = \frac{2}{1/2}\right\} = 4 = \frac{\bar{b}_3}{\alpha_{32}}$$

Thus $r = 3$, and the departing variable is $x_{B_3} = x_5$.

STEP5

(a) Because x_2 is the entering variable and x_5 is the departing variable, x_2 replaces x_5 in x_B as the basic variable in row 3.

(b) Row $r = 3$ of the new table is obtained by dividing row r of the proceeding tableau by $\alpha_{rk} = \frac{1}{2}$. Use elementary row operation on the new tableau, we obtained the third tableau as shown in Tableau 2.6. As before, the basis inverse can be identified as

$$\boldsymbol{B} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

TABLEAU 2.6

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	0	1	2	-3	10
x_1	1	0	0	1	-1	8
x_2	0	1	0	-1	2	4
$\boldsymbol{Z}$	0	0	0	1	2	44

(c) Return to Step 1.

Because all $z_j - c_j \geq 0$, the solution given in Tableau 2.6 is optimal. In fact, because $z_j - c_j > 0$ for the nonbasic variables x_4 and x_5, then this tableau represents the unique optimal solution. Thus the optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*;\ \boldsymbol{Z}^*) = (8,4,10,0,0; 44)$.

2.4. The Big-*M* Method [6],[7],[11]

Introduction to the Big-*M* method

In each of the previous examples, a starting basic feasible solution was quite apparent. For example, if we look at the initial tableau of Example 2.3 (Tableau 2.4) we see that there is an imbedded $m \times m$ identity matrix $\boldsymbol{I}$, and the starting basic variables are readily identified by letting $\boldsymbol{B} = \boldsymbol{I}$. And because the right-hand side (RHS) vector $\boldsymbol{b}$ is nonnegative, the resulting solution is clearly feasible because $\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{I}\boldsymbol{b} = \boldsymbol{b} \geq 0$. However, such a starting basic feasible solution is not always available. For example, consider the following problem:

Example 2.4:

LPP8: maximize $\boldsymbol{Z} = -2x_1 + 5x_2 - x_3$ (2.19)

subject to

$$-x_1 + x_2 + x_3 \geq 5$$
$$x_1 + x_2 - x_3 = 1$$
$$5x_1 + 3x_2 - x_3 \leq 9$$
$$x_1, x_2, x_3 \geq 0$$

Now, converting the problem to standard equality form by adding the appropriate slack/surplus variables yields

maximize $\boldsymbol{Z} = -2x_1 + 5x_2 - x_3$
subject to

$$-x_1 + x_2 + x_3 - x_4 = 5 \qquad (2.20)$$
$$x_1 + x_2 - x_3 = 1 \qquad (2.21)$$
$$5x_1 + 3x_2 - x_3 + x_5 = 9 \qquad (2.22)$$
$$x_j \geq 0 \text{ for all } j$$

Therefore, the coefficient matrix is given by

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 5 & 3 & -1 & 0 & 1 \end{bmatrix}$$

Observe the matrix $\boldsymbol{A}$ does not contain the identity as a submatrix. In fact, $\boldsymbol{A}$ contains only the second column of the identity matrix. Thus, in its present state, we cannot use $\boldsymbol{B} = \boldsymbol{I}$ as a convenient starting basis. Artificial variable techniques were developed to find a starting basis feasible solution in this all-too-common situation when a nice starting basis is not available. Here , we present one of the artificial-variable techniques, the Big-*M* method.

The general approach of the big-*M* method can be described as follows. First, we create an identity submatrix by adding the necessary artificial variables to the original constraints. For example 2.4 it would be necessary to add two artificial variables, say, x_6 and x_7, to constraints (2.20) and (2.21), respectively. This would result in the following system of constraints.

$$-x_1 + x_2 + x_3 - x_4 + x_6 = 5$$
$$x_1 + x_2 - x_3 + x_7 = 1$$
$$5x_1 + 3x_2 - x_3 + x_5 = 9$$
$$x_j \geq 0 \quad \text{for all } j$$

Thus, the coefficient matrix becomes

$$\boldsymbol{A} = [a_1, a_2, a_3, a_4, a_5, a_6, a_7] = \begin{bmatrix} -1 & 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 1 \\ 5 & 3 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Clearly, the identity submatrix is now available with $\boldsymbol{I} = [\boldsymbol{a_6}, \boldsymbol{a_7}, \boldsymbol{a_5}]$. Note that, the starting basic feasible solution is found by setting the nonbasic variable x_1, x_2, x_3 and x_4 equal to zero. That is, $x_6 = 5$, $x_7 = 1$ and $x_5 = 9$.

Now, to prevent an artificial variables from becoming part of an optimal solution to the original problem, a very large penalty is choosing a positive constant M so large that the artificial variable is forced to be zero in any final optimal solution of the original problem. We then add the terms $-Mx_6$ and $-Mx_7$ to the objective function:

$$\boldsymbol{Z} = -2x_1 + 5x_2 - x_3 - Mx_6 - Mx_7$$

We now have a new problem, called the modified problem. Where M is large positive number. This leads to the following sequence of tableaux.

TABLEAU 2.7

P8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**RHS**
x_6	-1	1	1	-1	0	1	0	5
x_7	1	1	-1	0	0	0	1	1
x_5	5	3	-1	0	1	0	0	9
$\boldsymbol{Z}$	2	-5	1	0	0	M	M	0

Multiply rows 1 and 2 by ($-M$) and add to cost row.

TABLEAU 2.8

P8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**RHS**
x_6	-1	1	1	-1	0	1	0	5
x_7	1	1	-1	0	0	0	1	1
x_5	5	3	-1	0	1	0	0	9
Z	2	-2M-5	1	0	0	0	0	-6M

Since $z_2 - c_2 = -2M - 5 < 0$. Thus x_2 entering the basis and x_7 departing the basis.

TABLEAU 2.9

P8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**RHS**
x_6	-2	0	2	-1	0	1	-1	4
x_2	1	1	-1	0	0	0	1	1
x_5	2	0	2	0	1	0	-3	6
Z	2M+7	0	-2M-4	M	0	0	2M+5	-4M+5

Since $z_3 - c_3 = -2M - 4 < 0$. Thus x_3 entering the basis and x_6 departing the basis.

TABLEAU 2.10

P8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**RHS**
x_3	-1	0	1	-1/2	0	1/2	-1/2	2
x_2	0	1	0	-1/2	0	1/2	1/2	3
x_5	4	0	0	1	1	-1	-2	2
Z	3	0	0	-2	0	M+2	M+3	13

Since x_6 and x_7 equals zero, so deleting its columns of Tables. We have Tableau 2.11.

TABLEAU 2.11

P8	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	-1	0	1	-1/2	0	2
x_2	0	1	0	-1/2	0	3
x_5	4	0	0	1	1	2
$\boldsymbol{Z}$	3	0	0	-2	0	13

Since $z_4 - c_4 = -2 < 0$. Thus x_4 entering the basis and x_5 departing the basis.

TABLEAU 2.12

P8	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	1	0	1	0	1/2	3
x_2	2	1	0	-1/2	1/2	4
x_4	4	0	0	1	1	2
$\boldsymbol{Z}$	11	0	0	0	2	17

Since $z_j - c_j \geq 0$ for each nonbasic variable, the last tableau is optimal and the optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*;\ \boldsymbol{Z}^*) = (0,4,3,2,0; 17)$.

Remark: The use of the penalty M will not force an artificial variable to zero level in the final simplex iteration if the LPP does not have a feasible solution will include at least one artificial variable at a positive level (see reference [6])

2.5. Duality [7],[11]

Associated with each linear programming problem is another linear programming problem called the dual.

There are two important forms (definitions) of duality: the canonical form of duality and the standard form of duality.

2.5.1. Canonical form of duality [7]

Suppose that the primal linear program is given in the (canonical) form:

$$\text{LPP: maximize } \boldsymbol{Z} = c^T\text{x} \quad (2.23)$$
$$\text{subject to}$$
$$A\text{x} \leq b$$
$$\text{x} \geq 0$$

where A is an $m \times n$ matrix, c and x are $n \times 1$ column vectors, and b is $m \times 1$ column vector, then the dual linear program is defined by

$$\text{DLPP: minimize } \boldsymbol{W} = \text{b}^T y \quad (2.24)$$
$$\text{subject to}$$
$$A^T y \geq c$$
$$\text{y} \geq 0$$

Note that there is exactly one dual variable for each primal constraint (i.e., y is $m \times 1$ column vector) and exactly one dual constraint for each primal variable. We shall say more about this later [9].

2.5.2. Stander form of duality [11]

Another definition of duality may be given with primal liner program stated in the following standard form:

$$\text{LPP: maximize } \boldsymbol{Z} = c^T\text{x}$$
$$\text{subject to}$$
$$A\text{x} = b$$
$$\text{x} \geq 0$$

Then the dual linear program is defined by:

$$\text{DLPP: minimize } \text{b}^T y = W$$
$$\text{subject to}$$
$$A^T y \leq c$$
$$\text{y unrestricted (free)}$$

2.5.3. General form of Duality [7]

There are few requirements as to the general form of a linear programming problem. The objective may be either of maximizing or minimizing form, variables may be restricted or unrestricted, and the constraints may be form (≤,≥,=) and of any mixture of the forms. We utilizing the relationships in Tableau 2.13, to write the dual problem for a given linear program without given though the intermediate step of transforming the problem to canonical form (see reference [7]).

Primal Problem		Dual Problem
Maximization problem		Minimization problem
Constraints i	⇔	*Variables* y_i
≤	⇔	≥ 0
≥	⇔	≤ 0
=	⇔	*unrestricted (free)*
Variables x_j	⇔	*Constraints j*
≥ 0	⇔	≥
≤ 0	⇔	≤
unrestricted (free)	⇔	=

Tableau 2.13: Primal-Dual Relationships

Theorem 2.1:[3]
Given a primal problem as in (2.23), the dual of its dual problem is again the primal problem.

Proof

The dual problem as given by (2.24) is

$$\left.\begin{array}{r} \text{minimize} \quad b^T y \\ \text{subject to} \\ A^T y \geq c \\ y \geq 0 \end{array}\right\} \qquad (2.25)$$

We can rewrite (2.25) as

$$\left.\begin{array}{l} \text{maximize} \quad \mathbf{W} = -b^T y \\ \text{subject to} \\ \qquad -A^T y \leq -c \\ \qquad\quad y \geq 0. \end{array}\right\} \tag{2.26}$$

Now the dual problem to (2.26) is

$$\begin{array}{l} \text{maximize} \quad \boldsymbol{Z} = -c^T \text{x} \\ \text{subject to} \\ \qquad (-A^T \text{x})^T = -b \\ \qquad\quad \text{x} \geq 0. \end{array}$$

This problem can be rewritten as

$$\begin{array}{l} \text{maximize} \quad \boldsymbol{Z} = c^T \text{x} \\ \text{subject to} \\ \qquad A\text{x} \leq b \\ \qquad\quad \text{x} \geq 0, \end{array}$$

Which is the primal problem.

2.5.4. Primal-Dual Relationships [11]

There is a deep relationship between objective function value, feasibility and boundedness of the primal problem and the dual problem. We will explore some these relationships in the following theorems.

Theorem 2.2:[3] (*Weak Duality Theorem*) . If $\bar{\text{x}}$ is feasible solution to the primal problem

$$\left.\begin{array}{l} \text{maximize} \quad \boldsymbol{Z} = c^T \text{x} \\ \text{subject to} \\ \qquad A\text{x} \leq b \\ \qquad\quad \text{x} \geq 0 \end{array}\right\} \tag{2.27}$$

And if $\bar{\text{y}}$ is a feasible solution to the dual problem

$$\left.\begin{array}{l} \text{minimize} \quad \mathbf{W} = b^T y \\ \text{subject to} \\ \qquad A^T y \geq c \\ \qquad\quad y \geq 0. \end{array}\right\} \tag{2.28}$$

then

$$c^T\bar{x} \leq b^T\bar{y} \tag{2.29}$$

That is, the value of the objective function of the dual problem is always greater than or equal to the value of the objective function of the primal problem.

Proof

Since $\bar{x}$ is feasible to (2.27), we have

$$A\bar{x} \leq b. \tag{2.30}$$

It follows from (2.30) that

$$\bar{y}^T A\bar{x} \leq \bar{y}^T b = b^T\bar{y} \tag{2.31}$$

Since $\bar{y} \geq 0$, the equality in (2.31) comes from the fact that $\bar{y}^T b$ is a 1×1 matrix and consequently is equal to its transpose.

Since $\bar{y}$ is a feasible solution to (2.28), we have

$$A^T\bar{y} \geq c.$$

Or, taking transposes,

$$\bar{y}^T A \geq c^T.$$

Again we can multiply by $\bar{x}$, which is nonnegative, without changing the inequality. We get

$$\bar{y}^T A\bar{x} \geq c^T\bar{x}\ . \tag{2.32}$$

Combining inequalities (2.31) and (2.32) gives the desired result.

Notice that each feasible solution to the maximization problem provides a lower bound for the objective of the minimization problem, and, likewise, each feasible solution to the minimization problem provides an upper bound for the objective of the maximization problem.

Corollary (2.1):[7]

If the primal objective is unbounded, then the dual problem is infeasible.

Example 2.5:

Consider the LPP4 in example 1.4, this problem is unbounded as we show in Figure 1.5. The dual problem to LPP4 is

$$\begin{aligned} \text{DP4: minimize} \quad & \boldsymbol{W} = 2y_1 - 6y_2 \\ \text{subject to} \quad & \\ & y_1 - y_2 \geq 2 \\ & -2y_1 - y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

The constraints are in Figure 2.3. There are no feasible solutions to the problem, since the second constraint can never hold for nonnegative values of y_1 and y_2.

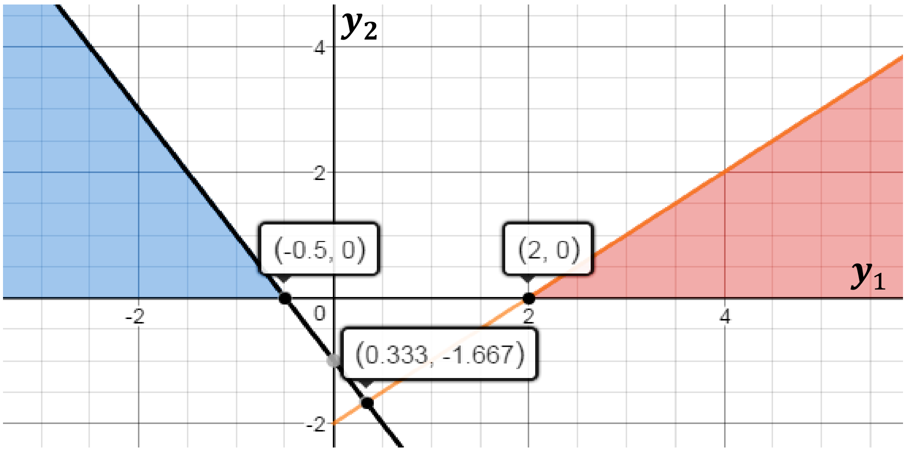

Figure 2.3

Corollary (2.2):[7]

If the dual objective is unbounded, then the primal problem is infeasible.

The converse of corollary (2.1) and (2.2) is not true. Because if one problem is infeasible, it is also possible for the other to be infeasible. This is illustrated via the following example.

Example 2.6: (*Infeasible Primal and Dual*)

Consider the following canonical primal- dual pair:

LPP10: max $\boldsymbol{Z} = x_1 + 3x_2$

subject to

$$-x_1 + 3x_2 \leq -3$$
$$x_1 - 3x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

DLPP10: min $\boldsymbol{W} = -3y_1 - 3y_2$

subject to

$$-y_1 + y_2 \geq 1$$
$$3y_1 - 3y_2 \geq 3$$
$$y_1, y_2 \geq 0$$

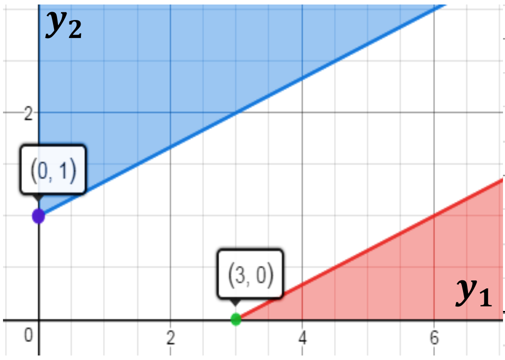

Figure 2.4

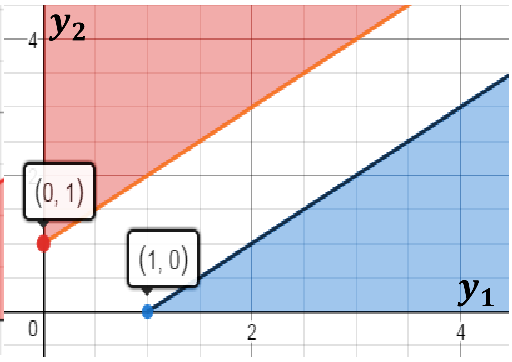

Figure 2.5

Upon graphing, it is clear from Figures 2.4 and 2.5 that neither the primal nor the dual possesses a feasible solution.

Corollary (2.3): [3]

If $\bar{x}$ is feasible to PP(Primal Problem), and $\bar{y}$ is feasible to DP(Dual Problem), and $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is an optimal solution to PP and $\bar{y}$ is an optimal solution to DP.

Proof

Suppose x_1 is any feasible solution to the primal problem. Then from the inequality (2.29).

$$c^T x_1 \leq b^T\bar{y} = c^T\bar{x}$$

Hence, $\bar{x}$ is an optimal. Similarly, if y_1 is any feasible solution to the dual problem, then from the inequality (2.29).

$$b^T\bar{y} = c^T\bar{x} \leq b^T y_1$$

and we see that $\bar{y}$ is an optimal solution to the dual problem.

Theorem 2.3: (*strong Duality*) [13]

If the PP(2.23) has an optimal solution with basis matrix B. Then

1. $y = (B^{-1})^T c_B$ is feasible solution of dual problem.
2. $c^T\text{x} = b^T y$.
3. $y = (B^{-1})^T c_B$ is an optimal solution of dual problem (2.24).

Proof

1. Let $A = [B, N]$, and since $\text{x}_B = B^{-1}b$ is an optimal solution. Hence

$$c_B^T B^{-1} N - c_N^T \geq 0 \qquad (\textit{optimality condition})$$

Requirement

$$c_B^T B^{-1} N \geq c_N^T$$

Now, we shall show that $y = \text{C}_B B^{-1}$ is feasible solution of dual problem

$$A^T y = [B^T y, N^T y] \geq [\text{c}_B, \text{c}_N] = \text{c}$$

and hence

$$A^T y \geq \text{c} \, ;$$

So that $y = (B^{-1})^T c_B$ is feasible solution of dual problem (2.24).

2. $b^T y = b^T (B^{-1})^T c_B = (c_B^T B^{-1} b)^T = (c_B^T \text{x}_B)^T = c_B^T \text{x}_B = c^T \text{x}$
3. Since $c^T\text{x} = b^T y$, it follows from corollary (2.3) that y is an optimal solution of dual problem.

Note that the dual feasibility conditions are precisely the same as primal optimality conditions. Also observe that Theorem 2.3 provides a method for computing the values of the dual variables. That is, whereas the primal solution can be written as

$$\mathrm{x}_N = 0$$
$$\mathrm{x}_B = B^{-1}b$$

the dual solution is given by

$$y^T = c_B^T B^{-1} \quad (2.33)$$

$$t = y^T A - c^T = c_B^T B^{-1} A - c^T \quad (2.34)$$

Where t is the vector of dual surplus variables. Finally, the objective value of both problem is (see reference [7])

$$\boldsymbol{Z} = c^T \mathrm{x} = c_B^T B^{-1} b = y^T b = \boldsymbol{W} \quad (2.35)$$

Primal-Dual Tableau Relationships [7]

Consider the initial simplex tableau corresponding to problem (2.23), and the optimal tableau as shown in Tableau 2.14 and Tableau 2.15 respectively.

