Linköping Studies in Science and Technology. Dissertations.
No. 1283

The Double Obstacle Problem on Metric Spaces

Zohra Farnana

Division of Applied Mathematics
Department of Mathematics

Linköping 2009

Linköping Studies in Science and Technology. Dissertations.
No. 1283

The Double Obstacle Problem on Metric Spaces

Zohra Farnana

Linköping University
INSTITUTE OF TECHNOLOGY

Division of Applied Mathematics
Department of Mathematics

Linköping 2009

The Double Obstacle Problem on Metric Spaces

Matematiska institutionen
Linköpings universitet
SE-581 83 Linköping, Sweden
zofar@mai.liu.se

Linköping Studies in Science and Technology
Dissertations, No 1283

ISBN 978-91-85831-00-5
ISSN 0280-7971

Printed by LiU-Tryck, Linköping 2009

Abstract

In this thesis we investigate the double obstacle problem for p-harmonic functions on metric spaces. We minimize the p-energy integral among all functions which have prescribed boundary values and lie between two given obstacles. This is a generalization of the Dirichlet problem for p-harmonic functions, in which case the obstacles are $-\infty$ and ∞.

We show the existence and uniqueness of solutions, and their continuity when the obstacles are continuous. Moreover we show that the continuous solution is p-harmonic in the open set where it does not touch the continuous obstacles. If the obstacles are not continuous, but satisfy a Wiener type regularity condition, we prove that the solution is still continuous. The Hölder continuity for solutions is shown, when the obstacles are Hölder continuous. Boundary regularity of the solutions is also studied.

Furthermore we study two kinds of convergence problems for the solutions. First we let the obstacles and the boundary values vary and show the convergence of the solutions. We also consider generalized solutions for insoluble obstacle problems, using the convergence results. Moreover we show that for soluble obstacle problems the generalized solution coincides, locally, with the standard solution.

Second we consider an increasing sequence of open sets, with union Ω, and fix the obstacles and the boundary values. We show that the solutions of the obstacle problems in these sets converge to the solution of the corresponding problem in Ω.

Acknowledgements

First of all I would like to thank my supervisor docent Jana Björn and co-supervisor docent Anders Björn for introducing me to this topic, very useful discussions, reading my papers carefully and for helping me with LaTeX. Their patience, their encouragement and their enthusiasm have been invaluable to me.

I would also like to thank my second co-supervisor Prof. Lars-Erik Andersson for giving me the opportunity to study at the Department of Mathematics, Linköping University. Thanks to our Director of postgraduate studies Dr Bengt Ove Turesson for all help. Thanks also to the Libyan Higher Education Ministry for financial support.

Finally, I would like to thank my family for their support and encouragement. Especially you Ali, without you I would not be where I am now.

Linköping, 30 October 2009

Zohra Farnana

Dubbelhinderproblemet

Låt oss betrakta följande exempel: Vi skulle vilja måla ett hus och har en massa möbler som måste täckas så att de inte blir nedsmutsade. Vi använder ett specifikt material som måste fixeras för att täcka bordet, stolen eller vad det nu är för sorts möbel. Givetvis vill vi inte använda för mycket material och vi vill göra täckningen så slät så möjligt.

Om möblerna flyttas ihop till en plats kan vi täcka dem alla utan att skära i materialet. Om å andra sidan möblerna står på olika platser behöver vi förmodligen dela det täckande materialet i flera mindre bitar. Det är klart att om vi täcker två stycken likadana möbler, t.ex. två stolar på olika platser, så kan vi täcka dem med likadana bitar.

Ovanstående är ett exempel på ett enkelhinderproblem, där hindret är en möbel eller grupp av möbler och lösningen är det täckande materialet. I dubbelhinderproblemet har vi ett hinder nerifrån, som möbeln/möblerna ovan, och därtill ett hinder uppifrån, t.ex. tak, lampor och dörrkarmar i situationen ovan.

I den här avhandlingen studerar vi enkel- och dubbelhinderproblemen. Detta görs i väldigt abstrakta och generella sammanhang, så kallade metriska rum. Hindren tillåts också vara väldigt generella, och behöver speciellt inte vara kontinuerliga. Vi har visat att det alltid finns en optimal entydig lösning och att lösningen är kontinuerlig om hindren är kontinuerliga. Det visas också i avhandlingen att lösningarna är kontinuerliga även om hindren inte är kontinuerliga, under förutsättning att visa andra villkor är uppfyllda.