TABLEAU 2.14

	x	$\mathbf{x}_S$	**RHS**
$\mathbf{x}_B$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$\boldsymbol{Z}$	$-\boldsymbol{c}$	$\mathbf{0}$	0

TABLEAU 2.15

	x	$\mathbf{x}_S$	**RHS**
$\mathbf{x}_B$	$\boldsymbol{B^{-1}A}$	$\boldsymbol{B^{-1}I}$	$\boldsymbol{B^{-1}b}$
$\boldsymbol{Z}$	$\boldsymbol{c_B^T B^{-1} A - c^T}$	$\boldsymbol{c_B^T B^{-1} I} - \mathbf{0}$	$\boldsymbol{c_B^T B^{-1} b}$

Note that the tableaux depicted in Tables 2.14 and 2.15 also establish some relationships between the primal and dual variables. To see more clearly, let us rewrite Tableau 2.15 utilizing the fact that

$$y^T = c_B^T B^{-1}$$
$$t^T = y^T A - c^T = c_B^T B^{-1} A - c^T$$

This results in Tableau 2.16

TABLEAU 2.16: *Primal Simplex Tableau*

	$\mathbf{x}$	$\mathbf{x}_S$	**RHS**
$\mathbf{x}_B$	$\boldsymbol{B^{-1}A}$	$\boldsymbol{B^{-1}}$	$\boldsymbol{B^{-1}b}$
$\boldsymbol{Z}$	t^T	y^T	$\mathbf{c}_{\boldsymbol{B}}\boldsymbol{B^{-1}b}$

First, note that $z_j - c_j$ values for the primal decision variables x are given by the dual surplus variables t. Just as $\boldsymbol{B^{-1}}$ resides in the portion of the tableau that was occupied by the original identity, $y^T = \boldsymbol{c_B^T B^{-1}}$ is located in two rows immediately above $\boldsymbol{B^{-1}}$. However, as we saw in Tableau 2.15, this is only true if the original objective coefficients of the corresponding slack variables are zero. Thus, the $z_j - c_j$ values for the zero-cost primal slack variables x_s are given by the dual decision variables y.

Thus, given a simplex tableau, it is possible to read the solution to both problems directly from the tableau. This idea is demonstrated further via the following example

Example 2.7:

Consider the problem in Example 2.1.

LPP7: maximize $\boldsymbol{Z} = 4x_1 + 3x_2$ (2.36)

subject to

$$-x_1 + x_2 \leq 6 \quad (2.37)$$

$$2x_1 + x_2 \leq 20 \quad (2.38)$$

$$x_1 + x_2 \leq 12 \quad (2.39)$$

$$x_1, x_2 \geq 0 \quad (2.40)$$

The optimal solution of this example is in the following final tableau

TABLEAU 2.17

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	0	1	2	-3	10
x_1	1	0	0	1	-1	8
x_2	0	1	0	-1	2	4
$\boldsymbol{Z}$	0	0	0	1	2	44

Slack variables x_3, x_4 and x_5 add to constraint (2.37),(2.38) and (2.39) respectively. The Tableau 2.17 indicates that the optimal primal solution is given by (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*; $\boldsymbol{Z}^*$)=(8, 4, 10, 0, 0; 44)

Now, denote the dual decision variables by y_1 , $y_2$and y_3 corresponding to constraints (2.37), (2.38) and (2.39) respectively. Also, let t_1 and t_2 represent the respective surplus variables for the two dual constraints. Then, by using the tableau relationships established in Tableau 2.16, the two row of the tableau will be in the following form:

	x_1	x_2	x_3	x_4	x_5	**RHS**
$\boldsymbol{Z}$	t_1	t_2	y_1	y_2	y_3	44

By comparing this with Tableau 2.17 it immediately follows that the solution is given by

$$\boldsymbol{W}^* = \boldsymbol{Z}^* = 44$$
$$y_1^* = z_3 - c_3 = 0$$
$$y_2^* = z_4 - c_4 = 1$$
$$y_3^* = z_5 - c_5 = 2$$
$$t_1^* = z_1 - c_1 = 0$$
$$t_2^* = z_2 - c_2 = 0$$

2.5.5. The Dual Simplex Method

When we use the simplex method to solve a max problem (we will refer to the max problem as a primal), we begin with a primal feasible solution (because each constraint in the initial tableau has a nonnegative right-hand side). It least one variable in objective row of the initial tableau has a negative coefficient, so our initial primal solution is not dual feasible. Through a sequence of simplex pivots, we maintain primal feasibility and obtain an optimal solution when dual feasibility (a nonnegative objective row) is attained. In many situations, however, it is easier to solve an LP by beginning with a tableau in which each variable in objective row has nonnegative coefficient (so the tableau is dual feasible) and at least one constraint has a negative right-hand said (so the tableau is primal infeasible). The dual simplex method maintains a nonnegative objective row (dual feasibility) and eventually obtains a tableau in which each right-hand said is nonnegative (primal feasibility). At this point, an optimal tableau has been obtained. Because this technique maintains dual feasibility, it is called ***the dual simplex method*** [13].

Algorithm 2: (Dual Simplex Method for Max Problem) [7]

STEP1 To employ this method, the problem must be dual feasible, that is, all $\boldsymbol{z_j - c_j \geq 0}$. If this condition is met, go to Step2.

STEP2 Determine the departing variable. If $b_i \geq 0$, for all i , then the current solution is optimal; stop. Otherwise, select the row associated with the most negative b_i. Denote this row as row r. The basic variable x_{B_r} associated with this row is the departing variable.

STEP3 Check for primal feasibility. If $\alpha_{rj} \geq 0$ for all j, then the primal problem is infeasible and the dual problem has an unbounded objective; stop. Otherwise, go to Step4.

STEP4 Determine the entering variable. Use the following minimum ratio test to determine the entering basic variable. That is, let

$$\frac{z_k - c_k}{-\alpha_{rk}} = minimum\{\frac{z_j - c_j}{-\alpha_{rj}} : \alpha_{rj} < 0\}$$

Column k is the pivot column, α_{rk} is the pivot element, and the nonbasic variable x_k associate with column k is the entering variable. Go to Step5.

STEP5 Pivot and establish a new tableau

a) The entering variable x_k is the new basic variable in row r.
b) Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consists of all zero elements except for a 1 at the pivot position α_{rk}.
c) Return to Step2.

The following example is considered to illustrate the dual simplex method.

Example 2.8:

LPP10: maximize $\boldsymbol{Z} = -2x_1 - x_3$ (2.41)
subject to

$$x_1 + x_2 - x_3 \geq 5 \quad (2.42)$$
$$x_1 - 2x_2 + 4x_3 \geq 8 \quad (2.43)$$
$$x_1, x_2, x_3 \geq 0 \quad (2.44)$$

Multiply both constraints through by -1. Adding slack variables x_4 to constraint (2.42) and x_5 to constraint (2.43) yields maximize $\boldsymbol{Z} = -2x_1 - x_3$ (2.45)

subject to

$$-x_1 - x_2 + x_3 + x_4 = -5 \quad (2.46)$$

$$-x_1 + 2x_2 - 4x_3 + x_5 = -8 \quad (2.47)$$

$$x_j \geq 0 \ , \ j = 1, \dots, 5 \quad (2.48)$$

The initial Tableau for the resulting problem is given in Tableau 2.18. Notice that the initial basis is primal infeasible ($x_4 = -5$ and $x_5 = -8$) and dual feasible (all $z_j - c_j \geq 0$). Thus, the dual simplex method can be employed.

TABLEAU 2.18

P10	x_1	x_2	x_3	x_4	x_5	**RHS**
x_4	-1	-1	1	1	0	-5
x_5	-1	2	-4	0	1	-8
$\boldsymbol{Z}$	2	0	1	0	0	0

STEP2 The most negative b_i is $b_2 = -8$. Thus, $r = 2$ and $x_{B_2} = x_5$ is the departing variable. Go to Step3.

STEP3 Because α_{21} and $\alpha_{23} < 0$, the primal infeasibility condition is not satisfied. Go to Step4.

STEP4 we now examine the ratios $(z_j - c_j)/(-\alpha_{ij})$, where $\alpha_{ij} < 0$ are

$$minimum \left\{ \frac{z_1 - c_1}{-\alpha_{21}} = \frac{2}{-(-1)}, \ \frac{z_3 - c_3}{-\alpha_{23}} = \frac{1}{-(-4)} \right\} = \frac{1}{4} = \frac{z_3 - c_3}{-\alpha_{23}}$$

Thus, $k = 3$ and the entering variable is x_3.

STEP5 a) Because x_3 is the entering variable and x_5 is the departing variable, x_3 replaces x_5 in x_B as the basic variable in row 2.

b) Pivot as usual on α_{23}. This results in Tableau 2.19.

Note that the Tableau 2.19 is still primal optimality (dual feasibility), but is not primal feasible.

TABLEAU 2.19

P10	x_1	x_2	x_3	x_4	x_5	**RHS**
x_4	-1/4	-1/2	0	1	1/4	-7
x_3	1/4	-1/2	1	0	-1/4	2
Z	7/4	1/2	0	0	1/4	-2

c) Return to Step2.

STEP2 The only negative b_i is $b_1 = -7$.

Thus, $r = 1$ and $x_{B_1} = x_4$ is the departing variable. Go to Step3.

STEP3 Because α_{11} and $\alpha_{21} < 0$, the primal infeasibility condition is not satisfied. Go to Step4.

STEP4 The ratios $(z_j - c_j)/(-\alpha_{ij})$, where $\alpha_{ij} < 0$ are

$$minimum\left\{\frac{z_1-c_1}{-\alpha_{11}} = \frac{7/4}{-(-5/4)},\ \ \frac{z_2-c_2}{-\alpha_{21}} = \frac{1/2}{-(-1/2)}\right\} = 1 = \frac{z_2-c_2}{-\alpha_{21}}$$

Thus, $k = 2$ and the entering variable is x_2.

STEP5 a) x_2 replaces x_4 in x_B as the basic variable in row 1.

b) Pivot as usual on $\alpha_{21} = \frac{-1}{2}$ to obtain Tableau 2.20.

Tableau 2.20 represents the optimal solution because both primal and dual feasibility are satisfied.

TABLEAU 2.20

P10	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	5/2	1	0	-2	-1/2	14
x_3	6/4	0	1	-1	-1/2	9
Z	1/2	0	0	1	1/2	-9

2.6. Sensitivity Analysis

2.6.1. Introduction [11]

In this section we shall describe how to make use of the optimality conditions (Primal–dual relationships) in order to find a new optimal solution for the modified problem, and here we shall discuss the following variations in the problem.

Change in the cost vector $\boldsymbol{c}^{\boldsymbol{T}}$.

Change in the right-hand side vector $\boldsymbol{b}$**.**

Change in $\boldsymbol{A}$ (change in the coefficient a_{ij}).

Deletion of a variable.

Deletion of a constraint.

2.6.2. Change in the cost vector $\boldsymbol{c}$ [11]

Consider the following linear programming problem

$$\begin{aligned} \text{LPP:} \quad & \text{maximize } \boldsymbol{Z} = \boldsymbol{c}^{\boldsymbol{T}}\mathbf{x} \\ & \text{subject to} \\ & \boldsymbol{A}\,\mathbf{x} = \boldsymbol{b} \\ & \mathbf{x} \geq 0. \end{aligned}$$

Suppose we have found an optimal solution to above LPP by using the simplex algorithm, and suppose that the cost coefficient of one (or more) of the variables is changed from c_k to c_k'. Changing the objective coefficients will not affect the primal feasibility, but could possibly effect the dual feasibility. Consider the following two cases:

Case I: $\boldsymbol{x_k}$ Is Nonbasic

Note that $\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}$ is effected; thus, the only impact of such a change is on the single tableau element, $z_k - c_k$. By letting c_k' be the new value of c_k, then $z_k - c_k'$ will replace $z_k - c_k$ in the optimal tableau. If $z_k - c_k'$ remains nonnegative, then the current basis remains optimal. However, if $z_k - c'_k < 0$, the dual feasibility (Primal optimality) has been lost and must be restored by using the primal simplex method. The value of $z_k - c'_k$ can be computed using the following relationship.

$$\begin{aligned} z_k - c'_k &= \boldsymbol{c_B^T B}^{-1} a_k - c'_k \\ &= \boldsymbol{c_B^T B}^{-1} a_k - c_k + c_k - c'_k \\ &= (z_k - c_k) + (c_k - c'_k) \qquad (2.49) \end{aligned}$$

Case II: $\boldsymbol{x_k}$ Is Basic, Say $\boldsymbol{x_k \equiv x_{B_t}}$

Because x_k is a basic variable, a change in c_{B_t} results in a change in the $\boldsymbol{c_B}$ vector. Thus, such a change can affect any or all of the $z_j - c_j$ elements and the value of $\boldsymbol{Z}$. let c'_{B_t} be the new value of c_{B_t} and let $\boldsymbol{c_B^{T\prime}}$ denote the revised $\boldsymbol{c_B^T}$. The $z_j - c_j$ elements associated with the basic variables will remain zero, so we only need to update the $z_j - c_j$ for the nonbasic variables as follows:

$$\begin{aligned} z'_j - c_j &= \boldsymbol{c_B^{T\prime} B}^{-1} \boldsymbol{a_j} - c_j \\ &= \boldsymbol{c_B^{T\prime} \alpha_j} - c_j \\ &= \sum_{i=1, i\neq t}^{m} c_{B_i} \alpha_{ij} + c'_{B_t} \alpha_{tj} - c_j \\ &= \sum_{i=1, i\neq t}^{m} c_{B_i} \alpha_{ij} + c_{B_t} \alpha_{tj} - c_{B_t} \alpha_{tj} + c'_{B_t} \alpha_{tj} - c_j \\ &= \sum_{i=1}^{m} c_{B_i} \alpha_{ij} - c_j + (c'_{B_t} - c_{B_t}) \alpha_{tj} \\ &= \left(z_j - c_j\right) + (c'_{B_t} - c_{B_t}) \alpha_{tj} \quad \text{for all } j. \qquad (2.50) \end{aligned}$$

In addition, the updated of the objective function is given by

$$\boldsymbol{\bar{Z}'} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}\prime}(\boldsymbol{B}^{-1}\boldsymbol{b})$$

$$= \textstyle\sum_{i=1,i\neq t}^{m} c_{B_i}(B^{-1}b)_i + c'_{B_t}(B^{-1}b)_t + c_{B_t}(B^{-1}b)_t - c_{B_t}(B^{-1}b)_t$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}(\boldsymbol{B}^{-1}\boldsymbol{b}) + (c'_{B_t} - c_{B_t})\bar{b}_t \quad ; \quad \bar{b}_i = (B^{-1}b)_i$$

$$= \boldsymbol{\bar{Z}} + \left(c'_{B_t} - c_{B_t}\right)\bar{b}_t \qquad (2.51)$$

If some $z'_j - c_j$ is negative, then dual feasibility (Primal optimality) must be restored by using the primal simplex method.

Example 2.9:

LPP11: maximize $\boldsymbol{Z} = 2x_1 + 3x_2 + 5x_3$

subject to

$$x_1 + 2x_2 + 3x_3 \leq 8$$
$$x_1 - 2x_2 + 2x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

The initial Tableau and optimal Tableau as shown in Tableau 2.21 and Tableau 2.22 respectively.

TABLEAU 2.21

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_4	1	2	3	1	0	8
x_5	1	-2	2	0	1	6
$\boldsymbol{Z}$	-2	-3	-5	0	0	0

TABLEAU 2.22

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1/4	1/4	-1/4	1/2
x_1	1	0	10/4	1/2	1/2	7
$\boldsymbol{Z}$	0	0	3/4	7/4	1/4	31/2

Suppose that $c_3 = 5$ is replaced by 6. Since x_3 is nonbasic, then $z_3 - c_3' = (z_3 - c_3) + (c_3 - c_3') = -1/4$, and all other $z_j - c_j$ unaffected. Hence, x_3 entering and x_2 departing the basis.

TABLEAU 2.23

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1/4	1/4	-1/4	1/2
x_1	1	0	10/4	1/2	1/2	7
$\boldsymbol{Z}$	0	0	-1/4	7/4	1/4	31/2

TABLEAU 2.24

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	4	1	1	-1	2
x_1	1	-10	0	-2	3	2
$\boldsymbol{Z}$	0	1	0	2	0	16

The Tableau 2.24 is optimal, and the optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*; \boldsymbol{Z}^*) = (2, 0, 2, 0, 0; 16)$.

Next, suppose that $c_{B_2} = c_1 = 2$ is replaced by zero, that is, $c_1' = 0 = c_{B_2}'$. Since x_1 is basic ($x_1 = x_{B_2}$), then the new cost row, except $z_1 - c_1$ is obtained by using (2.47) and (2.48) or by multiplying the row of x_1 by the net change in c_1[that is, $0 - 2 = -2$] and adding to the old cost row. The new $z_1 - c_1$ remains zero and we have the following tableau.

TABLEAU 2.25

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1/4	1/4	-1/4	1/2
x_1	1	0	10/4	1/2	1/2	7
$\boldsymbol{Z}$	0	0	-17/4	3/4	-3/4	3/2

Note that the new $z_3 - c_3$ and $z_5 - c_5$ are now negative, since the new $z_3 - c_3$ is the most negative , and hence x_3 entering and x_2 departing the basis, we have the following tableau.

TABLEAU 2.26

P11	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	4	1	1	-1	2
x_1	1	-10	0	-2	3	2
$\boldsymbol{Z}$	0	17	0	5	-5	10

since x_5 is negative, then x_5 entering and x_1 departing the basis, we obtain the optimal Tableau 2.27.

TABLEAU 2.27

P11	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1/3	2/3	1	1/3	0	8/3
x_5	1/3	-10/3	0	-2/3	1	2/3
$\boldsymbol{Z}$	5/3	1/3	0	5/3	0	40/3

2.6.3. Change in the Right-Hand Side [7],[11]

If a change in a particular b_i is made, there is an impact on both the $\bar{\mathbf{x}}_{\boldsymbol{B}}$ vector and the value of $\boldsymbol{Z}$**.** Recalling that $\bar{\mathbf{x}}_{\boldsymbol{B}}$ is given by $\boldsymbol{B}^{-1}\boldsymbol{b}$ and recalling that $\boldsymbol{B}^{-1}$can be found from the tableau by a proper arrangement of the α_j column vectors, we have

$$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \overline{\boldsymbol{b}'} = \boldsymbol{B}^{-1}\boldsymbol{b}' \qquad (2.52)$$

Where

$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \overline{\boldsymbol{b}'}$ = new value of the basic variables in the tableau of interest

$\boldsymbol{B}^{-1}$ = inverse of the present basis matrix

$\boldsymbol{b}'$ = new set of right-hand side constants

Also

$$\overline{\boldsymbol{Z}}' = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\overline{\boldsymbol{b}'} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b}' \qquad (2.53)$$

The basis inverse $\boldsymbol{B}^{-1}$ may contain negative elements, and thus there is always a possibility that $\overline{\boldsymbol{b}'}$ may include some negative elements. However, because dual feasibility is not effected, this presents no real problem because the dual simplex algorithm may be used to regain primal feasibility. This is illustrated in the following Example:

Example 2.10:

Suppose that the right-hand side of Example 2.9 is replaced by $\begin{bmatrix}1\\4\end{bmatrix}$. Note that $\boldsymbol{B}^{-1} = \begin{bmatrix}\frac{1}{4} & \frac{-1}{4}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$, and hence $\overline{\boldsymbol{b}'} = \boldsymbol{B}^{-1}\boldsymbol{b}' = \begin{bmatrix}\frac{1}{4} & \frac{-1}{4}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}\frac{-3}{4}\\ \frac{5}{2}\end{bmatrix}$

$$\overline{\boldsymbol{Z}}' = \boldsymbol{c}_{\boldsymbol{B}}\boldsymbol{B}^{-1}\boldsymbol{b}' = (3 \quad 2)\begin{bmatrix}\frac{-3}{4}\\ \frac{5}{2}\end{bmatrix} = \frac{11}{4}.$$

Then, $\boldsymbol{B}^{-1}\boldsymbol{b}' \leq 0$, and hence the new solution is infeasible ($x_2 = \frac{-3}{4}$) and dual feasible (all $z_j - c_j \geq 0$). Thus, the dual simplex algorithm can be employed.

TABLEAU 2.28

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1/4	1/4	-1/4	-3/4
x_5	1	0	10/4	1/2	1/2	5/2
$\boldsymbol{Z}$	0	0	3/4	7/4	1/4	11/4

the only negative $\bar{b}_i$ is $\bar{b}_1 = -3/4$. Thus, $x_{B_r} = x_2$ is departing variable, and x_5 is entering variable. We have tableau 2.29

TABLEAU 2.29

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_5	0	-4	-1	-1	1	3
x_1	1	2	3	1	0	1
$\boldsymbol{Z}$	1	1	1	2	0	2

The Tableau 2.29 is optimal and the optimal solution is $(x_1^*,\ x_2^*,\ x_3^*,$ $x_4^*,\ x_5^*;\ \boldsymbol{Z}^*) = (1, 0, 0, 0, 3; 2)$.

2.6.4. Change in A (change in the coefficients $\boldsymbol{a_{ij}}$) [7],[11]

We now discuss the effect of changing some of the coefficients a_{ij} of $\boldsymbol{A}$. The changes in the coefficients are relatively easy to handle if the a_{ij} associated with a nonbasic variable. However, a change in a_{ij} associated with a basic considerably more involved, and thus, for such a case, we shall resort to simply resolving the problem from the beginning.

Restricting our attention then to changes in the coefficients of nonbasic variables, we note that any change in the a_k column for a nonbasic variable x_k will directly affect the associated α_k vector (and, indirectly, the value of $z_k - c_k$). At any iteration , the α_k column vector is given by $\boldsymbol{B}^{-1}a_k$, so we have

$$\alpha'_k = \boldsymbol{B}^{-1}a'_k \qquad (2.54)$$

Where

$\boldsymbol{B}^{-1}$ = inverse of the present basis matrix

a'_k = new vector of coefficients associated with nonbasic variable x_k

α'_k = new updated vector corresponding to x_k in the final simplex tableau.

Now, suppose that the nonbasic column a_k is modified to a'_k. Then, the new updated column is

$$\alpha'_k = \boldsymbol{B}^{-1} a'_k$$

and

$$z'_k - c_k = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \boldsymbol{B}^{-1} a'_k - c_k = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \alpha'_k - c_k. \qquad (2.53)$$

If $z'_k - c_k \geq 0$, then the old solution is still optimal. Otherwise, the primal optimality (dual feasibility) has been lost and the primal simplex must be applied.

Example 2.11:

Suppose that in Example 2.9, a_3 is changed from $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$\alpha'_3 = \boldsymbol{B}^{-1} a'_3 = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and}$$

$$z'_3 - c_3 = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \alpha'_3 - c_3 = (3 \quad 2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 5 = -3 < 0.$$

Since $z'_3 - c_3$ is negative, we must apply the simplex method to have an optimal solution. Note that x_3 is entering and x_1 departing the basis, we obtain the optimal solution in tableau 2.31.

TABLEAU 2.30

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	0	1/4	-1/4	1/2
x_1	1	0	1	1/2	1/2	7
$\boldsymbol{Z}$	0	0	-3	7/4	1/4	31/2

TABLEAU 2.31

P11	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	0	1/4	-1/4	1/2
x_3	1	0	1	1/2	1/2	7
$\boldsymbol{Z}$	3	0	0	13/4	7/4	73/2

The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*; \boldsymbol{Z}^*) = (0, \frac{1}{2}, 7, 0, 0; 73/2)$.

2.6.5. Deletion of a variable [5],[10]

There are two cases:

Case (a): Deletion a nonbasic variable $\boldsymbol{x_k}$

Deletion of nonbasic variable is a totally superfluous operation and does not affect the feasibility and/or optimality of the current optimal solution, but the tableau will lose the column of that variable.

Case (b): Deletion of basic variable $\boldsymbol{x_{B_t} \equiv x_k}$

Deletion of a basic variable may affect the optimality and a new optimum solution may have to be found out. For this, a heavy penalty $-M$(M in case of minimization problems) is assigned to the variable under consideration and the new optimum solution is obtained by applying regular simplex method to the (modified) current optimum tableau. Calculate revised values of $\boldsymbol{Z}$ and $z_j - c_j$ as in equations (2.50) and (2.51), where $(c_{B_t} \to c'_{B_t} = -M)$, so we have

$$z'_j - c_j = \left(z_j - c_j\right) + (-M - c_{B_t})\alpha_{tj}$$

and

$$\overline{\boldsymbol{Z}}' = \overline{\boldsymbol{Z}} + \left(-M - c_{B_t}\right)\bar{b}_t \;\; ; \text{ where } \; \bar{b}_t = (B^{-1}b)_t$$

Example 2.12:

Consider the optimal tableau 2.6 of Example 2.3. let $x_2 \equiv x_{B_3}$ making changes in $z_j - c_j$ and $\boldsymbol{Z}$ accordingly, we have following tableau 2.32

TABLEAU 2.32

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	0	1	2	-3	10
x_1	1	0	0	1	-1	8
x_2	0	1	0	-1	2	4
$\boldsymbol{Z}$	0	0	0	$M+4$	$-2M-4$	$-4M+32$

Since x_5 is most negative, then x_5 entering the basis and x_2 departing it, thus we have the following tableau.

TABLEAU 2.33

P7	x_1	x_2	x_3	x_4	x_5	**RHS**
x_3	0	3/2	1	1/2	0	16
x_1	1	1/2	0	1/2	0	10
x_5	0	1/2	0	-1/2	1	2
$\boldsymbol{Z}$	0	$M+2$	0	2	0	40

Now, since x_2 is nonbasic, so on deleting α_2. The solution in above tableau is optimal, thus the optimal solution of the perturbed problem is $\boldsymbol{Z}^* = 40$ and $(x_1^*, x_3^*, x_4^*, x_5^*) = (10, 16, 0, 2)$.

2.6.6. Deletion of a constraint [5],[10]

There are two cases:

Case(a): Deletion of inactive constraint

An inactive constraint is one that one which corresponding slack or surplus variable would be basic and at nonzero level. Suppose we want to delete ith constraint which is inactive. Then delete the row and column of the slack/surplus variable corresponding to ith constraint. There will be no change in the optimal solution.

Case(b): Deletion of an active constraint

An active constraint is that one which corresponding slack or surplus variable would be nonbasic and at zero level. let ith constraint is active and we want to delete it. For this, we make this constraint inactive and then proceed as in case(a). To make in active its slack/surplus must be introduce into basis at positive level. So give slake /surplus high positive cost $+M(-M$ in minimization case) and calculate $z_j - c_j'$ for this slack/surplus variable and enter slack/surplus variable into basis at next iteration. This makes the constraint inactive cut the row and column of corresponding slack/surplus variable.

Example 2.13:

Consider the optimal tableau 2.34 of LPP2 in Example 1.2, is

TABLEAU 2.34

P2	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1	0	1	4
x_4	0	0	1	1	0	4
x_1	1	0	0	0	1	3
$\boldsymbol{Z}$	0	0	2	0	3	11

Note that $x_4 > 0$, so the second constraint is inactive. So to find the optimal solution of the perturbed problem, we delete the column α_4 and the second row from tableau 2.34, and there will be no change in $\mathbf{x}_B$, $\boldsymbol{Z}$ and $z_j - c_j$, this shown in Figure 2.6.

TABLEAU 2.35

P2	x_1	x_2	x_3	x_5	**RHS**
x_2	0	1	1	1	4
x_1	1	0	0	1	3
$\boldsymbol{Z}$	0	0	2	3	11

The solution is optimal $(x_1^*,\ x_2^*,\ x_3^*,\ x_5^*; \boldsymbol{Z}^*) = (3,4,0,0; 11)$.

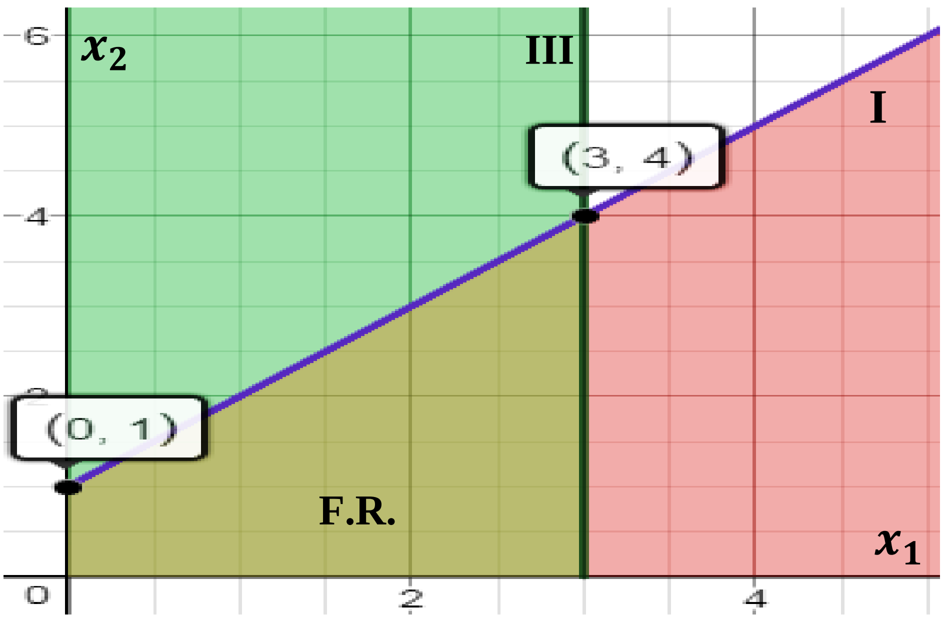

Figure 2.6

Example 2.14:

Consider the LPP2 in Example 1.2, with the optimal tableau 2.36.

TABLEAU 2.36

P2	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1	0	1	4
x_4	0	0	1	1	0	4
x_1	1	0	0	0	1	3
$\boldsymbol{Z}$	0	0	2	2	3	11

The third constraint is an active. To make it inactive, change $c_5(=0) \rightarrow c_5'(=M)$ and calculate $z_5 - c_5'$ as follows:

$$z_5 - c_5' = c_B^T \alpha_5 - c_5' = (2 \ \ 0 \ \ 1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - M = 3 - M \ ,$$

We have the tableau 2.37

TABLEAU 2.37

P2	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	0	1	1	0	1	4
x_4	0	0	1	1	0	4
x_1	1	0	0	0	1	3
$\boldsymbol{Z}$	0	0	2	2	$3 - M$	11

As $z_5 - c_5' < 0$, so x_5 undergoes change. Applying simplex algorithm repeatedly, we have

TABLEAU 2.38

P2	x_1	x_2	x_3	x_4	x_5	**RHS**
x_2	-1	1	1	0	0	1
x_4	0	0	1	1	0	4
x_5	1	0	0	0	1	3
$\boldsymbol{Z}$	$-3 + M$	0	0	0	0	$2 + 3M$

Note that $x_5 > 0$, so the third constraint is now inactive. On deleting α_5 and the third row in the above tableau and also making changes in $\boldsymbol{Z}$ and $z_j - c_j$, we have the following tableau.

TABLEAU 2.39

P7	x_1	x_2	x_3	x_4	RHS
x_1	-1	0	1	0	1
x_2	0	1	1	1	4
Z	-3	0	2	0	2

Note that the problem is infeasible. Thus the perturbed problem has not optimal solution, this shown in Figure 2.7.

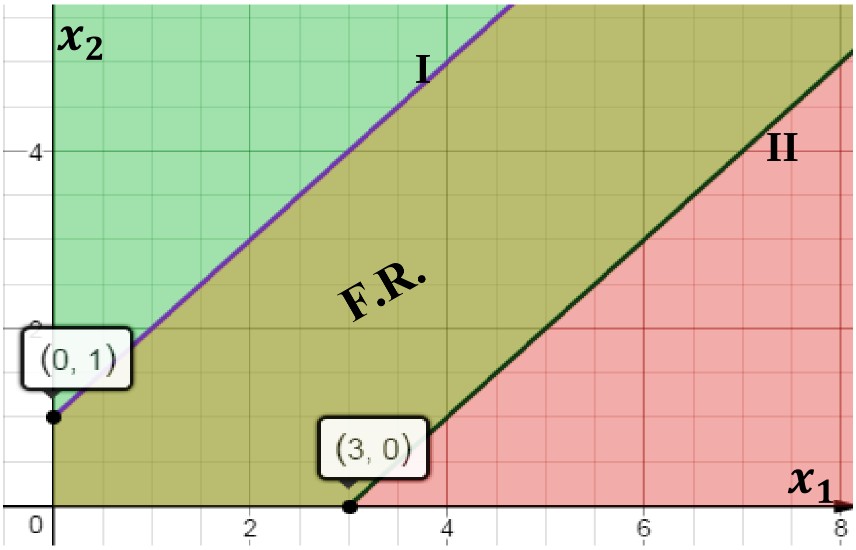

Figure 2.7

Chapter (3)

Bounded-Variable Simplex Method

3.1. Introduction

3.2. optimality conditions

3.3. The simplex algorithm for bounded variables

3.4. Finding an initial basic feasible solution to bounded variable

Chapter3

3.1. Introduction [7],[8]

In most practical problem the variables are usually bounded. A typical variable x_j is bounded from below by l_j and from above by u_j, respectively, we get the following linear programming with bounded variables (BLPP):

$$\text{BLPP : maximize } \boldsymbol{Z} = \boldsymbol{c}^T\,\mathbf{x} \tag{3.1a}$$

subject to

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b} \tag{3.1b}$$

$$l \leq \mathbf{x} \leq u \tag{3.1c}$$

where $\mathbf{x}$, $\boldsymbol{c}^T$, l, $u \in R^n$; $\boldsymbol{b} \in R^m$ and $\boldsymbol{A}$ is an $m \times n$ $(m \leq n)$ matrix .

We will make the following tow assumptions .

Assumption 1.The coefficient matrix $\boldsymbol{A}$ has full row rank i.e., rank($\boldsymbol{A}$) = m

Assumption 2. l is nonnegative vector .

Of course, it is possible to consider all the bounds as explicit constraint, however, this would effectively increase the size of the basis matrix from $m \times m$ to $(m + 2n)\times(m + 2n)$. Because operations involving the basis and basis inverse represent the largest part of the computational storage over-head, this is a very inefficient approach. The basic idea of bounded-variables simplex method is to handle the simplex bounds on the variables in an implicit manner (in a manner analogous to the handling of the nonnegativity restrictions in the standard simplex method). This allows us to maintain a standard $m \times m$ basis matrix, which is generally referred to as the *working basis*.

In the standard simplex method, nonbasic variables are those variables that are fixed at their lower bound value of zero. However, in the bounded-variables simplex method, a nonbasic variable represents a variable that is either fixed at its lower bound or upper bound. That is, the vector $\mathbf{x}$ will be partitioned into the basic variables $\mathbf{x}_{\boldsymbol{B}}$, the nonbasic variables at their lower bound $\mathbf{x}_{N_1}$, and the nonbasic variables at their upper bound $\mathbf{x}_{N_2}$ [7],[8].

Definition 3.1: (Basic Feasible Solution) [7],[8],[11]

The solution $\bar{\mathbf{x}}$ to the equation (3.1b) is a basic solution of this system if $\boldsymbol{A}$ can be partitioned into a nonsingular (working) basis matrix $\boldsymbol{B}$ and the matrices of nonbasic column N_1, and N_2. That is

$$\boldsymbol{A} = [\boldsymbol{B} \;\; N_1 \;\; N_2]$$

Now, the linear system $\boldsymbol{A}\mathbf{x} = \boldsymbol{b}$ can be rewritten to yield

$$\boldsymbol{B}\mathbf{x}_B + N_1\mathbf{x}_{N_1} + N_2\mathbf{x}_{N_2} = \boldsymbol{b}$$

This simplifies to

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1\mathbf{x}_{N_1} - \boldsymbol{B}^{-1}N_2\mathbf{x}_{N_2} \tag{3.2}$$

Now, setting $\mathbf{x}_{N_1} = l_{N_1}$ and $\mathbf{x}_{N_2} = u_{N_2}$, we see that (3.2) results in

$$\bar{\mathbf{x}}_B = \bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1 l_{N_1} - \boldsymbol{B}^{-1}N_2 u_{N_2} \tag{3.3}$$

The solution

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_{N_1} \\ \mathbf{x}_{N_2} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{b}} \\ l_{N_1} \\ u_{N_2} \end{bmatrix} \tag{3.4}$$

is called a basic solution. If, in addition, $l_B \le \bar{\mathbf{x}}_B \le u_B$ where $l_B(u_B)$ is a lower (upper) bound vector of basic variables, then the solution is a basic feasible solution, and if $l_B < \bar{\mathbf{x}}_B < u_B$, then $\bar{\mathbf{x}}$ is called a nondegenerate basic solution, otherwise, it is called a degenerate basic feasible solution (For more detail see reference [8]).

Chapter3

Improving a Basic Feasible Solution [7],[9],[11]

Suppose that we have a basis $\boldsymbol{B}$ and suppose that the nonbasic matrix N is decomposed into N_1, N_2, that is, $\boldsymbol{A} = [\boldsymbol{B}\ N_1\ N_2]$. Accordingly, the vector $\mathbf{x}$ is decomposed into $[\mathbf{x}_B\ \mathbf{x}_{N_1}\ \mathbf{x}_{N_2}]$ and $\boldsymbol{c}^{\boldsymbol{T}}$ is decomposed into $[\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\ \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\ \boldsymbol{c}_{N_2}^{\boldsymbol{T}}]$. Both the basic variables and the objective function can be represented in terms of the nonbasic vectors $\mathbf{x}_{N_1}$ and $\mathbf{x}_{N_2}$ as follows:

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1\mathbf{x}_{N_1} - \boldsymbol{B}^{-1}N_2\mathbf{x}_{N_2} \tag{3.5}$$

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\mathbf{x}_B + \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\mathbf{x}_{N_1} + \boldsymbol{c}_{N_2}^{\boldsymbol{T}}\mathbf{x}_{N_2}$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\left(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1\mathbf{x}_{N_1} - \boldsymbol{B}^{-1}N_2\mathbf{x}_{N_2}\right) + \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\mathbf{x}_{N_1} + \boldsymbol{c}_{N_2}^{\boldsymbol{T}}\mathbf{x}_{N_2}$$

$$= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \left(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_1 - \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\right)\mathbf{x}_{N_1} - (\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_2 - \boldsymbol{c}_{N_2}^{\boldsymbol{T}})\mathbf{x}_{N_2}. \tag{3.6}$$

Suppose we have current basic feasible solution where $\mathbf{x}_{N_1} = l_{N_1}$, $\mathbf{x}_{N_2} = u_{N_2}$, and $l_B \le \bar{\mathbf{x}}_B \le u_B$, then we have

$$\bar{\mathbf{x}}_B = \bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1 l_{N_1} - \boldsymbol{B}^{-1}N_2 u_{N_2} \tag{3.7}$$

$$\bar{\boldsymbol{Z}} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \left(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_1 - \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\right)l_{N_1} - (\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_2 - \boldsymbol{c}_{N_2}^{\boldsymbol{T}})u_{N_2}. \tag{3.8}$$

Letting J_1, J_2 denote the index sets of the variables that are nonbasic at their lower bounds, upper bounds, respectively, (3.5) and (3.6) can be rewritten as follows:

$$\boldsymbol{Z} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J_1}(z_j - c_j)\,x_j - \sum_{j\in J_2}(z_j - c_j)\,x_j \tag{3.9}$$

$$\mathbf{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J_1}(\boldsymbol{\alpha_j}\,x_j) - \sum_{j\in J_2}(\boldsymbol{\alpha_j}\,x_j) \tag{3.10}$$

Where $z_j = \boldsymbol{B}^{-1}\,\boldsymbol{\alpha_j}$; $\boldsymbol{\alpha_j} = \boldsymbol{B}^{-1}\boldsymbol{a_j}$; $\boldsymbol{a_j} \in N_j$.