I avhandlingen studeras också flera olika konvergensproblem för enkel- och dubbelhinderproblemen som visar hur lösningarna varierar när hindren varierar.

In this thesis we investigate the double obstacle problem on metric spaces. In particular we consider the existence, regularity and some convergence properties of the solutions.

The classical Dirichlet problem is to find a harmonic function (a solution of the Laplace equation) with prescribed boundary values. An equivalent variational formulation of this problem is the minimization problem

$$\int |\nabla u|^2 \, dx$$

among all functions which have the required boundary values. A more general nonlinear analogue of the classical Dirichlet problem is the p-energy minimization problem

$$\int |\nabla u|^p \, dx,$$

with $1 < p < \infty$. The minimizers are solutions of the corresponding Euler–Lagrange equation, which is the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

and continuous minimizers are called *p-harmonic functions*.

During the last decade, potential theory and p-harmonic functions have been developed in the setting of doubling metric measure spaces supporting a p-Poincaré inequality. This theory unifies, and has applications in several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds and Carnot groups, subelliptic differential operators and potential theory on graphs.

In a general metric measure space it is not clear how to employ partial differential equations. That led Heinonen–Koskela [10] to introduce the concept of an upper gradient as a substitute for the modulus of the usual gradient based on the following observation: It is well known from the fundamental theorem of calculus that, for $x, y \in \mathbf{R}^n$ and a smooth function u on $\mathbf{R}^n$, on the line segment $[x, y]$ we have

$$|u(y) - u(x)| \leq \int_{[x,y]} |\nabla u| \, ds.$$

In fact, for every rectifiable curve γ with end points x and y we have

$$|u(y) - u(x)| \leq \int_{\gamma} |\nabla u| \, ds. \tag{1}$$

Similarly, a nonnegative Borel function g is an *upper gradient* of u if (1) holds for all rectifiable curves when $|\nabla u|$ is replaced by g. It has many useful properties similar to those of the usual gradient. This makes the variational approach of the Dirichlet problem available in metric spaces and Sobolev spaces can then be extended to metric spaces.

There are many notions of Sobolev spaces in metric spaces; see for example Cheeger [6], Hajłasz [8] and Shanmugalingam [20], [21]. The definitions in these references are different but by [20] they give the same Sobolev spaces under mild assumptions. We shall follow the definition of Shanmugalingam [20], where the Sobolev space $N^{1,p}(X)$ (called the Newtonian space) was defined as the collection of p-integrable functions with p-integrable upper gradients.

In [21] it was shown that Newtonian spaces are lattices i.e. if $u, v \in N^{1,p}(X)$ then $\min\{u, v\}$ and $\max\{u, v\}$ belong to $N^{1,p}(X)$. Also it turns out that Newtonian spaces are Banach spaces when regraded as equivalence classes, where two functions belong to the same equivalent class if they differ only on a set of capacity zero.

On $\mathbf{R}^n$ it is well-known that every Sobolev function has a representative which is absolutely continuous on almost every line parallel to the coordinate axes. In this setting we have a stronger property for Newtonian functions, namely that they are absolutely continuous on almost every curve. One more improvement in the continuity properties of Newtonian functions is that a function in $N^{1,p}(\Omega)$ is continuous when restricted to the complement of a small set. This is a *Luzin type phenomenon.* In the present setting it is called *quasicontinuity* and the removed set has small capacity.

When specialized to $\mathbf{R}^n$, Newtonian spaces coincide with the usual Sobolev spaces in the sense that every $u \in N^{1,p}(\mathbf{R}^n)$ belongs to $W^{1,p}(\mathbf{R}^n)$ and every $u \in W^{1,p}(\mathbf{R}^n)$ has a representative in the Newtonian space $N^{1,p}(\mathbf{R}^n)$ which is quasicontinuous. This can be seen for example in the plane, where the real line has two-dimensional Lebesgue measure zero, we have $W^{1,p}(\mathbf{R}^2) \ni \chi_{\mathbf{R}} \notin N^{1,p}(\mathbf{R}^2)$ but $\chi_{\mathbf{R}} = 0$ a.e. in $\mathbf{R}^2$ and clearly $0 \in N^{1,p}(\mathbf{R}^2)$.