Now, we try to improving the objective by investigating the possibility of modifying the nonbasic variables. For $j \in J_1$ if $z_j - c_j < 0$, it would be to our benefit to increase x_j from its current value of l_j. Similarly, for $j \in J_2$, if $z_j - c_j > 0$, it would be to our benefit to decrease x_j from its current value of u_j . As in the simplex method, we shall modify the value of only one nonbasic variable while all other nonbasic variables are fixed. The index k of this nonbasic variable is determined as follows:

$$z_k - c_k = maximum\{max_{j\in J_1} -(z_j - c_j)\, , max_{j\in J_2}(z_j - c_j)\} \qquad (3.11)$$

If this maximum is positive, then let k be the $k \in J_1$, then x_k is increased from its current level of l_k , and if $j \in J_2$, then x_k is decreased from its current level of u_k . If the maximum is nonpositive, then $z_j - c_j \geq 0$ for all $j \in J_1$ and $z_j - c_j \leq 0$ for all $j \in J_2$. Examining (3.9), this indicates that the current solution is optimal.

3.2. Optimality condition [7],[11]

Given a basic feasible solution $\bar{x}$ to BLPP in (3.1)

$$\text{if} \quad z_j - c_j \geq 0 \quad \text{for all } j \in J_1 \qquad (3.12)$$

and

$$\text{if} \quad z_j - c_j \leq 0 \quad \text{for all } j \in J_2 \qquad (3.13)$$

then the current solution is optimal.

The idea of the simplex method for BLPP is to move from basic feasible solution to basic feasible solution until the optimality conditions (3.12) and (3.13) are satisfied.

Increasing a Nonbasic Variable x_k From Its Lower Bound l_k

Suppose that $z_k - c_k < 0$ and x_k is currently nonbasic at its lower bound l_k.Then the solution can be improved by increasing x_k. Let $\Delta_k \geq 0$ be the increase in x_k , that is, the new value of x_k will be given by

$$x_k = l_k + \Delta_k \tag{3.14}$$

Because all other nonbasic variables remain fixed at either their lower or upper bounds, substituting into (3.5) and (3.6) yields

$$\hat{\mathbf{x}}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1 l_{N_1} - \boldsymbol{B}^{-1}N_2 u_{N_2} - \boldsymbol{B}^{-1}a_k\Delta_k$$

$$= \bar{\mathbf{x}}_B - \alpha_k\Delta_k \tag{3.15}$$

and

$$\widehat{\boldsymbol{Z}} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \left(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_1 - \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\right)l_{N_1} - \left(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_2 - \boldsymbol{c}_{N_2}^{\boldsymbol{T}}\right)u_{N_2} -$$

$$(z_k - c_k)\Delta_k$$

$$= \overline{\boldsymbol{Z}} - (z_k - c_k)\Delta_k \tag{3.16}$$

Maintain feasibility

To maintain feasibility, the value of Δ_k must be chosen to satisfy the following conditions.

$$l_k \leq x_k \leq u_k \Rightarrow l_k \leq l_k + \Delta_k \leq u_k \tag{3.17}$$

$$l_B \leq \hat{\mathbf{x}}_B \leq u_B \Rightarrow l_B \leq \bar{\mathbf{x}}_B - \alpha_k\Delta_k \leq u_B$$

$$\Rightarrow l_{B_i} \leq \bar{\mathbf{x}}_{B_i} - \alpha_{ik}\Delta_k \leq u_{B_i}, \quad \text{for all } i = 1, \dots, m. \tag{3.18}$$

Because $\Delta_k \geq 0$, it is follows from (3.17) that

$$\Delta_k \leq u_k - l_k \tag{3.19}$$

Now consider (3.18). If $\alpha_{ik} > 0$, then the basic variable x_{B_i} is decreasing and it follows that we must enforce

$$l_{B_i} \leq \bar{\mathbf{x}}_{B_i} - \alpha_{ik}\Delta_k \text{ ; for all } i \text{ such that } \alpha_{ik} > 0 \quad (3.20)$$

which yields

$$\Delta_k \leq \frac{\bar{\mathbf{x}}_{B_i} - l_{B_i}}{\alpha_{ik}} \quad ; \text{ for all } i \text{ such that } \alpha_{ik} > 0. \quad (3.21)$$

On the other hand, if $\alpha_{ik} < 0$, then the basic variable x_{B_i} is increasing and we must enforce

$$\bar{\mathbf{x}}_{B_i} - \alpha_{ik}\Delta_k \leq u_{B_i} \text{ ; for all } i = 1, \dots, m. \quad (3.22)$$

and, thus,

$$\Delta_k \leq \frac{u_{B_i} - \bar{\mathbf{x}}_{B_i}}{-\alpha_{ik}} \quad ; \text{ for all } i \text{ such that } \alpha_{ik} < 0. \quad (3.23)$$

Therefore. To determine the largest value of Δ_k that will result in a feasible solution, we use

$$\Delta_k = min\{\delta_1, \delta_2, u_k - l_k\} \quad (3.24)$$

where

$$\delta_1 = \begin{Bmatrix} \infty & , if\ \alpha_{ik} \leq 0 \\ min\left\{\frac{\bar{\mathbf{x}}_{B_i} - l_{B_i}}{\alpha_{ik}} : \alpha_{ik} > 0\right\} & , otherwise \end{Bmatrix} \quad (3.25)$$

$$\delta_2 = \begin{Bmatrix} \infty & , if\ \alpha_{ik} \geq 0 \\ min\left\{\frac{u_{B_i} - \bar{\mathbf{x}}_{B_i}}{-\alpha_{ik}} : \alpha_{ik} < 0\right\} & , otherwise \end{Bmatrix}. \quad (3.26)$$

<u>Note that</u> :

1. If $\Delta_k = \delta_1 = (\bar{\mathbf{x}}_{B_r} - l_{B_r})/\alpha_{rk}$, then the departing variable is x_{B_r} which becomes nonbasic at its lower bound.

2. If $\Delta_k = \delta_2 = (u_{B_r} - \bar{\mathbf{x}}_{B_r})/(-\alpha_{rk})$, then x_{B_r}departs at its upper bound.
3. $\Delta_k = u_k - l_k$, then the entering variable x_k blokes itself and x_k moves from nonbasic at its lower bound to nonbasic at its upper bound. In this case, the basis matrix remains the same; the only changes are $\bar{\boldsymbol{Z}}$ and $\bar{\mathbf{x}}_B$ according to (3.15) and (3.16).
4. If these computation result $\Delta_k = \infty$, then x_k can be increased without bound, consequently, no finite optimal solution exists.

Decreasing a Nonbasic Variable x_k From Its Upper Bound u_k

Now consider the case when $z_k - c_k > 0$ and x_k is currently nonbasic at its upper bound u_k. Then decreasing x_k will improve the objective value. Let $\Delta_k \geq 0$ be the amount by which x_k is decreased, that is, the value of x_k will be given by

$$x_k = u_k - \Delta_k. \tag{3.27}$$

Now, as in the previous case, (3.5) and (3.6) yield

$$\hat{\mathbf{x}}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}N_1 l_{N_1} - \boldsymbol{B}^{-1}N_1 u_{N_2} + \alpha_k \Delta_k$$

$$= \bar{\mathbf{x}}_B + \alpha_k \Delta_k \tag{3.28}$$

and

$$\hat{\boldsymbol{Z}} = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{b} - \left(\boldsymbol{c}_B \boldsymbol{B}^{-1}N_1 - \boldsymbol{c}_{N_1}^{\boldsymbol{T}}\right)l_{N_1} - \left(\boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}N_2 - \boldsymbol{c}_{N_2}^{\boldsymbol{T}}\right)u_{N_2} +$$

$$(z_k - c_k)\,\Delta_k$$

$$\hat{\boldsymbol{Z}} = \bar{\boldsymbol{Z}} + (z_k - c_k)\,\Delta_k \tag{3.29}$$

Maintain Feasibility

To maintain feasibility, the value of Δ_k must be chosen to satisfy the following conditions:

$$l_k \leq x_k \leq u_k \Rightarrow l_k \leq u_k - \Delta_k \leq u_k \quad (4.30)$$

$$l_B \leq \hat{\mathbf{x}}_B \leq u_B \Rightarrow l_B \leq \bar{\mathbf{x}}_B + \alpha_k \Delta_k \leq u_B$$

$$\Rightarrow l_{B_i} \leq \bar{\mathbf{x}}_{B_i} + \alpha_{ik}\Delta_k \leq u_{B_i} ; \quad \text{for all } i = 1, \dots, m. \quad (3.31)$$

Following the same logic as before, we see that Δ_k is defined as follows:

$$\Delta_k = min\{\delta_1, \delta_2, u_k - l_k\} \quad (3.32)$$

where

$$\delta_1 = \begin{Bmatrix} \infty & , if\ \alpha_{ik} \leq 0 \\ min\left\{\frac{u_{B_i} - \bar{\mathbf{x}}_{B_i}}{\alpha_{ik}} :\ \alpha_{ik} > 0\right\} & , otherwise \end{Bmatrix} \quad (3.33)$$

$$\delta_2 = \begin{Bmatrix} \infty & , if\ \alpha_{ik} \geq 0 \\ min\left\{\frac{\bar{\mathbf{x}}_{B_i} - l_{B_i}}{-\alpha_{ik}} :\ \alpha_{ik} < 0\right\} & , otherwise \end{Bmatrix} \quad (3.34)$$

Note that :

1. If $\Delta_k = \delta_1 = (u_{B_r} - \bar{\mathbf{x}}_{B_r})/\alpha_{rk}$, then the departing variable is x_{B_r} , which becomes nonbasic at its upper bound.
2. If $\Delta_k = \delta_2 = (\bar{\mathbf{x}}_{B_r} - l_{B_r})/(-\alpha_{rk})$, then x_{B_r} departs at its lower bound.
3. If $\Delta_k = u_k - l_k$, then the entering variable x_k blocs itself and x_k moves from nonbasic at its upper bound to nonbasic at its lower bound, and $\bar{\boldsymbol{Z}}$, $\bar{\mathbf{x}}_B$ being updated according to (3.28) and (3.29).

4. If these computations result $\Delta_k = \infty$, then x_k can be decreased without bound, and, consequently, no finite optimal solution exists.

3.3. The Simplex Algorithm for Bounded Variables (Maximization Problem) [7],[11]

Initialization step

Find a starting basic feasible solution. Let x_B be the basic variables and let x_{N_1} and x_{N_2} be the nonbasic variables at their lower and upper bounds, respectively. Form the following tableau 3.1, where

$$\bar{\boldsymbol{Z}} = \boldsymbol{c_B^T B^{-1} b} - (\boldsymbol{c_B^T B^{-1}} N_1 - \boldsymbol{c_{N_1}^T}) l_{N_1} - (\boldsymbol{c_B^T B^{-1}} N_2 - \boldsymbol{c_{N_2}^T}) u_{N_2}$$

and

$$\bar{\mathbf{x}}_B = \bar{\boldsymbol{b}} = \boldsymbol{B^{-1} b} - \boldsymbol{B^{-1}} N_1 l_{N_1} - \boldsymbol{B^{-1}} N_2 u_{N_2}$$

	$\mathbf{x}_B$	x_{N_1}	x_{N_2}	RHS
$\mathbf{x}_B$	**I**	$\boldsymbol{B^{-1}} N_1$	$\boldsymbol{B^{-1}} N_2$	$\bar{\boldsymbol{b}}$
$\boldsymbol{Z}$	0	$\boldsymbol{c_B^T B^{-1}} N_1 - \boldsymbol{c_{N_1}^T}$	$\boldsymbol{c_B^T B^{-1}} N_2 - \boldsymbol{c_{N_2}^T}$	$\bar{\boldsymbol{Z}}$

Tableau 3.1

Algorithm 3:(The Simplex Method for Bounded Variables)

STEP1 Check for possible improvement . Example the $z_j - c_j$ value for the nonbasic variables. If $z_j - c_j \geq 0$, for all $j \in J_1$, and $z_j - c_j \leq 0$, for all $j \in J_2$, then the current basic feasible solution is optimal; stop. Otherwise, select the nonbasic variable x_k as the entering variable with

$$z_k - c_k = maximum\{max_{j \in J_1} -(z_j - c_j), max_{j \in J_2} (z_j - c_j)\}.$$

If x_k is currently at its lower bound (i.e., $k \in J_1$), then go Step 2. If x_k is currently at its upper bound (i.e., $k \in J_2$), then go Step 3.

STEP2 Increase x_k from its current value of l_k, let $x_k = l_k + \Delta_k$

a) Compute Δ_k using (3.24-3.26). If $\Delta_k = \infty$, then the problem has an unbounded objective value; stop.

b) If $\Delta_k = u_k - l_k$, then x_k becomes nonbasic at its upper bound. Update the right-hand side of the tableau relationships defined by (3.15) and (3.16). The basis does not change and the remainder of the tableau remains the same. Return to Step 1.

c) If $\Delta_k = \delta_1$, then the departing variable x_{B_r} becomes nonbasic at its lower bound.

If $\Delta_k = \delta_2$, then the departing variable x_{B_r} becomes nonbasic at its upper bound.

The entering variable x_k is the new basic variable in row r with value $x_k = l_k + \Delta_k$. Update the remainder of right-hand side using the relationships defined by (3.15) and (3.16). update the remainder of the tableau by pivoting the usual manner on α_{rk}. Return to Step 1.

STEP3 Decrease x_k from its current value of u_k. Let $x_k = u_k - \Delta_k$.

a) Compute Δ_k using (3.32-3.34). If $\Delta_k = \infty$, then the problem has an unbounded objective value; stop.

b) If $\Delta_k = u_k - l_k$, then x_k becomes nonbasic at its lower bound. Update the right-hand side of the tableau using the relationships defined by (3.28) and (3.29), the basis does not change and the remainder of the tableau remains the same. Return to Step 1.

c) If $\Delta_k = \delta_1$, then the departing variable x_{B_r} becomes nonbasic at its upper bound.

If $\Delta_k = \delta_2$, then the departing variable x_{B_r} becomes nonbasic at its lower bound.

The entering variable x_k is the new basic variable in row *r* with value $x_k = u_k - \Delta_k$. Update the remainder of the right-hand side using the relationships defined by (3.28) and (3.29). Update the remainder of the tableau by pivoting in the usual manner on α_{rk}. Return to Step 1.

We illustrating this Algorithm by the following example.

Example 3.1:

BLPP1: maximize $\boldsymbol{Z} = 2x_1 + 3x_2$

subject to

$$x_1 + 2\,x_2 \leq 23$$
$$x_1 - \;\; x_2 \leq 2$$
$$0 \leq \; x_1 \leq 7$$
$$2 \leq \; x_2 \leq 10$$

Adding slack variables x_3 and x_4, the problem can be recast in the following form:

$$\text{maximize } \boldsymbol{Z} = 2x_1 + 3x_2 + 0x_3 + 0x_4 \qquad (3.35)$$

subject to

$$x_1 + 2\,x_2 + x_3 = 23$$
$$x_1 - \;\; x_2 + x_4 = 2$$
$$0 \leq \; x_1 \leq 7$$
$$2 \leq \; x_2 \leq 10$$
$$0 \leq \; x_3 \leq \infty$$
$$0 \leq \; x_4 \leq \; \infty$$

Notice that the coefficient matrix contains an imbedded identity, and thus it is possible that a nice starting basis is available. But, first, we fix $x_1$and

x_2 at either of their bounds and compute $\bar{\mathbf{x}}_B$. Arbitrarily set $x_1 = 0$ (lower bound) and $x_2 = 2$ (lower bound). Then

$$\bar{\mathbf{x}}_B = \begin{bmatrix} \bar{x}_{B_1} \\ \bar{x}_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 23 - 0 - 2(2) \\ 2 - 0 + 1(2) \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix} \geq 0.$$

Because $\bar{\mathbf{x}}_B \geq 0$, then x_3 and x_4 form a convenient starting basic variables with

$$\bar{\mathbf{x}}_B = \begin{bmatrix} \bar{x}_{B_1} \\ \bar{x}_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$$

If $\bar{\mathbf{x}}_B$ had not been nonnegative, then it would have been necessary to add artificial variables to form a starting basis. The big-M method could then be applied in attempt to drive the artificial variables to zero. [For an example of getting started under these conditions, (see Example 3.2)].

The current value of the objective can be computed from (3.35):

$$\bar{\boldsymbol{Z}} = 2(0) + 3(2) = 6$$

The initial tableau is depicted in tableau 3.2. Note that the nonbasic variables have been labeled to identify that they are presently nonbasic at their lower bounds.

TABLEAU 3.2

	l_1	l_2			
BP1	x_1	x_2	x_3	x_4	RHS
x_3	1	2	1	0	19
x_4	1	-1	0	1	4
$\boldsymbol{Z}$	-2	-3	0	0	6

STEP1 The current solution is clearly not optimal because x_1, x_2 are nonbasic at their lower bound and $z_1 - c_1 < 0$ and $z_2 - c_2 < 0$.

By using (3.11), x_2 is chosen as the entering variable (i.e., $k = 2$). Because x_2 is currently at its lower bound, go Step 2.

STEP2 Let $x_2 = l_2 + \Delta_2 = 2 + \Delta_2$

a) Compute Δ_2 using (3.24-3.26).

$$\delta_1 = \frac{\bar{\mathbf{x}}_{B_1} - l_{B_1}}{\alpha_{12}} = \frac{19-0}{2} = \frac{19}{2}$$

$$\delta_2 = \frac{u_{B_2} - \bar{x}_{B_2}}{-\alpha_{22}} = \frac{\infty - 4}{1} = \infty$$

$$u_2 - l_2 = 10 - 2 = 8$$

$$\Delta_2 = minimum\ \{\frac{19}{2}, \infty, 8\} = 8$$

b) $\Delta_2 = u_2 - l_2 = 8$; therefore, x_2 goes from nonbasic at its lower bound to nonbasic at its upper bound (i.e., $x_2 = 2 + 8 = 10$). Update the right-hand side using (3.15) and (3.16).

$$\hat{\boldsymbol{Z}} = 6 - (z_2 - c_2)\Delta_2 = 6 - (-3)8 = 30$$

$$\hat{\mathbf{x}}_{\mathrm{B}} = \begin{bmatrix} 19 \\ 4 \end{bmatrix} - \alpha_2\Delta_2 = \begin{bmatrix} 19 \\ 4 \end{bmatrix} - (8)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

The updated tableau is shown in Tableau 3.3. Note that the basic variables did not change. Return to Step 1.

TABLEAU 3.3

	l_1	u_2			
BP1	x_1	x_2	x_3	x_4	RHS
x_3	1	2	1	0	3
x_4	1	-1	0	1	12
$\boldsymbol{Z}$	-2	-3	0	0	30

STEP1 Select x_1 as the entering variable because $z_1 - c_1 < 0$ and x_1 is nonbasic at its lower bound. Go to Step 2.

STEP2 Let $x_1 = l_1 + \Delta_1 = \Delta_1$.

a) Compute Δ_1 using (3.24-3.26).

$$\delta_1 = min\left\{\frac{\bar{x}_{B_1} - l_{B_1}}{\alpha_{11}} = \frac{3-0}{1} = 3, \frac{\bar{x}_{B_2} - l_{B_2}}{\alpha_{21}} = \frac{12-0}{1} = 12\right\} = 3$$

$\delta_2 = \infty$

$u_1 - l_1 = 7 - 0 = 7$

$\Delta_1 = min\{3, \infty, 7\} = 3$

c) $\Delta_1 = \delta_1 = 3$; therefore, the departing variable is $x_{B_1} = x_3$, which becomes nonbasic at its lower bound. x_1 becomes the basic variable in row 1.

$$x_1 = \Delta_1 = 3$$

$$\hat{\boldsymbol{Z}} = 30 - (z_1 - c_1)\Delta_1 = 30 - (-2)3 = 36$$

$$\hat{x}_{B_2} = 12 - (3)1 = 9$$

The remainder of tableau is update by performing a standard pivot operation on $\alpha_{11} = 1$. Tableau 3.4 summarizes the results. Return to Step 1.

TABLEAU 3.4

		u_2	l_3		
BP1	x_1	x_2	x_3	x_4	RHS
x_1	1	2	1	0	3
x_4	0	-3	-1	1	9
$\boldsymbol{Z}$	0	1	2	0	36

STEP1 Select x_2 as the entering variable because $z_2 - c_2 > 0$ and x_2 is nonbasic at its upper bound. Go to Step 3.

STEP3 let $x_2 = u_2 - \Delta_2 = 10 - \Delta_2$

a) Compute Δ_2 using (3.32-3.34)

$$\delta_1 = \frac{u_{B_1} - \bar{x}_{B_1}}{\alpha_{12}} = \frac{7-3}{2} = 2$$

$$\delta_2 = \frac{\bar{x}_{B_2} - l_{B_2}}{-\alpha_{22}} = \frac{9-0}{3} = 3$$

$$u_2 - l_2 = 10 - 2 = 8$$
$$\Delta_2 = min\{2, 3, 8\} = 2$$

c) $\Delta_2 = \delta_1 = 2$; therefore, the departing variable is $x_{B_1} = x_1$, which becomes nonbasic at its Upper bound, and x_2 becomes basic in row 1.

$$x_2 = 10 - 2 = 8$$
$$\hat{\boldsymbol{Z}} = 36 + (1)2 = 38$$
$$\hat{x}_{B_2} = 9 + (-3)2 = 3$$

The remainder of the tableau is updated by performing a standard pivot operation on $\alpha_{12} = 2$. Tableau 3.5 summarizes the results. Return to Step 1.

TABLEAU 3.5

	u_1		l_3		
BP1	x_1	x_2	x_3	x_4	RHS
x_2	1/2	1	1/2	0	8
x_4	3/2	0	1/2	1	3
$\boldsymbol{Z}$	-1/2	0	3/2	0	38

STEP1 Tableau 3.5 represent the optimal solution, which be summarized as follows: $(x_1^*, x_2^*, x_3^*, x_4^*; \boldsymbol{Z}^*) = (7, 8, 0, 3; 38)$.

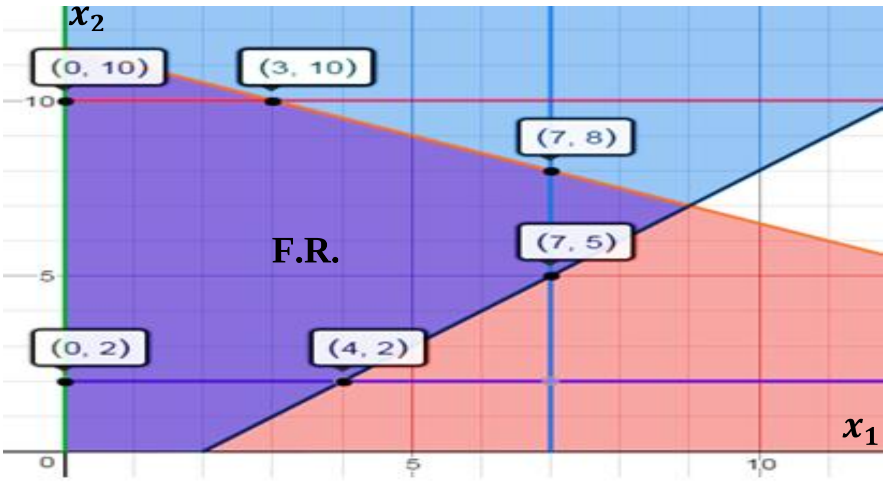

Figure 3.1

3.4. Finding an Initial Basic Feasible Solution to Bounded Variables [2]

If no basic feasible solution is conveniently available, we may start the lower-upper bound simplex method with artificial variables. This is accomplished by:

(1) Setting all of the original variables to one of their bounds;

(2) Adjusting the **RHS** values accordingly;

(3) Multiplying rows, as necessary, by -1 to get $\bar{\mathbf{x}}_B \geq 0$,

(4) Adding artificial columns.

Here we will use the big-*M* method to drive the artificial variables out of the basis, there is another method is could two-phase method may be employed to drive the artificial variables out of the basis.

Example 3.2:

BLPP2: maximize $\boldsymbol{Z} = 7x_1 + 9x_2$

subject to

$$
\begin{aligned}
-x_1 + x_2 &\leq 3 \\
x_1 + x_2 &\leq 8 \\
-x_1 + 2x_2 &\geq 1 \\
1 \leq x_1 &\leq 4 \\
2 \leq x_2 &\leq 6
\end{aligned}
$$

Now, placing the problem in standard form by adding slack variables yields

$$\text{maximize } \boldsymbol{Z} = 7x_1 + 9x_2 + 0x_3 + 0x_4 + 0x_5 \qquad (3.36)$$

subject to

$$-x_1 + x_2 + x_3 = 3 \qquad (3.37)$$

$$x_1 + x_2 + x_4 = 8 \qquad (3.38)$$

$$-x_1 + 2x_2 - x_5 = 1 \qquad (3.39)$$

$$
\begin{aligned}
1 \leq x_1 &\leq 4 \\
2 \leq x_2 &\leq 6 \\
0 \leq x_3, x_4, x_5 &\leq \infty
\end{aligned}
$$

Suppose we set $x_1 = 1$ (nonbasic at lower bound) and $x_2 = 6$ (nonbasic at upper bound) in (3.37 – 3.39) yield

$$x_3 = 3 + 1 - 6 = -2$$
$$x_4 = 8 - 1 - 6 = 1$$
$$-x_5 = 1 + 1 - 2(6) = -10$$

Because x_3 and x_5 are negative, we can multiply the first and third constraints by -1 to force x_3 and x_5 to become positive. Note that x_3 is negative (and thus infeasible), whereas in the resulting system, x_4 and x_5 are both positive and provide part of a starting basis. Therefore, we need to add an artificial variable x_6 to the constraint (3.37) after multiplying by -1. These operations result in the following problem.

$$\text{maximize } \boldsymbol{Z} = 7x_1 + 9x_2 - Mx_6 \qquad (3.40)$$

subject to

$$x_1 - x_2 - x_3 + x_6 = -3$$
$$x_1 + x_2 + x_4 = 8$$
$$x_1 - 2x_2 + x_5 = -1$$
$$1 \le x_1 \le 4$$
$$2 \le x_2 \le 6$$
$$0 \le x_3, x_4, x_5, x_6 \le \infty$$

Now, letting $x_1 = 1$ (lower bound), $x_2 = 6$ (upper bound), and $x_3 = 0$ (lower bound) yield

$$\bar{\mathbf{x}}_B = \begin{bmatrix} \bar{x}_{B_1} \\ \bar{x}_{B_2} \\ \bar{x}_{B_3} \end{bmatrix} = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 - 1 + 6 + 0 \\ 8 - 1 - 6 \\ -1 - 1 + 2(6) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 10 \end{bmatrix} \ge 0$$

Because $\bar{\mathbf{x}}_B \ge 0$, then $x_6, x_4,$ and x_5 form a convenient starting basis with

$$\bar{\mathbf{x}}_B = \begin{bmatrix} \bar{x}_{B_1} \\ \bar{x}_{B_2} \\ \bar{x}_{B_3} \end{bmatrix} = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 10 \end{bmatrix}$$

The initial objectives can be computed from (3.40).

$$\overline{\boldsymbol{Z}} = 7(1) + 9(6) - M(2)$$
$$= 61 - 2(M)$$

The initial tableau is shown in Tableau 3.6.

TABLEAU 3.6

	l_1	u_2	l_3				
BP2	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_6	1	-1	-1	0	0	1	2
x_4	1	1	0	1	0	0	1
x_5	1	-2	0	0	1	0	10
$\boldsymbol{Z}$	-7	-9	0	0	0	M	$61-2M$

Multiply row 1 by $(-M)$ and add to cost row, except $\bar{\mathbf{x}}_B$ and $\overline{\boldsymbol{Z}}$, we have tableau 3.7.

TABLEAU 3.7

	l_1	u_2	l_3				
BP2	x_1	x_2	x_3	x_4	x_5	x_6	**RHS**
x_6	1	-1	-1	0	0	1	2
x_4	1	1	0	1	0	0	1
x_5	1	-2	0	0	1	0	10
$\boldsymbol{Z}$	$(-7-M)$	$(-9+M)$	M	0	0	0	$61-2M$

STEP1 Since x_1 is nonbasic at its lower bound and $z_1 - c_1 < 0$, x_2 is nonbasic at its upper bound and $z_2 - c_2 > 0$. By using (3.11), x_1 is the entering variable. Because x_1 is currently at its lower bound, go Step 2.

STEP2 Let $x_1 = l_1 + \Delta_1 = 1 + \Delta_1$

a) Compute Δ_1 using (3.24-3.26)

$$\delta_1 = min\left\{\frac{\bar{\mathbf{x}}_{B_1}-l_{B_1}}{\alpha_{11}} = 2, \frac{\bar{x}_{B_2}-l_{B_2}}{\alpha_{21}} = 1, \frac{\bar{\mathbf{x}}_{B_3}-l_{B_3}}{\alpha_{31}} = 10\right\} = 1$$

$$\delta_2 = \infty$$

$$u_1 - l_1 = 4 - 1 = 3$$

$$\Delta_1 = min\{1, \infty, 3\} = 1 = \delta_1$$

c) $\Delta_1 = \delta_1 = 1$; therefore the departing variable is $x_{B_2} = x_4$, which becomes nonbasic at its lower bound. x_1 becomes the basic variable in row 2.

$$\hat{\mathbf{x}}_B = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 10 \end{bmatrix} - (1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}, \quad x_1 = 2$$

$$\hat{\boldsymbol{Z}} = (61 - 2M) - (-7 - M)(1) = 68 - M$$

The remainder of tableau is updating by performing a standard pivot operation on $\alpha_{21} = 1$. We have Tableau 3.8.

TABELAU 3.8

		u_2	l_3	l_4			
BP2	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_6	0	-2	-1	-1	0	1	1
x_1	1	1	0	1	0	0	2
x_5	0	-3	0	-1	1	0	9
$\boldsymbol{Z}$	0	$-2+2M$	M	$7+M$	0	0	$68-M$

STEP1 Select x_2 as the entering variable because $z_2 - c_2 > 0$, and x_2 is nonbasic at its upper bound, go Step 3.

STEP3 let $x_2 = u_2 - \Delta_2 = 6 - \Delta_2$.

a) Compute Δ_2 using (3.32-3.34)

$$\delta_1 = \frac{u_{B_2} - \bar{x}_{B_2}}{\alpha_{22}} = \frac{4-2}{1} = 2$$

$$\delta_2 = min\{ \frac{\bar{x}_{B_1} - l_{B_1}}{-\alpha_{12}} = \frac{1-0}{2} = \frac{1}{2}, \frac{\bar{x}_{B_3} - l_{B_3}}{-\alpha_{32}} = \frac{9-0}{3} = 3\} = \frac{1}{2}$$

$$u_2 - l_2 = 6 - 2 = 4$$

$$\Delta_2 = min\{2, \frac{1}{2}, 4\} = \frac{1}{2}$$

c) $\Delta_2 = \delta_2 = \frac{1}{2}$, therefore, the departing variable is $x_{B_1} = x_6$, which becomes nonbasic at its lower bound. x_2 becomes the basic variable in row 1.

$$\hat{\mathbf{x}}_B = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} + \left(\frac{1}{2}\right) \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2} \\ \frac{15}{2} \end{bmatrix}, \quad x_2 = \frac{11}{2}$$

$$\widehat{\boldsymbol{Z}} = (68 - M) + (-2 + 2M)\left(\frac{1}{2}\right) = 67 .$$

The remainder tableau is updating by performing a standard pivot operation on $\alpha_{12} = -2$, we have Tableau 3.9.

TABLEAU 3.9

BP2	x_1	x_2	x_3 (l_3)	x_4 (l_4)	x_5	x_6	RHS
x_2	0	1	1/2	1/2	0	-1/2	11/2
x_1	1	0	-1/2	1/2	0	1/2	5/2
x_5	0	0	3/2	1/2	1	-3/2	15/2
$\boldsymbol{Z}$	0	0	1	8	0	$-1+M$	67

Since $z_j - c_j \geq 0$ for each nonbasic variable at its lower bound. The last tableau gives an optimal solution, so the optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*; \boldsymbol{Z}^*) = (\frac{5}{2}, \frac{11}{2}, 0, 0, \frac{15}{2}; 67)$.

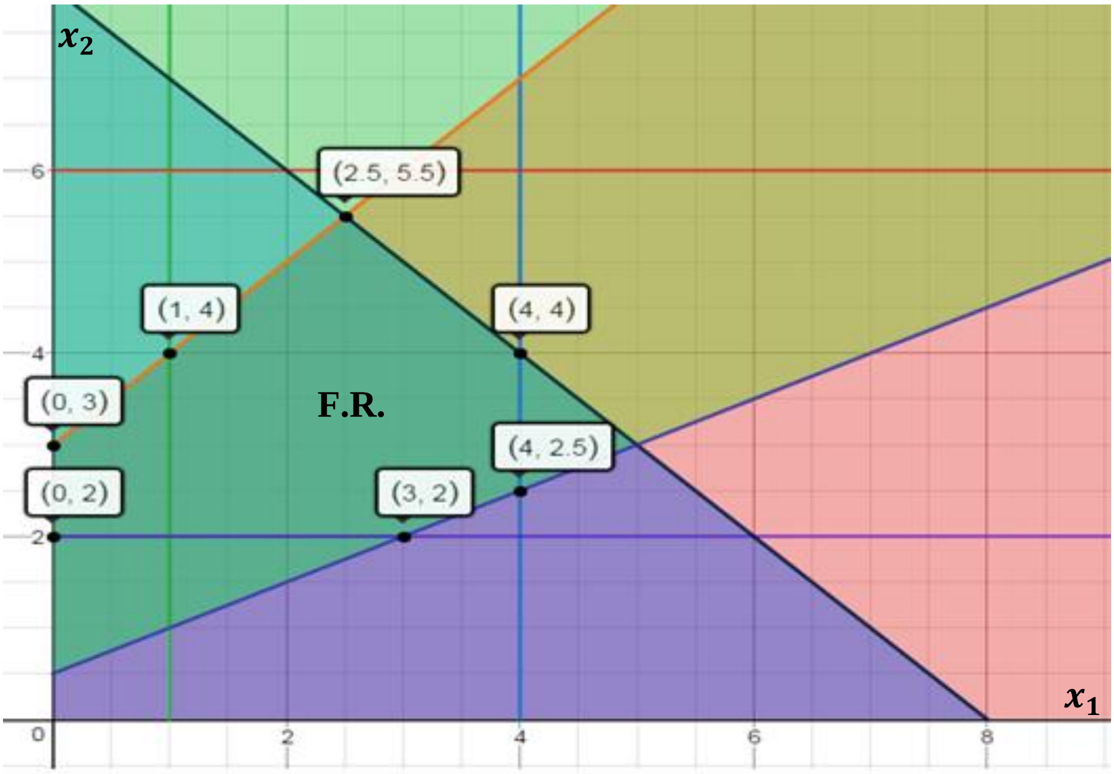

Figure 3.2

Chapter (4)

The Dual Simplex Method and Sensitivity Analysis

4.1. Duality

4.2. Dual Simplex Method

4.3. Sensitivity Analysis

Chapter4

4.1. Duality [1],[2],[9]

Consider the primal linear program with bounded variable in the (standard) form:

$$\text{BLPP:}\quad \text{maximize } \boldsymbol{Z} = \boldsymbol{c}^T\mathbf{x} \qquad (4.1a)$$

subject to

$$\boldsymbol{A}\,\mathbf{x} = \boldsymbol{b} \qquad (4.1b)$$

$$l \le \mathbf{x} \le u. \qquad (4.1c)$$

Then we can treat individual bounds (4.1c) like constraints and introduce dual variables $\boldsymbol{y} \in R^m$ for the constraints (4.1b), $v \in R^n$ for the lower bound constraints and $h \in R^n$ for the upper bound constraints in (4.1c). The constraints (4.1b) can be dualized by using the

Primal Problem	Dual Problem
Maximization problem	Minimization problem
Constraints i	Variables y_i
$a^i x \le b_i$	$y_i \ge 0$
$a^i x \ge b_i$	$y_i \le 0$
$a^i x = b_i$	y_i free
Variables x_j	Constraints j
$x_j \ge 0$	$a_j^T y \ge c_j$
$x_j \le 0$	$a_j^T y \le c_j$
x_j free	$a_j^T y = c_j$

Tableau 4.1: *Primal-Dual Transformation Rules*

third transformation rule, the lower bounds by using the first and the upper bound by using the second rule, where a^i are the rows vector of $\boldsymbol{A}$.

This leads to the following dual for BLPP :

DBLPP : minimize $\boldsymbol{W} = b^T\boldsymbol{y} + l^T v + u^T h$ (4.2a)

subject to

$$\boldsymbol{A}^T\boldsymbol{y} + v + h = \boldsymbol{c} \quad (4.2b)$$

$$v \leq 0, \quad h \geq 0 \text{ and } \boldsymbol{y} \text{ free} \quad (4.2c)$$

or as the form

DBLPP : minimize $\boldsymbol{W} = \boldsymbol{b}^T\boldsymbol{y} - l^T v + u^T h$ (4.3a)

subject to

$$\boldsymbol{A}^T\boldsymbol{y} - v + h = \boldsymbol{c} \quad (4.3b)$$

$$v \geq 0, \quad h \geq 0 \text{ and } \boldsymbol{y} \text{ free.} \quad (4.3c)$$

We call v_j and h_j dual slack variables. If the upper bound $u_j = \infty$ (infinite value) in the primal BLLP then the dual problem is in the form

DBLPP : minimize $\boldsymbol{W} = \boldsymbol{b}^T\boldsymbol{y} + l^T v + \sum_{\{j:u_j<\infty\}} u_j^T h_j$ (4.4a)

subject to

$$a_j^T\boldsymbol{y} + v_j + h_j = c_j \quad \text{if } l_j \geq 0 \text{ and } u_j < \infty \quad (4.4b)$$

$$a_j^T\boldsymbol{y} + v_j = c_j \quad \text{if } l_j \geq 0 \text{ and } u_j = \infty \quad (4.4c)$$

$$v_j \leq 0 \quad \text{if } l_j \geq 0 \quad (4.4d)$$

$$h_j \geq 0 \quad \text{if } u_j < \infty \quad (4.4e)$$

Example 4.1:

BLPP3: maximize $\boldsymbol{Z} = 2x_1 + 4x_2 + x_3$

subject to

$$2x_1 + x_2 - x_3 \leq 10$$
$$x_1 + x_2 - x_3 \leq 4$$
$$1 \leq x_1 \leq 4$$
$$0 \leq x_2 \leq 6$$
$$1 \leq x_3 \leq 4.$$

To formulate this problem in standard form, we must introduce the slack variables x_4 and x_5. These are bounded below by zero and bounded above by ∞, we have

maximize $\boldsymbol{Z} = 2x_1 + 4x_2 + x_3 + 0x_4 + 0x_5$
subject to

$$2x_1 + x_2 - x_3 + x_4 = 10$$
$$x_1 + x_2 - x_3 + x_5 = 4$$
$$1 \le x_1 \le 4$$
$$0 \le x_2 \le 6$$
$$1 \le x_3 \le 4$$
$$0 \le x_4 \le \infty$$
$$0 \le x_5 \le \infty$$

DBP3: minimize $\boldsymbol{W} = 10y_1 + 4y_2 + v_1 + v_3 + 4h_1 + 6h_2 + 4h_3$
subject to

$$2\,y_1 + y_2 + v_1 + h_1 = 2$$
$$y_1 + y_2 + v_2 + h_2 = 4$$
$$-\,y_1 - y_2 + v_3 + h_3 = 1$$
$$y_1 \qquad + v_4 \qquad = 0$$
$$y_2 \qquad + v_5 \qquad = 0$$

y_i free, $v_j \le 0$, $h_j \ge 0$, where $i = 1, 2$; $j = 1, 2, 3, 4, 5$.