Newtonian spaces enable us to study variational integrals and potential theory can be built on minimizers of the p-Dirichlet integral

$$\int g_u^p \, d\mu, \tag{2}$$

where g_u denotes the minimal p-weak upper gradient of u, whose existence and uniqueness was proved in Shanmugalingam [20]. Although potential theory of minimizers of the p-Dirichlet integral in the Euclidean case is linear for $p = 2$ our theory has nonlinear features for all $p > 1$. The reason for this is that the operation of taking an upper gradient is not linear. Several results concerning solubility of the Dirichlet problem for p-harmonic functions have been obtained in metric spaces in e.g. Cheeger [6], Björn–Björn [1], [2], Björn–Björn–Shanmugalingam [4], [5], Kinnunen–Shanmugalingam [14] and Shanmugalingam [21], [22]. The existence and uniqueness of minimizers of (2) were proved in [21]. Then it was shown in [14] that, under certain assumptions, the minimizers of (2) satisfy the Harnack inequality, the maximum principle and are locally Hölder continuous.

The single obstacle problem has been studied in the setting of metric spaces in Kinnunen–Martio [13] where it was shown that solutions of the single obstacle problem are superminimizers, satisfy a weak Harnack inequality and have lower semicontinuous representatives. The geometric interpretation of the superminimizing property is that the solutions of the single obstacle problem locally lie above the corresponding minimizer with the same boundary values. In particular when specialized to $\mathbf{R}$ superminimizers are concave functions. Further results about the single obstacle problem can be found in Björn–Björn [1] and Björn–Björn–Parviainen [3].

In this thesis we study the double obstacle problem on metric spaces. One significant difference between the single and the double obstacle problems is that the solution of the single obstacle problem turns out to be a superminimizer whereas this is no longer true in the double obstacle situation. This does not allow for the use of the weak Harnack inequality for superminimizers, which was a main tool in the analysis of the single obstacle problem. Therefore new arguments are needed. However we are still able to obtain many useful results for the double obstacle problem.

The standard assumption for the theory and for this thesis is that of a complete metric space X endowed with a metric d and a *doubling* Borel measure μ, i.e. there exists a constant $C \geq 1$ such that for all balls $B = B(x,r) := \{y \in X : d(x,y) < r\}$ in X we have

$$0 < \mu(2B) \leq C\mu(B) < \infty,$$

where $\tau B = B(x, \tau r)$. The doubling property implies that X is complete if and only if X is proper, i.e., closed bounded sets are compact. We also require the space X to support a p-Poincaré inequality, which means that the mean oscillation of every function is locally controlled by the L^p-norm of its upper gradient. More specifically, there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(z,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \leq Cr \left(\fint_{B(z,\lambda r)} g^p \, d\mu \right)^{1/p},$$

where $u_{B(z,r)} := \fint_{B(z,r)} u \, d\mu := \mu(B(z,r))^{-1} \int_{B(z,r)} u \, d\mu$.

Let Ω be a bounded open subset of X. We minimize the p-Dirichlet integral (2) on Ω among all functions which have prescribed boundary values f and lie between two given obstacles ψ_1 and ψ_2. A minimizer is called a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}$-problem. This generalizes the Euclidean obstacle problem based on equations of p-Laplace type as e.g. in Kinderlehrer–Stampaccia [12] and Malý–Ziemer [18]. In particular existence and regularity for the solutions were shown. For historical account, see also Section 5.3 in [18] and the references therein.

Further results about the obstacle problem in $\mathbf{R}^n$ can be found in Heinonen–Kilpeläinen–Martio [9], which concerns the single obstacle problem, Li–Martio [16], [17] and Olek–Szczepaniak [19]. In Dal Maso–Mosco–Vivaldi [7] the double obstacle problem in $\mathbf{R}^n$ was considered, for $p = 2$ and $f \equiv 0$. The main tools in the proofs used therein are connected with the Euler–Lagrange equation for the minimizing function of the given problem. In the general setting of metric spaces we do not have an analogue of the Euler–Lagrange equation, and therefore our proofs use variational techniques.