<u>Notation</u>:

1. Any vector $(\boldsymbol{y}^T, v^T, h^T)^T \in R^{m+2n}$ that satisfies the dual constraints (4.4b), (4.4c) is called a dual solution.
2. If a dual solution additionally satisfies constraints (4.4d), (4.4e), it is called a dual feasible solution.
3. If no dual feasible solution exists, BLPP is said to be a dual infeasible. Otherwise feasible.
4. If for every M ∈ R there is a dual feasible solution $(\boldsymbol{y}^T, v^T, h^T)^T$ such that $\boldsymbol{b}^T\boldsymbol{y} + l^T v + \sum_{\{j:u_j<\infty\}} u_j^T\, h_j >$M, then BLPP is dual unbounded.

4.2. Dual Simplex Method For Bounded Variables: [8]

The dual simplex method starts with a dual feasible basis but primal infeasible and walks to a terminal basis by moving along adjacent dual feasible basis. At each pivot step, this method tries to reduce primal infeasibility while retaining dual feasibility.

Let $\boldsymbol{B}$ be a known dual feasible basis and $\mathbf{x}_B = (x_{B_1}, x_{B_2}, \ldots, x_{B_n})$ be the associated basic vector. Suppose rth basic variable x_{B_r} is not within its boundes, so we depart this basic variable and enter some nonbasic variable say $\boldsymbol{a_k} \notin \boldsymbol{B}$.

There are two possibilities. Either x_{B_r} is below its lower bound or above its upper bound.

Case(I): If $\bar{x}_{B_r}$ is below its lower bound. While applying dual simplex iteration in this case, our aim is to increase x_{B_r} till its attains its lower bound. Again there are two possibilities

(i) $\boldsymbol{a_k} \in N_1$ (ii) $\boldsymbol{a_k} \in N_2$.

(i) Let $\boldsymbol{a_k} \in N_1$, which is currently nonbasic and at its lower bound with $z_k - c_k \geq 0$, is selected for replacing B_r, where $\boldsymbol{B} = (B_1\ B_2 \ldots B_m)$.

Let $\hat{x}_k = l_k + \Delta_k$; where Δ_k is nonnegative and determined by $\hat{x}_{B_r} = \bar{x}_{B_r} - \alpha_{rk}\Delta_k = l_{B_r}$; where α_{rk} is the pivot element.

Note that $\alpha_{rk} \leq 0$. Since $\Delta_k > 0$ and for increasing x_{B_r} , α_{rk} should be negative.

$$\Rightarrow \quad \Delta_k = \frac{l_{B_r} - \bar{x}_{B_r}}{-\alpha_{rk}} \quad . \qquad (4.5)$$

When $\boldsymbol{a_k}$ enters and B_r departs, then

$$\hat{z}_j - c_j = (z_j - c_j) - \frac{\alpha_{rj}}{\alpha_{rk}}(z_k - c_k), \quad \forall \quad \boldsymbol{a_j} \notin \boldsymbol{B} . \tag{4.6}$$

For maintain optimality, $\hat{z}_j - c_j \geq 0$, $\forall$ $\boldsymbol{a_j} \in N_1$ and $\hat{z}_j - c_j \leq 0$, $\forall$ $\boldsymbol{a_j} \in N_2$ **.**

For $\alpha_{rj} < 0, \hat{z}_j - c_j \geq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \geq \frac{z_j - c_j}{\alpha_{rj}} \ \forall \ \boldsymbol{a_j} \in N_1. \tag{4.7}$$

Clearly from (4.6), for $\alpha_{rj} \leq 0$, $\hat{z}_j - c_j \leq 0$, for $\boldsymbol{a_j} \in N_2$.

For $\alpha_{rj} > 0$, $\hat{z}_j - c_j \leq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \geq \frac{z_j - c_j}{\alpha_{rj}} \ \forall \ \boldsymbol{a_j} \in N_2. \tag{4.8}$$

Relations (4.7) and (4.8) imply that

$$\frac{z_k - c_k}{\alpha_{rk}} = max\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} < 0 \,, \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} > 0\right\} \tag{4.9}$$

(ii) Let $\boldsymbol{a_k} \in N_2$, which is currently nonbasic and at its upper bound with $(z_k - c_k) \leq 0$ is selected for replacing B_r .

Let $\hat{x}_k = u_k - \Delta_k$, where Δ_k is nonnegative and determined by $\hat{x}_{B_r} = \bar{x}_{B_r} + \alpha_{rk}\,\Delta_k = l_{B_r}$, where α_{rk} is the pivot element.

Note that $\alpha_{rk} \geq 0$. Since $\Delta_k > 0$ and for increasing x_{B_r}, α_{rk} should be positive.

$$\Rightarrow \ \Delta_k = \frac{l_{B_r} - \bar{x}_{B_r}}{\alpha_{rk}} \tag{4.10}$$

For maintaining optimality, $\hat{z}_j - c_j \geq 0$, $\forall$ $a_j \in N_1$ and $\hat{z}_j - c_j \leq 0$, $\forall$ $\boldsymbol{a_j} \in N_2$.

Form (4.6), for $\boldsymbol{a_j} \in N_1$, $\hat{z}_j - c_j \geq 0$, for $\alpha_{rj} \geq 0$.

For $\alpha_{rj} < 0$, $\hat{z}_j - c_j \geq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \geq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall\, \boldsymbol{a_j} \in N_1. \tag{4.11}$$

Form (4.6), for $\boldsymbol{a_j} \in N_2$, $\hat{z}_j - c_j \leq 0$, for $\alpha_{rj} \leq 0$.

For $\alpha_{rj} < 0$, $\hat{z}_j - c_j \leq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \geq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall\, \boldsymbol{a_j} \in N_2. \tag{4.12}$$

Relations (4.11) and (4.12) imply that

$$\frac{z_k - c_k}{\alpha_{rk}} = max\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} < 0, \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} > 0\right\} \tag{4.13}$$

Which is same as (4.9).

Case(II): if $\bar{x}_{B_r}$ is above its upper bound. While applying dual simplex iteration in this case, our aim is to decrease $\bar{x}_{B_r}$ till it attains its upper bound. Again there are two possibilities

(i) $\boldsymbol{a_k} \in N_1$ (ii) $\boldsymbol{a_k} \in N_2$.

(i) Let $a_k \in N_1$, which is currently nonbasic and at its lower bound with $z_k - c_k \geq 0$, is selected for replacing B_r , where $\boldsymbol{B} = (B_1\ B_2 \ldots B_m)$.

Let $\hat{x}_k = l_k + \Delta_k$, where Δ_k is nonnegative and determined by $\hat{x}_{B_r} = \bar{x}_{B_r} - \alpha_{rk}\, \Delta_k = u_{B_r}$, where α_{rk} is the pivot element.

Note that $\alpha_{rk} \geq 0$. Since $\Delta_k > 0$ and for decreasing x_{B_r}, α_{rk} should be positive.

$$\Rightarrow \quad \Delta_k = \frac{\bar{x}_{B_r} - u_{B_r}}{\alpha_{rk}} \tag{4.14}$$

As in case(I), for maintaining optimality, $\hat{z}_j - c_j \geq 0, \forall\ \boldsymbol{a_j} \in N_1$ and $\hat{z}_j - c_j \leq 0,\ \forall\ \boldsymbol{a_j} \in N_2$.

Form (4.6), for $\boldsymbol{a_j} \in N_1$, $\hat{z}_j - c_j \geq 0$, for $\alpha_{rj} \leq 0$.

For $\alpha_{rj} > 0, \hat{z}_j - c_j \geq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \leq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall\ \boldsymbol{a_j} \in N_1. \tag{4.15}$$

Form (4.6), for $\boldsymbol{a_j} \in N_2$, $\hat{z}_j - c_j \leq 0$, for $\alpha_{rj} \geq 0$.

For $\alpha_{rj} < 0, \hat{z}_j - c_j \leq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \leq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall\ \boldsymbol{a_j} \in N_2. \tag{4.16}$$

Relations (4.15) and (4.16) imply that

$$\frac{z_k - c_k}{\alpha_{rk}} = min\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} > 0 , \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} < 0\right\} \tag{4.17}$$

(ii) Let $\boldsymbol{a_k} \in N_2$, which is currently nonbasic and at its upper bound whit $(z_k - c_k) \leq 0$ is selected for replacing B_r.

Let $\hat{x}_k = u_k - \Delta_k$, where Δ_k is nonnegative and determined by

$\hat{x}_{B_r} = \bar{x}_{B_r} + \alpha_{rk}\, \Delta_k = u_{B_r}$, where α_{rk} is the pivot element.

Note that $\alpha_{rk} \leq 0$. Since $\Delta_k > 0$ and for decreasing x_{B_r} , α_{rk} should be negative.

$$\Rightarrow \quad \Delta_k = \frac{\bar{x}_{B_r} - u_{B_r}}{-\alpha_{rk}} \tag{4.18}$$

For maintaining optimality, $\hat{z}_j - c_j \geq 0$, $\forall\, \boldsymbol{a_j} \in N_1$ and $\hat{z}_j - c_j \leq 0$, $\forall\ \boldsymbol{a_j} \in N_2$ **.**

Form (4.6), for $\boldsymbol{a_j} \in N_1$, $\hat{z}_j - c_j \geq 0$, for $\alpha_{rj} \leq 0$.

For $\alpha_{rj} > 0, \hat{z}_j - c_j \geq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \leq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall \ \boldsymbol{a_j} \in N_1. \tag{4.19}$$

Form (4.6), for $\boldsymbol{a_j} \in N_2$, $\hat{z}_j - c_j \leq 0$, for $\alpha_{rj} \geq 0$.

For $\alpha_{rj} < 0, \hat{z}_j - c_j \leq 0$ if

$$\frac{z_k - c_k}{\alpha_{rk}} \leq \frac{z_j - c_j}{\alpha_{rj}} \quad \forall \ \boldsymbol{a_j} \in N_2. \tag{4.20}$$

Relations (4.19) and (4.20) imply that

$$\frac{z_k - c_k}{\alpha_{rk}} = min\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} > 0, \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} < 0\right\} \tag{4.21}$$

which is same as (4.17).

Result 1 (Primal infeasibility criterion) : The original BLPP is infeasible if corresponding to a dual feasible basis $\boldsymbol{B}$, there exists an i such that either

(I) $x_{B_i} = \bar{b}_i < l_{B_i}$ and $\alpha_{ij} \geq 0 \ \ \forall \ \boldsymbol{a_j} \in N_1$ and $\alpha_{ij} \leq 0 \ \ \forall \boldsymbol{a_j} \in N_2$,

or

(II) $x_{B_i} = \bar{b}_i > u_{B_i}$ and $\alpha_{ik} \leq 0 \ \ \forall \ \boldsymbol{a_j} \in N_1$ and $\alpha_{ij} \geq 0 \ \ \forall \boldsymbol{a_j} \in N_2$.

Result 2 (Dual simplex entering criterion) : If some x_{B_r} ($< l_{B_r}$) is chosen to leave the basis then the variable x_k enters the basis if

$$\frac{z_k - c_k}{\alpha_{rk}} = max\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} < 0, \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} > 0\right\},$$

and if some x_{B_r} ($> u_{B_r}$) is chosen to leave the basis then the variable x_k enters the basis if

$$\frac{z_k - c_k}{\alpha_{rk}} = min\left\{\frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_1, \alpha_{rj} > 0, \frac{z_j - c_j}{\alpha_{rj}} : \boldsymbol{a_j} \in N_2, \alpha_{rj} < 0\right\}.$$

If any of the criterion is not applicable, then there exists no feasible solution to the BLPP (by Result 1).

Algorithm 4: (The Dual Simplex Method For bounded Variables)

STEP1 Convert the minimization problem into maximization if it is minimization form. Convert the $\geq$ type inequalities, representing the constraints of the given linear programming, if any, into those of $\leq$ type. Call this problem as (BLPP).

STEP2 Introduce slack variables in the constraints of the given problem and obtain an initial basic dual feasible solution and consider the corresponding starting dual simplex table.

STEP3 Test the nature of $(z_j - c_j)$ in the starting simplex table.

a) If $l_j \leq x_j \leq u_j \ \ \forall$ j = 1, 2,…, n and $z_j - c_j \geq 0 \ \ \forall \ \boldsymbol{a_j} \in N_1$ and $z_j - c_j \leq 0 \ \ \forall \ \boldsymbol{a_j} \in N_2$, then an optimal basis feasible solution of (BLPP) has been obtained.

b) If $z_j - c_j \geq 0 \ \ \forall \ \boldsymbol{a_j} \in N_1$ and $z_j - c_j \leq 0 \ \ \forall \ \boldsymbol{a_j} \in N_2$ and at least one basic variable say x_{B_i} is not within its bounds, then go to step 4(a) or 4(b) accordingly as $x_{B_i} < l_{B_i}$ or $x_{B_i} > u_{B_i}$.

STEP4 a) Select that basic variable x_{B_i} for which $\left|x_{B_i} - l_{B_i}\right|$ is maximum. Let $x_{B_r} \equiv x_k$ be such that $|x_k - l_k|$ is maximum so that $\boldsymbol{\alpha_k}$ leaves the basis. Go step 5(a).

b) Select that basic variable x_{B_i} for which $\left|x_{B_i} - u_{B_i}\right|$ is maximum. Let $x_{B_r} \equiv x_k$ be such that $|x_k - u_k|$ is maximum so that $\boldsymbol{\alpha_k}$ leaves the basis. Go to step 5(b).

STEP5 a) Test the nature of $\alpha_{kj}, j = 1,2,\dots,n$.

(I) If $\alpha_{kj} \geq 0 \;\; \forall \;\; \boldsymbol{a_j} \in N_1$ and $\alpha_{kj} \leq 0 \;\; \forall \;\; \boldsymbol{a_j} \in N_2$, there does not exist any Feasible solution to the given problem (by Result 1).

(II) If at least one α_{kj} is negative for some $\boldsymbol{a_j} \in N_1$ or α_{kj} is positive for some $\boldsymbol{a_j} \in N_2$, compute the replacement ratios

$$\left\{\left(\frac{z_j - c_j}{\alpha_{kj}} : \boldsymbol{a_j} \in N_1, \alpha_{kj} < 0\right), \left(\frac{z_j - c_j}{\alpha_{kj}} : \boldsymbol{a_j} \in N_2, \alpha_{kj} > 0\right)\right\}$$

And choose the maximum of these. The corresponding column vector, say $\boldsymbol{\alpha_r}$, then enter the basis.

b) Test the nature of α_{kj} , $j = 1,2,\dots,n$.

(I) If $\alpha_{kj} \leq 0 \;\; \forall \;\; \boldsymbol{a_j} \in N_1$ and $\alpha_{kj} \geq 0 \;\; \forall \;\; \boldsymbol{a_j} \in N_2$,there does not exist any feasible solution to the given problem (by Result 1).

(II) If at least one $\alpha_{kj}$is positive for some $\boldsymbol{a_j} \in N_1$or $\alpha_{kj}$is negative for some $\boldsymbol{a_j} \in N_2$, compute the replacement ratios

$$\left\{\left(\frac{z_j - c_j}{\alpha_{kj}} : \boldsymbol{a_j} \in N_1, \alpha_{kj} > 0\right), \left(\frac{z_j - c_j}{\alpha_{kj}} : \boldsymbol{a_j} \in N_2, \alpha_{kj} < 0\right)\right\}$$

and choose the minimum of these. The corresponding column vector, say $\boldsymbol{\alpha_k}$, enter the basis.

STEP6 Test the new iterated dual simplex table for dual optimality. Repeat the method until either an optimum feasible solution has been obtained (in a finite number of steps) or there is an indication of nonexistence of a primal feasible solution.

<u>Remark 1</u>: If upper bounds of all the decision variables are finite then primal problem is bounded. If not so, then the dual problem is known to be feasible, the primal problem cannot be unbounded, by weak duality theorem. The algorithm discussed here will terminate with a basis that satisfies either the optimality criterion or primal infeasibility criterion.[13]

4.3. Sensitivity Analysis

Consider the following problem:

BLPP: maximize $\boldsymbol{Z} = \boldsymbol{c}^T\mathbf{x}$

subject to

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{b}$$
$$l \leq \mathbf{x} \leq u.$$

Suppose that the **Algorithm3** produces an optimal basis **B**. we shall describe how to make use of the optimality conditions to find a new optimal solution, if some of problem data change. In particular, the following variations in the problem will be considered.

Change in the cost vector $\boldsymbol{c}^T$.

Change in the right-hand-side vector $\boldsymbol{b}$**.**

Change in the bounded of the variables.

Change in $\boldsymbol{A}$ (change in the coefficient matrix a_{ij}).

Deletion of a variable.

Deletion of a constraint.

4.3.1. Change in the Cost Vector $\boldsymbol{c}^T$

Given an optimal basic feasible solution, suppose that the cost coefficient of one (or more) of the variables is changed from c_k to $\grave{c}_k$. The effect of this change on the final tableau will occur in the cost row; that is, optimality (dual feasibility) might be lost. Consider the following two cases:

Case(I): x_k Is Nonbasic

In this case, $\boldsymbol{c_B^T}$ is not affected, and hence, $z_j = \boldsymbol{c_B^T \alpha_j}$ is not changed for any j .Thus, $z_k - c_k$ is replaced by $z_k - c'_k$. Now we have two possibilities: Either $c_k \in c_{N_1}$ or $c_k \in c_{N_2}$.
Case (a): If $c_k \in c_{N_1}$, then

$$z_k - c'_k = (z_k - c_k) + (c_k - c'_k) \tag{4.22}$$

and

$$\begin{aligned} \overline{\boldsymbol{Z}}' &= \boldsymbol{c_B^T}(\boldsymbol{B}^{-1}\boldsymbol{b}) - \textstyle\sum_{j\in J_1}[(z_j - c_j) + (z_k - c'_k)]\, l_j - \sum_{j\in J_2}(z_j - c_j)u_j \\ &= \boldsymbol{c_B^T}(\boldsymbol{B}^{-1}\boldsymbol{b}) - \textstyle\sum_{j\in J_1, j\neq k}(z_j - c_j)l_j - (z_k - c'_k)\, l_k - \sum_{j\in J_2}(z_j - c_j)u_j \\ &= \boldsymbol{c_B^T}(\boldsymbol{B}^{-1}\boldsymbol{b}) - \textstyle\sum_{j\in J_1}(z_j - c_j)l_j - (c_k - c'_k)\, l_k - \sum_{j\in J_2}(z_j - c_j)u_j \\ &= \overline{\boldsymbol{Z}} - (c_k - c'_k)l_k \,. \end{aligned} \tag{4.23}$$

Note that, if $z_k - c'_k < 0$, then x_k must be introduced into the basis and the **Algorithm3** is continued as usual. Otherwise, the old solution is still optimal.

Case(b): if $c_k \in c_{N_2}$, then,

$$z_k - c'_k = (z_k - c_k) + (c_k - c'_k) \tag{4.24}$$

and

$$\overline{\boldsymbol{Z}}' = \overline{\boldsymbol{Z}} - (c_k - c'_k)u_k \,. \tag{4.25}$$

If $z_k - c'_k > 0$, then x_k must be introduced into the basis. Otherwise, the old solution is still optimal with respect to the new problem.