The primary example of this obstacle problem is given by the definition of the *variational capacity*. For a compact set $K \subset \Omega$ the variational capacity is obtained as the solution of the $\mathcal{K}_{\chi_K,1,0}$-problem, i.e. by

$$\inf_u \int_\Omega g_u \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $\chi_K \le u \le 1$.

This thesis is organized in four papers. In Paper 1, we define the double obstacle problem, and prove that there exists a unique solution (up to sets of capacity zero) of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. We also show that there is a continuous solution of the double obstacle problem provided the two obstacles are continuous, in this case we also prove that the solution is a minimizer in the open set where the continuous solution does not touch the two obstacles. Furthermore we study the boundary regularity for the double obstacle problem, and prove that under certain conditions the solution of the obstacle problem is continuous up to the boundary. We also give two new characterizations of regular boundary points. Our work in this paper extends some results from Björn–Björn [1] and Kinnunen–Martio [13] in which similar investigations were undertaken for the case of a single obstacle problem.

In Paper 2, we investigate the continuity at a given point x_0 of the solutions of the double obstacle problem. The obstacles in this context are to be regarded as quite general and irregular. In particular, they may be discontinuous. We show that if the obstacles are not continuous, but satisfy a Wiener type regularity condition, the solution is still continuous.

Since the p-harmonic functions are solutions of special obstacle problems, we can expect at most Hölder continuity for the regularity for our solution. Indeed, we show that the continuous solution of the single obstacle problem, with locally Hölder continuous obstacle, is locally Hölder continuous. For the double obstacle problem we prove that if the obstacles are locally Hölder continuous, then the continuous solution u is Hölder continuous at every point $x_0 \in \Omega$.

In Paper 3, we study various convergence properties of the obstacle problem. First we consider two sequences of obstacles $\{\psi_j\}_{j=1}^\infty$ and $\{\varphi_j\}_{j=1}^\infty$ converging to ψ and φ, respectively. We assume that the sequence $\{\psi_j\}_{j=1}^\infty$ converges to ψ q.e. from below while the sequence $\{\varphi_j\}_{j=1}^\infty$ converges to φ q.e. from above. We prove that the solutions of the $\mathcal{K}_{\psi_j,\varphi_j,f}$-problem, with $f \in N^{1,p}(\Omega)$, converge to the solution of the $\mathcal{K}_{\psi,\varphi,f}$-problem. In the

Euclidean case, a similar result was proved in Olek–Szczepaniak [19], by a completely different method. Also we show that if one of the assumption was omitted (if the two sequences are decreasing) then the convergence of the obstacles does not imply convergence of the solutions to the solution of the limit problem. Hence more assumptions are needed for the convergence of the solutions when the two sequences of obstacles are decreasing.

Second we assume that $\{\psi_j\}_{j=1}^\infty$, $\{\varphi_j\}_{j=1}^\infty$ and $\{f_j\}_{j=1}^\infty$ are decreasing and converge to ψ, φ and f, respectively, such that $\psi_j \to \psi$ in the Newtonian space $N^{1,p}(\Omega)$. We also assume that $\{f_j\}_{j=1}^\infty$ to be bounded in $N^{1,p}(\Omega)$ and $\psi_j - f_j \in N_0^{1,p}(\Omega)$, $j = 1, 2, \ldots$. Then the solutions of the $\mathcal{K}_{\psi_j,\varphi_j,f_j}$-problems converge to the solution of the $\mathcal{K}_{\psi,\varphi,f}$-problem monotonically and in $L^p(\Omega)$. We also give an example illustrating that the assumption that $\psi_j \in N^{1,p}(\Omega)$, $j = 1, 2, \ldots$, cannot be omitted.

Finally, as an application of the convergence properties of solutions of the obstacle problems, we consider *generalized solutions* of the obstacle problem $\{\psi_1, \psi_2\}$ for more general boundary values $f \notin N^{1,p}(\Omega)$ or in the case where there is no Newtonian function between the obstacles with the given Newtonian boundary values. Solutions of the $\mathcal{K}_{\psi_1,\psi_2,f}$-problems belonging to the Newtonian space $N^{1,p}(\Omega)$ exist provided the given obstacles ψ_1 and ψ_2 are separated by some $N^{1,p}(\Omega)$ function with the given, Newtonian, boundary values. The generalized solution is defined as a limit of variational solutions, by only requiring the separating function to be a uniform limit of Newtonian functions.

In Paper 4 we continue our study of convergence properties of the obstacle problem. In particular, we consider an increasing sequence of open sets Ω_j whose union is Ω. We analyze the convergence of the solutions u_j of the obstacle problems corresponding to the sets Ω_j. Our purpose here is to give sufficient conditions on the obstacles and the boundary values which imply that the sequence of solutions u_j converges to the solution of the obstacle problem corresponding to the set Ω. We give several generalizations of Theorem 4.3 in Björn–Björn [2].

We have shown that the p-harmonic extensions of $f \in N^{1,p}(\Omega)$ to Ω_j converge locally uniformly to the p-harmonic extension of f to Ω. This extends Theorem 4.3 from Björn–Björn [2] where a similar result was proved for $f \in C(\overline{\Omega})$. A corresponding results for the double obstacle problem are obtained under the assumption $f \in N^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $\psi_1 \le f \le \psi_2$.

References

[1] A. Björn and J. Björn, Boundary regularity for p-harmonic functions and solutions of the obstacle problem on metric spaces, *J. Math. Soc. Japan* **58** (2006), 1211–1232.

[2] A. Björn and J. Björn , Approximations by regular sets and Wiener solutions in metric spaces, *Comment. Math. Univ. Carolin.* **48** (2007), 343–355.

[3] A. Björn, J. Björn and M. Parviainen, Lebesgue points and the fundamental convergence theorem for superharmonic functions on metric spaces, to appear in *Rev. Mat. Iberoamericana.*

[4] A. Björn, J. Björn and N. Shanmugalingam, The Dirichlet problem for p-harmonic functions on metric spaces, *J. Reine Angew. Math.* **556** (2003), 173–203.

[5] A. Björn, J. Björn and N. Shanmugalingam, The Perron method for p-harmonic functions in metric spaces, *J. Differential Equation* **195** (2003), 398–429.

[6] J. Cheeger, Differentiability of Lipschitz functions on metric spaces, *Geom. Funct. Anal.* **9** (1999), 428–517.

[7] G. Dal Maso, U. Mosco and M. A. Vivaldi, A pointwise regularity theory for the two-obstacle problem, *Acta Math.* **163** (1989), 57–107.

[8] P. Hajłasz, Sobolev spaces on an arbitrary metric spaces, *Potential Anal.* **5** (1995), 403–415.

[9] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.

[10] J. Heinonen and P. Koskela, Qquasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.

[11] T. Kilpeläinen and W. P. Ziemer, Pointwise regularity of solutions to nonlinear double obstacle problems, *Ark. Mat.* **29** (1991), 83–106.

[12] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.

[13] J. Kinnunen and O. Martio, Nonlinear potential theory on metric spaces, *Illinois Math. J.* **46** (2002), 857–883.

[14] J. Kinnunen and N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, *Manuscripta Math.* **105** (2001), 401–423.

[15] P. Koskela and P. MacManus, Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.

[16] G. Li and O. Martio, Stability in obstacle problems, *Math. Scand.* **75** (1994), 87–100.

[17] G. Li and O. Martio, Stability and higher integrability of derivatives of solutions in double obstacle problems, *J. Math. Anal. Appl.* **272** (2002), 19–29.

[18] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys and Monographs **51**, Amer. Math. Soc., Providence, RI, 1997.

[19] A. Olek and K. Szczepaniak, Continuous dependence on obstacles in double global obstacle problems, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 89–97.

[20] N. Shanmugalingam, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoamericana* **16** (2000), 243–279.

[21] N. Shanmugalingam, Harmonic functions on metric spaces, *Illinois Math. J.* **45** (2001), 1021–1050.

[22] N. Shanmugalingam, Some convergence results for p-harmonic function on metric spaces, *Proc. London Math. Soc.* **87** (2003), 226–246.

Paper 1

The double obstacle problem on metric spaces

The article have been removed due to Copyright restircitons.

Paper 2

Pointwise regularity for solutions of double obstacle problems on metric spaces

The article has been removed due to Copyright restircitons.

Paper 3

Continuous dependence on obstacles for the double obstacle problem on metric spaces

The article has been removed due to Copyright restircitons.

Paper 4

Convergence results for obstacle problems on metric spaces

The article has been removed due to Copyright restircitons.