Case(II): $\boldsymbol{x_k}$ Is Basic, Say $\boldsymbol{x_k \equiv x_{B_t}}$

Here, c_{B_t} is replaced by c'_{B_t}. Let the new values of z_j be z'_j and $\overline{\boldsymbol{Z}}$ be $\overline{\boldsymbol{Z}}'$. Then $z'_j - c_j$ and $\overline{\boldsymbol{Z}}'$ are calculated as follows:

$$z'_j - c_j = \boldsymbol{c_B^{T\prime}} \boldsymbol{\alpha_j} - c_j$$

$$= \sum_{i=1, i\neq t}^{m} c_{B_i} \alpha_{ij} + c'_{B_t} \alpha_{tj} - c_j$$

$$= \sum_{i=1, i\neq t}^{m} c_{B_i} \alpha_{ij} + c_{B_t} \alpha_{tj} - c_{B_t} \alpha_{tj} + c'_{B_t} \alpha_{tj} - c_j$$

$$= \sum_{i=1}^{m} c_{B_i} \alpha_{ij} - c_j + (c'_{B_t} - c_{B_t})\, \alpha_{tj}$$

$$= \left(z_j - c_j\right) + (c'_{B_t} - c_{B_t}) \alpha_{tj} \quad \text{for all } j\,. \tag{4.26}$$

In particular, for j = k, $z_k - c_k = 0$, and $\alpha_{tk} = 1$, and hence, $z'_k - c_k = c'_k - c_k$, so

$$z'_k - c'_k = z'_k - c_k + c_k - c'_k$$

$$= (z'_k - c_k) - (c'_k - c_k)$$

$$= (c'_k - c_k) - (c'_k - c_k) = 0. \tag{4.27}$$

That is mean $z'_k - c'_k$ is still equal to zero. Therefore, the cost row can be updated by adding the net change in the cost of $x_{B_t} \equiv x_k$ times the current t row of the final tableau, to the original cost row. Then, $z'_k - c_k$ is updated to $z'_k - c'_k = 0$.

$$\overline{\boldsymbol{Z}}' = \boldsymbol{c_B^{T\prime}}(\boldsymbol{B^{-1}b}) - \sum_{j\in J_1}(z'_j - c_j) l_j - \sum_{j\in J_2}(z'_j - c_j) u_j$$

$$= \sum_{i=1, i\neq t}^{m} c_{B_i} (B^{-1}b)_i + c'_{B_t}(B^{-1}b)_t - \sum_{j\in J_1} \left(z_j - c_j\right) l_j -$$

$$\sum_{j\in J_1}\left(c'_{B_t} - c_{B_t}\right) \alpha_{tj} l_j - \sum_{j\in J_2}\left(z_j - c_j\right) u_j - \sum_{j\in J_2}\left(c'_{B_t} - c_{B_t}\right) \alpha_{tj} u_j$$

$$+ c_{B_t}(B^{-1}b)_t - c_{B_t}(B^{-1}b)_t$$

$$= \overline{\boldsymbol{Z}} + \left(c'_{B_t} - c_{B_t}\right) \overline{b}_t; \tag{4.28}$$

Where $\overline{b_i} = (B^{-1}b)_i - \sum_{j\in J_1} \alpha_{ij}\, l_j - \sum_{j\in J_2} \alpha_{ij}\, u_j$.

Example 4.2:

BLLP4: maximize $\boldsymbol{Z} = 4x_1 - 2x_2 + x_3 + 2x_4 + x_5$
subject to

$$-x_1 - 2x_2 + x_3 + 2x_4 - x_5 \le 3$$
$$x_1 + x_2 + x_3 + x_4 + 2x_5 \le 4$$
$$0 \le x_j \le 1 \qquad \forall \quad j = 1, 2, 3, 4, 5.$$

The initial and optimal tableau are showing in Tableau 4.1 and Tableau 4.2 respectively.

TABLEAU 4.1

	l_1	l_2	l_3	l_4	l_5			
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1	-2	1	2	-1	1	0	3
x_7	1	1	1	1	2	0	1	4
$\boldsymbol{Z}$	-4	2	-1	-2	-1	0	0	0

TABLEAU 4.2

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

Suppose that $c_2 =$ -2 is replaced by 1. Since x_2 is nonbasic at its lower bound, the $z_2 - c_2' = (z_2 - c_2) + (c_2 - c_2') = \frac{5}{2} - 3 = -\frac{1}{2}$, and all other $z_j - c_j$ are unaffected. The new objective value

$$\overline{\boldsymbol{Z}}' = \overline{\boldsymbol{Z}} - (c_2 - c_2')l_2 = \frac{15}{2}.$$

we have Tableau 4.3 .

TABLEAU 4.3

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	-1/2	-1/2	-3/2	0	0	1/2	15/2

Since x_2 is nonbasic at its lower bound and $(z_2 - c_2) < 0$, x_2 must be entering the basis. Let $x_2 = l_2 + \Delta_2 = \Delta_2$. Compute Δ_2 by using (3.24-3.26), we have $\Delta_2 = \delta_1 = 1$, therefore, the departing variable is $x_{B_2} = x_5$, which becomes nonbasic at its lower bound. x_2 becomes basic variable in row 2.

$$\hat{\mathbf{x}}_{\boldsymbol{B}} = \begin{bmatrix} x_6 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} - (1) \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \hat{\boldsymbol{Z}} = \frac{15}{2} - \left(-\frac{1}{2}\right) = \frac{16}{2} = 8,\ x_2 = 1.$$

The remainder of the tableau is updating by performing a standard pivot operation on $\alpha_{22} = \frac{1}{2}$. we obtained the Tableau 4.4.

TABLEAU 4.4

	u_1		u_3	u_4	l_5		l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	1	0	3	4	3	1	2	3
x_2	1	1	1	1	2	0	1	1
$\boldsymbol{Z}$	-3	0	0	-1	1	0	1	8

The Tableau 4.4 is optimal, and the optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*,$ $x_6^*\ , x_7^*; \boldsymbol{Z}^*) = (1,1,1,1,0,3,0; 8)$

Next, suppose that $c_5 = 1$ is replaced by -2. Since x_5 is basic (in tableau 4.2), then the new cost row, except $z_5 - c_5$, is obtained by multiplying the row of x_5 by the net change in c_5 [that is, $-2 - 1 = -3$] and adding to the old cost row. The new $z_5 - c_5$ remains zero, and the new objective value $\overline{\boldsymbol{Z}}' = \overline{\boldsymbol{Z}} + (c_5' - c_5)\, x_{B_5} = \frac{15}{2} + (-3)\left(\frac{1}{2}\right) = 6$. Note that the new $z_7 - c_7$ is now positive and x_7 nonbasic at its lower bound, so x_7 entering the basic.

TABLEAU 4.5

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-5	1	-2	-3	0	0	-1	6

Let $x_7 = l_7 + \Delta_7 = \Delta_7$.

Compute Δ_7 using (3.24-3.26), we have $\Delta_7 = \delta_1 = 1$, so $x_{B_2} = x_5$ departing the basis at its lower bound.

$$\hat{\mathbf{x}}_{\boldsymbol{B}} = \begin{bmatrix} x_6 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} - (1)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_7 = 1, \text{ and } \ \hat{\boldsymbol{Z}} = 6 - 1(-1) = 7$$

The reminder of the Tableau is updating by performing a standard pivot operation on $\alpha_{27} = \frac{1}{2}$. We have optimal Tableau.

TABLEAU 4.6

	u_1	l_2	u_3	u_4	l_5			
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1	-2	1	2	-1	1	0	1
x_7	1	1	1	1	2	0	1	1
$\boldsymbol{Z}$	-4	2	-1	-2	2	0	0	7

The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*, x_6^*, x_7^*; \boldsymbol{Z}^*) = (1,0,1,1,0,1,1; 7)$

4.3.2. Change in the Right-Hand Side [8],[11]

If the right-hand side vector $\boldsymbol{b}$ is replaced by $\boldsymbol{b}'$, then $\bar{\mathbf{x}}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \sum_{j\in J_1} \boldsymbol{\alpha_j} l_j - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j$ will be replaced by

$$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b}' - \sum_{j\in J_1} \boldsymbol{\alpha_j} l_j - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j\ . \qquad (4.29)$$

The new right-hand side can be calculated without explicitly evaluating $\boldsymbol{B}^{-1}\boldsymbol{b}'$. This is evident by noting that $\boldsymbol{B}^{-1}\boldsymbol{b}' = \boldsymbol{B}^{-1}\boldsymbol{b} + \boldsymbol{B}^{-1}(\boldsymbol{b}' - \boldsymbol{b})$.

Hence

$$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \bar{\mathbf{x}}_{\boldsymbol{B}} + \boldsymbol{B}^{-1}(\boldsymbol{b}' - \boldsymbol{b}). \qquad (4.30)$$

Since $z_j - c_j \geq 0$ for all nonbasic variables at its lower bound, and $z_j - c_j \leq 0$ for all nonbasic variables at its upper bound, the only possible violation of optimality is that the new vector $\bar{\mathbf{x}}'_{\boldsymbol{B}}$ my have some entries are not within them bounds. If $\boldsymbol{l_B} \leq \bar{\mathbf{x}}'_{\boldsymbol{B}} \leq \boldsymbol{u_B}$, then the same basis remains optimal. Otherwise, the **Algoritm4** can be used to find a new optimal solution by restoring primal feasibility. The new value of the objective function is

$$\bar{\boldsymbol{Z}}' = \bar{\boldsymbol{Z}} + \boldsymbol{c_B^T}\boldsymbol{B}^{-1}(\boldsymbol{b}' - \boldsymbol{b}). \qquad (4.31)$$

Example 4.3:

Suppose that the right-hand-side of Example 4.2 is replaced by $\begin{bmatrix}6\\2\end{bmatrix}$.

Note that, $\boldsymbol{B}^{-1}(\boldsymbol{b}'-\boldsymbol{b}) = \begin{bmatrix}1 & \frac{1}{2}\\0 & \frac{1}{2}\end{bmatrix}\left(\begin{bmatrix}6\\2\end{bmatrix}-\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}1 & \frac{1}{2}\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

and hence $\bar{\mathbf{x}}_{\boldsymbol{B}}' = \begin{bmatrix}\frac{3}{2}\\\frac{1}{2}\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}\frac{7}{2}\\\frac{-1}{2}\end{bmatrix}$, also $\bar{\boldsymbol{Z}}' = \frac{15}{2} + (0 \quad 1)\begin{bmatrix}2\\-1\end{bmatrix} = \frac{13}{2}$.

We obtain Tableau 4.7.

TABLEAU 4.7

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	7/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	-1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	13/2

Note that $x_5 = -\frac{1}{2} < l_5 = 0$, this means the new solution is not feasible, x_5 departing the basis. Applying **Algorithm4**, first compute the replacement ratios by using the relation (4.9).

$$max\left\{\frac{z_1-c_1}{\alpha_{21}} = -7, \frac{z_3-c_3}{\alpha_{23}} = -1, \frac{z_4-c_4}{\alpha_{24}} = -3\right\} = -1 = \frac{z_3-c_3}{\alpha_{23}}$$

and hence x_3 enters the basis, since x_3 nonbasic at its upper bound.

Let $x_3 = u_3 - \Delta_3 = 1 - \Delta_3$, and computing Δ_3 as follows:

$$\Delta_3 = \frac{l_{B_2} - \bar{x}_{B_2}}{\alpha_{23}} = \frac{0-\left(-\frac{1}{2}\right)}{\frac{1}{2}} = 1$$

Hence $\quad x_3 = 0, \hat{\mathbf{x}}_{\boldsymbol{B}} = \begin{bmatrix} x_6 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \end{bmatrix} + (1) \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$

And $\quad \widehat{\boldsymbol{Z}} = \frac{13}{2} + 1\left(-\frac{1}{2}\right) = 6.$

We obtain the optimal Tableau 4.8.

TABLEAU 4.8

	u_1	l_2		u_4	l_5		l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-2	-3	0	1	-3	1	-1	5
x_3	1/2	1/2	1/2	1/2	1	0	1/2	0
$\boldsymbol{Z}$	-3	3	0	-1	1	0	1	6

The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*, x_6^*, x_7^*; \boldsymbol{Z}^*) = (1,0,0,1,0,5,0; 6)$.

4.3.3. Change in the bounded of variables

Given an optimal basic feasible solution, suppose that the lower(upper) bound or both of one (or more) of the variable is changed where $0 \le l_j' < u_j'$. In this case the optimality is maintained but feasibility may be hampered. Consider the following two cases:

Case(a): $\boldsymbol{x_k}$ Is Nonbasic

In this case there are two possibilities

(i) $\quad x_k = l_k$

(ii) $\quad x_k = u_k$

(i) let $x_k = l_k$ and the lower bound l_k was changed into l_k' , then the new values of $\mathbf{x}_{\boldsymbol{B}}$ and $\boldsymbol{Z}$ can be calculate as follows

$$\bar{\mathbf{x}}_{\boldsymbol{B}}' = \bar{\mathbf{x}}_{\boldsymbol{B}} + (l_k - l_k')\alpha_k \tag{4.32}$$

and

$$\bar{\boldsymbol{Z}}' = \bar{\boldsymbol{Z}} + (z_k - c_k)(l_k - l_k') \tag{4.33}$$

If $\bar{\mathbf{x}}_{\boldsymbol{B}}'$ feasible, then the new solution is optimal. Otherwise apply **Algorithm4** and proceed.

(ii) $x_k = u_k$, parallel to (i).

Case (b): $\boldsymbol{x_k}$ Is Basic say $\boldsymbol{x_k \equiv x_{B_r}}$

If the lower bound l_k of x_k is replaced by l_k' and/or upper bound u_k of x_k is replaced by u_k' and if $l_k \leq l_k' \leq x_k \leq u_k \leq u_k'$, then the solution is still feasible and it is optimal. Otherwise applying **Algorithm4** and proceed.

Example 4.4:

Consider the BLPP4 in Example 4.2 with the optimal Tableau 4.9.

TABLEAU 4.9

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

Suppose that the lower bound $l_2 = 0$ of x_2 is replaced by $l_2' = 2$, and the upper bound $u_2 = 1$ is replaced by $u_2' = 4$. The new value of basic variables is $\bar{\mathbf{x}}_{\boldsymbol{B}}' = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + (0-2)\begin{bmatrix} \frac{-3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ \frac{-1}{2} \end{bmatrix}$, and the new objective value of objective function is

$$\bar{\boldsymbol{Z}}' = \frac{15}{2} + \left(\frac{5}{2}\right)(-2) = \frac{5}{2}$$

we obtain new Tableau 4.10.

TABLEAU 4.10

	u_1	l_2'	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	9/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	-1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	5/2

As $x_5 = \frac{-1}{2} < l_5 = 0$, this means the new solution is not feasible, x_5 departing the basis. Applying **Algorithm4**, first compute the replacement ratios by using the relation (4.9).

$$max\left\{\frac{z_1-c_1}{\alpha_{21}} = -7, \frac{z_3-c_3}{\alpha_{23}} = -1, \frac{z_4-c_4}{\alpha_{24}} = -3\right\} = -1 = \frac{z_3-c_3}{\alpha_{23}}$$

and hence x_3 enters the basis, since x_3 is nonbasic at its upper bound.

Let $x_3 = u_3 - \Delta_3 = 1 - \Delta_3$, compute Δ_3.

$$\Delta_3 = \frac{l_{B_2} - \bar{x}_{B_2}}{\alpha_{23}} = \frac{0-(\frac{-1}{2})}{\frac{1}{2}} = 1, \text{ hence } x_3 = 0.$$

$$\hat{\mathbf{x}}_{\boldsymbol{B}} = \begin{bmatrix} x_6 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ \frac{-1}{2} \end{bmatrix} + (1)\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{Z}} = \frac{5}{2} + (1)\left(\frac{-1}{2}\right) = 2.$$

We obtain the optimal Tableau 4.10.

TABLEAU 4.11

	u_1	l_2		u_4	l_5		l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-2	-3	0	1	-3	1	-1	6
x_3	1	1	1	1	2	0	1	0
$\boldsymbol{Z}$	-3	3	0	-1	1	0	1	2

The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*, x_6^*, x_7^*; \boldsymbol{Z}^*) = (1,0,0,1,0,6,0; 2)$

4.3.4. Change in the Coefficient Matrix *A* [3]

The changes in the coefficients are relatively easy to handle if the a_k to be changed are associated with a nonbasic variable. However, a change in an associated with a basic variable is considerably more involved, and thus, for such a case, we shall resolving the problem for the beginning.

Let $\boldsymbol{B}$ be the optimal feasible basis for the original problem and $\mathbf{A} = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \boldsymbol{a_k}\ \ldots \boldsymbol{a_n}]$ and a_k undergoes change, and let $\boldsymbol{a_k} \notin \boldsymbol{B}$. There are two cases:

Case I : $\boldsymbol{a_k} \in N_1$, Case II : $\boldsymbol{a_k} \in N_2$.

If $\boldsymbol{a_k} \in N_1$, then $x_k = l_k$.

$$\text{Let } \boldsymbol{a_k} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{rk} \\ \vdots \\ a_{mk} \end{pmatrix} \Rightarrow \boldsymbol{a'_k} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a'_{rk} \\ \vdots \\ a_{mk} \end{pmatrix} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{rk} \\ \vdots \\ a_{mk} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (a'_{rk} - a_{rk}) \\ 0 \\ \vdots \end{pmatrix}.$$

This means $\boldsymbol{a'_k} = \boldsymbol{a_k} + e_r\ (a'_{rk} - a_{rk})$,

so $\boldsymbol{\alpha'_k} = \boldsymbol{B}^{-1}\boldsymbol{a'_k} = \boldsymbol{\alpha_k} + \overleftarrow{\beta}_k\ (a'_{rk} - a_{rk})$, where $\boldsymbol{B}^{-1} = (\overleftarrow{\beta}_1 \quad \overleftarrow{\beta}_2 \quad \ldots \quad \overleftarrow{\beta}_k \quad \ldots \quad \overleftarrow{\beta}_m)$.

and

$$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \sum_{j\in J_1, j\neq k}(\boldsymbol{\alpha_j} l_j) - \boldsymbol{\alpha'_k} l_k - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j$$

$$= \boldsymbol{B}^{-1}\boldsymbol{b} - \sum_{j\in J_1, j\neq k}(\boldsymbol{\alpha_j} l_j) - [\boldsymbol{\alpha_k} + \overleftarrow{\beta}_k\,(a'_{rk} - a_{rk})] l_k - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j$$

$$= \boldsymbol{B}^{-1}\boldsymbol{b} - \sum_{j\in J_1}(\boldsymbol{\alpha_j} l_j) - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j - \overleftarrow{\beta}_k\,(a'_{rk} - a_{rk}) l_k$$

$$\bar{\mathbf{x}}'_{\boldsymbol{B}} = \bar{\mathbf{x}}_B - \overleftarrow{\beta}_k\,(a'_{rk} - a_{rk}) l_k. \qquad (4.34)$$

Also $z'_j - c_j = z_j - c_j \quad \forall\ j \in (J_1 \cup J_2);\ j \neq k$

and

$$z'_k - c_k = \boldsymbol{c_B^T}\boldsymbol{\alpha'_k} - c_k$$

$$= \boldsymbol{c_B^T}[\boldsymbol{\alpha_k} + \overleftarrow{\beta}_k\,(a'_{rk} - a_{rk})] - c_k$$

$$= \boldsymbol{c_B^T}\boldsymbol{\alpha_k} - c_k + \boldsymbol{c_B^T}\,\overleftarrow{\beta}_k\,(a'_{rk} - a_{rk})$$

$$= (z_k - c_k) + \boldsymbol{c_B^T}\,\overleftarrow{\beta}_k\,(a'_{rk} - a_{rk}) \qquad (4.35)$$

$$\bar{\boldsymbol{Z}}' = \boldsymbol{c_B^T}(\boldsymbol{B}^{-1}\boldsymbol{b}) - \sum_{j\in J_1}(z'_j - c_j)\, l_j - \sum_{j\in J_2}(z'_j - c_j)\, u_j$$

$$= \boldsymbol{c_B^T}(\boldsymbol{B}^{-1}\boldsymbol{b}) - \sum_{j\in J_1, j\neq k}(z_j - c_j)\, l_j - \sum_{j\in J_2}(z_j - c_j)\, u_j -$$

$$[(z_k - c_k) + \boldsymbol{c_B^T}\,\overleftarrow{\beta}_k\,(a'_{rk} - a_{rk})] l_k$$

$$= \bar{\boldsymbol{Z}} - \boldsymbol{c_B^T}\,\overleftarrow{\beta}_k\,(a'_{rk} - a_{rk}) l_k. \qquad (4.36)$$

So change in $\boldsymbol{a_k} \in N_1$ affects both optimality as well as feasibility, similarly, if $\boldsymbol{a_k} \in N_2$ undergoes change parallel results will be obtained by replacing l_k by u_k .

<u>Remark</u> 2:(Solution of the problem when some $\boldsymbol{a_k} \in \mathrm{N}_1$ undergoes change)

If only optimality is hampered, apply **Algorithm3** and solve. If only feasibility is hampered, applying **Algorithm4** and solve. If both optimality as well as feasibility are hampered, then x_k which is currently at lower bound, set at its upper bound $x_k = u_k$ and calculate

$$\hat{\mathbf{x}}_{\boldsymbol{B}} = \bar{\mathbf{x}}'_{\boldsymbol{B}} - \boldsymbol{\alpha}'_{\boldsymbol{k}} u_k,$$

and

$$\widehat{\boldsymbol{Z}} = \overline{\boldsymbol{Z}}' - (z'_k - c'_k) u_k$$

All other relative cost coefficients, basis and $\boldsymbol{\alpha_j}$ remain unaltered during this change. Now x_k is at its upper bound and $z_k - c_k < 0$. So this solution is optimal but need not be feasible. If $\bar{\mathbf{x}}_{\boldsymbol{B}}$ is feasible, then it is optimal basic feasible solution, otherwise apply **Algorithm4** and solve. Similarly we can solve for $\boldsymbol{a_k} \in N_2$.

Example 4.5:

Suppose that in Example 4.2, $\boldsymbol{a_1}$ is changed from $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then,

$$\boldsymbol{\alpha}'_{\boldsymbol{1}} = \boldsymbol{\alpha_1} + \overleftarrow{\beta}_1 (a'_{11} - a_{11}) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (3) = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$z'_1 - c_1 = (z_1 - c_1) + \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \overleftarrow{\beta}_1 (a'_{11} - a_{11})$$

$$= \left(-\frac{7}{2}\right) + (0 \quad 1) \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\frac{7}{2}$$

$$\bar{\mathbf{x}}'_B = \bar{\mathbf{x}}_B - \overleftarrow{\beta}_1 (a'_{11} - a_{11})u_s = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} (3)(1) = \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}.$$

We obtained Tableau 4.12.

TABLEAU 4.12

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	5/2	-3/2	3/2	5/2	0	1	1/2	-3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

Note that this solution is optimal (dual feasible), but not feasible, since $x_6 = -\frac{3}{2} < l_6 = 0$. Apply **Algorithm4**, compute the replacement ratios by using the relation (4.9).

$$max\left\{\frac{z_1 - c_1}{\alpha_{11}} = \frac{-7/2}{5/2}, \ \frac{z_3 - c_3}{\alpha_{13}} = \frac{-1/2}{3/2}, \ \frac{z_4 - c_4}{\alpha_{14}} = \frac{-3/2}{5/2}\right\} = \frac{-1}{3} = \frac{z_3 - c_3}{\alpha_{13}} \ .$$

That is mean x_3 entering the basis, and x_6 departing the basis at its lower bound.

Let $x_3 = u_3 - \Delta_3 = 1 - \Delta_3$, where $\Delta_3 = \frac{l_{B_1} - \bar{\mathbf{x}}_{B_1}}{\alpha_{13}} = \frac{0 - \left(-\frac{3}{2}\right)}{\frac{3}{2}} = 1$

$$\bar{\mathbf{x}}_B = \begin{bmatrix} x_6 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} (1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad x_3 = 1 - 1 = 0$$

and $\bar{\boldsymbol{Z}} = \frac{15}{2} + \left(-\frac{1}{2}\right)(1) = 7.$

We obtained the optimal Tableau 4.13.

TABLEAU 4.13

	u_1	l_2		u_4		l_6	l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	5/3	-1	1	5/3	0	2/3	1/3	0
x_5	-1/3	1	0	-1/3	1	-1/3	1/3	1
$\boldsymbol{Z}$	-8/3	2	0	-3/2	0	1/2	2/3	7

The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*, x_6^*, x_7^*; \boldsymbol{Z}^*) = (1,0,0,1,1,0,0; 7)$

4.3.5. Deletion of a Variable [5],[13]

There are two cases:

Case (a): Deletion a nonbasic variable $\boldsymbol{x_k}$

When a nonbasic variable say x_k is dropped, then basis and $z_j - c_j$ will not change, only $\mathbf{x}_{\boldsymbol{B}}$ and $\boldsymbol{Z}$ will undergo change, $\boldsymbol{\alpha_k}$ will be taken away. Let $\boldsymbol{a_k}$ is not belong to B and it is dropped. There are two possibilities:

(i) $\boldsymbol{a_k} \in N_1$

(ii) $\boldsymbol{a_k} \in N_2$

(i) If $\boldsymbol{a_k} \in N_1$, then we calculate the new value of $\mathbf{x}_{\boldsymbol{B}}$ and $\boldsymbol{Z}$ as follows:

$$\bar{\mathbf{x}}_{\boldsymbol{B}}' = \boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J_1} \boldsymbol{\alpha_j} l_j - \sum_{j\in J_2} \boldsymbol{\alpha_j} u_j + \boldsymbol{\alpha_k} l_k$$

$$= \overline{\boldsymbol{X}}_{\boldsymbol{B}} + \boldsymbol{\alpha}_k l_k \qquad (4.37)$$

$$\overline{\boldsymbol{Z}}' = \boldsymbol{c_B^T}\boldsymbol{B}^{-1}\boldsymbol{b} - \textstyle\sum_{j\in J_1}(z_j - c_j)l_j - \sum_{j\in J_2}(z_j - c_j)u_j + (z_k - c_k)l_k$$

$$= \boldsymbol{Z} + (z_k - c_k)l_k \qquad (4.38)$$

In this case optimality is maintained but feasibility may be hampered. If $\boldsymbol{l_B} \leq \bar{\mathbf{x}}'_{\boldsymbol{B}} \leq \boldsymbol{u_B}$, then new solution is optimal as well as feasible. If $\bar{\mathbf{x}}'_{\boldsymbol{B}}$ is not feasible then we apply **Algorithm4** and proceed.

(i) $\boldsymbol{a_k} \in N_2$, parallel to (i).

Example 4.6:

Consider the following problem.

BLPP5: maximize $\boldsymbol{Z} = 2x_1 + 3x_2$
subject to

$$x_1 + 2\,x_2 \leq 23$$
$$x_1 - x_2 \leq 2$$
$$0 \leq x_1 \leq 7$$
$$2 \leq x_2 \leq 10$$

the initial and the optimal Tableaus are shown in tableaus 4.14 and 4.15 respectively .

TABLEAU 4.14

	l_1	l_2			
BP5	x_1	x_2	x_3	x_4	RHS
x_3	1	2	1	0	19
x_4	1	-1	0	1	4
$\boldsymbol{Z}$	-2	-3	0	0	6

TABLEAU 4.15

	u_1		l_3		
BP5	x_1	x_2	x_3	x_4	RHS
x_2	1/2	1	1/2	0	8
x_4	3/2	0	1/2	1	3
$\boldsymbol{Z}$	-1/2	0	3/2	0	38

Suppose that $x_1 = u_1$ be dropped, then

$$\bar{\mathbf{x}}'_B = \bar{\mathbf{x}}_B + \boldsymbol{\alpha_1} u_1 = \begin{bmatrix} 8 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} (7) = \begin{bmatrix} \frac{23}{2} \\ \frac{27}{2} \end{bmatrix},$$

and $$\bar{\boldsymbol{Z}}' = \bar{\boldsymbol{Z}} + (z_1 - c_1)u_1 = 38 + \left(\frac{-1}{2}\right)(7) = \frac{69}{2}.$$

Here feasibility is hampered as $x_2 = \frac{23}{2} > u_2$. As x_1 is nonbasic so on deleting the column $\boldsymbol{\alpha_1}$ in Tableau 4.15, we have Tableau 4.16.

TABLEAU 4.16

		l_3		
BP5	x_2	x_3	x_4	RHS
x_2	1	1/2	0	23/2
x_4	0	1/2	1	27/2
$\boldsymbol{Z}$	0	3/2	0	69/2

As $x_2 = \frac{23}{2} > u_2$. So it departs at its upper bound and x_3 enters the basis. Applying **Algorithm4** repeatedly optimality solution of perturbed problem is given by

Let $x_3 = l_3 + \Delta_3 = \Delta_3$; where $\Delta_3 = \frac{\bar{\mathbf{x}}_{B_1} - u_{B_1}}{\alpha_{13}} = \frac{\frac{23}{2} - 10}{\frac{1}{2}} = 3,$

then $x_3 = 3$, $x_4 = \frac{27}{2} - \frac{1}{2}(3) = 12$ and $\bar{\boldsymbol{Z}}' = \frac{69}{2} - \left(\frac{3}{2}\right)(3) = 30.$

We have the optimal Tableau 4.17.

TABLEAU 4.17

BP5	x_2 (u_2)	x_3	x_4	RHS
x_3	2	1	0	3
x_4	-1	0	1	12
$\boldsymbol{Z}$	-3	0	0	30

The optimal solution is $(\ x_2^*,\ x_3^*,\ x_4^*; \boldsymbol{Z}^*) = (10,3,12; 30)$

Case (b): Deletion of basic variable $\boldsymbol{x_{B_t}} \equiv \boldsymbol{x_k}$

Deletion of basic variable may affect the optimality as well as feasibility. For deletion of x_{B_t}, we make x_{B_t} a nonbasic, give it a high negative cost $-M(+M$ in minimization case) in optimal Tableau of BLPP and also change its bounds $l_{B_t} = 0$, $u_{B_t} = \infty$. The resaved value of $\boldsymbol{Z}$ and $z_j - c_j$ can be calculate by using (4.26) and (4.28), where $c'_{B_t} = -M$, we have

$$z'_j - c_j = (z_j - c_j) + (-M - c_{B_t})\alpha_{tj} \text{ for all } j \tag{4.39}$$

and

$$\bar{\boldsymbol{Z}}' = \bar{\boldsymbol{Z}} + (-M - c_{B_t})\bar{b}_t \tag{4.40}$$

Also, the cost row can be updated by the net change in the cost of $x_{B_t} \equiv x_k$ times the current t row of the final tableau, to the original cost row. then, $z'_k - c_k$ is updated to $z'_k - c'_k = 0$.

Now, x_{B_t} serves as an artificial variable, while making these changes, only optimality can hamper. If optimality is hampered, then we applying **Algorithm3** and find optimal solution. In the optimal tableau check, whether x_{B_t} is basic or nonbasic.

If x_{B_t} is nonbasic then it will be at its lower bound and now delete its column, no change in objective function value and basic variables. If x_{B_t} is basic and not replaceable the problem will be infeasible (Result 2), otherwise replace it and proceed as discussed above.

Example 4.7:

Consider the BLPP4 in example 4.2 with the optimal Tableau 4.18.

TABLEAU 4.18

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

let $\boldsymbol{a_5} \in B$ be deleted. Here we consider that $x_5 \geq 0$, so it serves as an artificial variable. Also $c_5 = 1$ is replaced by $-M$. From Tableau 4.18, making changes in $z_j - c_j$ and $\boldsymbol{Z}$ accordingly, we have following Tableau 4.19.

TABLEAU 4.19

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	$\frac{-M}{2} - 4$	$\frac{-M}{2} + 2$	$\frac{-M}{2} - 1$	$\frac{-M}{2} - 2$	0	0	$\frac{-M}{2}$	$\frac{-M}{2} + 7$

By using the relation (3.11), x_7 undergoes change, applying **Algorithm3** as follows:

Let $x_7 = l_7 + \Delta_7 = \Delta_7$,

compute Δ_7 by using (3.24-3.26)

$$\delta_1 = min\left\{\frac{\bar{\mathrm{x}}_{B_1} - l_{B_1}}{\alpha_{17}} = \frac{3/2}{1/2} = 3,\ \frac{\bar{\mathrm{x}}_{B_2} - l_{B_2}}{\alpha_{27}} = \frac{1/2}{1/2} = 1\right\} = 1$$

$$\delta_2 = \infty$$

$$u_7 - l_7 = \infty$$

$\Delta_7 = min\{\delta_1, \delta_2, u_7 - l_7\} = \delta_1 = 1$, then the departing variable is $x_{B_2} = x_5$ which becomes nonbasic at its lower bound.

$$\hat{\mathbf{x}}_{B_1} = \bar{\mathbf{x}}_{B_1} - \alpha_{17}\Delta_7$$

$$= \frac{3}{2} - \left(\frac{1}{2}\right)(1) = 1$$

$$\widehat{\boldsymbol{Z}} = \overline{\boldsymbol{Z}} - (z_7 - c_7)\Delta_7$$

$$= \left(\frac{-M}{2} + 7\right) - \left(\frac{-M}{2}\right)(1) = 7.$$

The remainder of the Tableau 4.19 is updating by performing a standard pivot operation on $\alpha_{27} = \frac{1}{2}$. We obtained the Tableau 4.20.

TABLEAU 4.20

	u_1	l_2	u_3	u_4	l_5			
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1	-2	1	2	-1	1	0	1
x_7	1	1	1	1	2	0	1	1
$\boldsymbol{Z}$	-4	2	-1	-2	M	0	0	7

The solution in above Tableau is optimal and x_5 is nonbasic at its lower bound, so on deleting $\boldsymbol{\alpha_5}$, the optimal solution of the perturbed problem is $\mathrm{x}^* = (1,0,1,1)$ and $\boldsymbol{Z}^* = 7$.

4.3.6. Deletion of a constraint [5],[13]

There are two cases:

Case(a): Deletion of inactive constraint

An inactive constraint is that one which is satisfied as strict inequality. So its corresponding slack or surplus variable would be basic and at nonzero level. Suppose we want to delete the row ith constraint which is inactive. Then delete the row and column of the slack/surplus variable corresponding to ith constraint. There will be no change in $\mathbf{x}_{\boldsymbol{B}}$, $\boldsymbol{Z}$ and $z_j - c_j$.

Example 4.8:

Consider the BLPP4 in Example 4.2 with the optimal Tableau 4.21.

TABLEAU 4.21

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

As $x_6 = \frac{3}{2} > 0$, so the first constraint is inactive. So to find the optimal solution of the perturbed problem, we delete the column $\boldsymbol{\alpha_6}$ and first row from Tableau 4.21 and there will be no change in $\mathbf{x}_{\boldsymbol{B}}$, $\boldsymbol{Z}$ and $z_j - c_j$.

TABLEAU 4.22

	u_1	l_2	u_3	u_4		l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_7	RHS
x_5	1/2	1/2	1/2	1/2	1	1/2	1/2
$\boldsymbol{Z}$	-7/5	5/2	-1/2	-3/2	0	1/2	15/2

The optimal solution is $(x_1^*,\ x_2^*,\ x_3^*,\ x_4^*,\ x_5^*,\ x_7^*; \boldsymbol{Z}^*) = (1,0,1,1,\frac{1}{2},0;\frac{15}{2})$

Case(b): Deletion of an active constraint

A constraint, which is satisfied as an equation, is called an active constraint. Let ith constraint is active and we want to delete it. For this, we make this constraint inactive and then proceed as in case(a). To make it inactive its slack/surplus must be introduce into basis at positive level . so give slack/surplus high positive cost $+M(-M$ in minimization case) and calculate $z_j - c_j'$ for this slack/surplus variable and enter slack/surplus variable into basis at next iteration. This makes the constraint inactive, cut the row and column of corresponding slack/surplus variable.

<u>Note</u>:

let x_{n+i} be the slack variable in ith constraint, which is active in optimal Tableau. As $x_{n+i} \geq 0$ and has no finite upper bound. So, if x_{n+i} is nonbasic, then it will be at its lower bound only and when $c_{n+i} \rightarrow M,\ z_{n+i} - c_{n+i}' = \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} \alpha_{n+i} - M < 0$, so it will always enter the basis and make constraint inactive.

Example 4.9:

Consider the BLPP4 in example 4.2, with the optimal Tableau 4.23.

TABLEAU 4.23

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	1/2	15/2

The second constraint is active. To make it inactive, change $c_7 = 0$ into $c'_7 = M$ and calculate $z_7 - c'_7$ as follows

$$z_7 - c'_7 = \boldsymbol{c_B}\alpha_7 - c'_7 = (0 \quad 1)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - M = \frac{1}{2} - M < 0$$

We have the Tableau 4.24

TABLEAU 4.24

	u_1	l_2	u_3	u_4			l_7	
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1/2	-3/2	3/2	5/2	0	1	1/2	3/2
x_5	1/2	1/2	1/2	1/2	1	0	1/2	1/2
$\boldsymbol{Z}$	-7/2	5/2	-1/2	-3/2	0	0	$\frac{1}{2} - M$	15/2

As $z_7 - c'_7 < 0$, so x_7 undergoes change. Applying **Algorithm3** repeatedly.

Let $x_7 = l_7 + \Delta_7 = \Delta_7$,

Compute Δ_7 by using (3.24-3.26),

$$\delta_1 = min\left\{ \frac{\bar{\mathbf{x}}_{B_1} - l_{B_1}}{\alpha_{17}} = \frac{(3/2)-0}{1/2} = 3,\ \frac{\bar{\mathbf{x}}_{B_2} - l_{B_2}}{\alpha_{27}} = \frac{(1/2)-0}{1/2} = 1 \right\} = 1$$

$$\delta_2 = \infty$$

$$u_7 - l_7 = \infty$$

$\Delta_7 = \min\{\delta_1, \delta_2, u_7 - l_7\} = \delta_1 = 1$, then the departing variable is $x_{B_2} = x_5$ which becomes nonbasic at its lower bound.

$$\hat{\mathbf{x}}_{B_1} = \bar{\mathbf{x}}_{B_1} - \alpha_{17}\Delta_7$$

$$= \frac{3}{2} - \left(\frac{1}{2}\right)(1) = 1$$

$$\widehat{\boldsymbol{Z}} = \overline{\boldsymbol{Z}} - (z_7 - c'_7)\Delta_7 = \frac{15}{2} - \left(\frac{1}{2} - M\right)(1) = 7 + M$$

TABLEAU 4.25

	u_1	l_2	u_3	u_4	l_5			
BP4	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_6	-1	-2	1	2	-1	1	0	1
x_7	1	1	1	1	2	0	1	1
Z	$M-4$	$M+2$	$M-1$	$M-2$	$2M-1$	0	0	$M+7$

Note that $x_7 > 0$, so second constraint is inactive. On deleting α_7 and second row in the above Tableau and also making changes in ***Z*** and $z_j - c_j$, we have the following Tableau.

TABLEAU 4.26

	u_1	l_2	u_3	u_4	l_5		
BP4	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_6	-1	-2	1	2	-1	1	1
Z	-4	2	-1	-2	-1	0	7

This Tableau is feasible but not optimal. So applying **Algorithm3** repeatedly.

Let $x_5 = l_5 + \Delta_5 = \Delta_5$,

Compute Δ_5 by using (3.24-3.26)

$\delta_1 = \infty$

$\delta_2 = \infty$

$u_5 - l_5 = 1$

$\Delta_7 = min\{\delta_1, \delta_2, u_5 - l_5\} = u_5 - l_5 = 1$, then x_5 moves from nonbasic at its lower bound to nonbasic at its upper bound and Tableau 4.26 remains the same, but $\bar{\boldsymbol{Z}}$ and $\bar{\mathbf{x}}_B$ are changes.

$\hat{\mathbf{x}}_{B_1} = \bar{\mathbf{x}}_{B_1} - \alpha_5 \Delta_5 = 1 - (-1)(1) = 2$

$$\hat{\boldsymbol{Z}} = \bar{\boldsymbol{Z}} - (z_5 - c_5)\Delta_5$$

$$= 7 - (-1)(1) = 8.$$

TABLEAU 4.27

	u_1	l_2	u_3	u_4	u_5		
BP4	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_6	-1	-2	1	2	-1	1	2
$\boldsymbol{Z}$	-4	2	-1	-2	-1	0	8

The Tableau 4.27 is optimal, and the optimal solution is $(x_1^*,\ x_2^*,\ x_3^*,\ x_4^*,\ x_5^*,\ x_6^*; \boldsymbol{Z}^*) = (1,0,1,1,1,2; 8)$.

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