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Matematiska institutionen
Linköpings universitet
SE-581 83 Linköping, Sweden
`zofar@mai.liu.se`

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Abstract

During the last decade, potential theory and p-harmonic functions have been developed in the setting of doubling metric measure spaces supporting a p-Poincaré inequality. This theory unifies, and has applications in several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds and Carnot groups, subelliptic differential operators and potential theory on graphs.

In this thesis we investigate the double obstacle problem for p-harmonic functions on metric spaces. We show the existence and uniqueness of solutions and their continuity when the obstacles are continuous. Moreover the solution is p-harmonic in the open set where it does not touch the continuous obstacles. The boundary regularity of the solutions is also studied.

Furthermore we study two kinds of convergence problems for the solutions. First we let the obstacles vary and fix the boundary values and show the convergence of the solutions. Second we consider an increasing sequence of open sets, with union Ω, and fix the obstacles and the boundary values. We show that the solutions of the obstacle problems in these sets converge to the solution of the corresponding problem in Ω.

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Populärvetenskaplig beskrivning

Låt oss börja med att betrakta följande situation: Vi vill förflytta oss från en plats vid ena sidan av en äng till en viss punkt på andra sidan ängen. På båda sidor om ängen finns skogsområden som vi inte får gå in i. Ängen är tyvärr inte homogen utan består av olika sorters mark som vi har noggrant beskrivet på en karta. Vi vill göra förflyttningen på smidigast sätt, men då ängen inte är homogen ska vi förmodligen inte gå rakaste vägen utan ska anpassa vägen optimalt efter terrängen. Detta är ett exempel på ett dubbelhinderproblem där hindren är skogsområdena på sidorna som vi måste hålla oss utanför.

Mer abstrakt vill man minimiera energin hos funktioner som tar vissa givna randvärden (de givna start- och slutpunkterna i exemplet ovan) och som håller sig mellan ett undre och ett övre hinder. I denna avhandling studeras detta dubbelhinderproblem i väldigt allmänna situationer.

För att kunna lösa hinderproblemet krävs det att vi tillåter icke-kontinuerliga lösningar och då visas i avhandlingen att hinderproblemet är entydigt lösbart. Ett huvudresultat i avhandlingen är att om våra hinder är kontinuerliga så blir även lösningen kontinuerlig. Vidare visas diverse konvergenssatser som visar hur lösningarna varierar när hindren eller området i vilket problemet löses varierar.

Hinderproblem har utöver eget intresse viktiga tillämpningar i potentialteorin, bland annat för att studera motsvarande energiminimeringsproblem utan hinder.

1. Introduction

The object of this thesis is the double obstacle problem in metric spaces. In particular we consider the existence, regularity and some convergence problems for the solutions.

Elliptic partial differential equations describe many phenomena in physics and natural sciences. The chemical concentration, temperature distribution and electrostatic potential are described by the linear Laplace equation. Other physical phenomena are described by the nonlinear p-Laplace equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, whose solutions are p-harmonic functions, and which is the Euler–Lagrange equation of the p-energy minimization problem

$$\min \int_\Omega |\nabla u|^p \, d\mu$$

among all functions with given boundary values. In some instances it is physically needed for the solution to be between two impediments, which leads to the study of the double obstacle problem.

During the last decade, potential theory and p-harmonic functions have been developed in the setting of doubling metric measure spaces supporting a p-Poincaré inequality. This theory unifies, and has applications in several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds and Carnot groups, subelliptic differential operators and potential theory on graphs.

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a positive complete Borel measure μ which is *doubling*, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X, we have

$$0 < \mu(2B) \le C\mu(B) < \infty,$$

where $2B = B(x_0, 2r)$.

In a metric space the gradient has no obvious meaning as in domains in $\mathbf{R}^n$. Therefore the concept of an upper gradient was introduced in Heinonen–Koskela [7] as a substitute for the modulus of the usual gradient, based on the following observation: It is well known from the fundamental theorem of calculus that, for a smooth function u on $\mathbf{R}^n$ and $x, y \in \mathbf{R}^n$, on the line segment $[x, y]$ we have

$$|u(y) - u(x)| \le \int_{[x,y]} |\nabla u| \, ds$$

and in fact for every rectifiable curve γ with end points x and y we have

$$|u(y) - u(x)| \le \int_\gamma |\nabla u| \, ds. \tag{1}$$

Similarly a nonnegative Borel function g on a metric space is an upper gradient of u if (1) holds when $|\nabla u|$ is replaced by g. It has many useful properties similar to those of the usual gradient. This makes it possible to define and study the Sobolev type spaces $N^{1,p}(X)$ (called Newtonian spaces) in metric spaces which enables us to study variational integrals in metric spaces and to build a nonlinear potential theory for minimizers of the variational integral

$$\int g_u^p \, d\mu, \tag{2}$$

where g_u denotes the minimal p-weak upper gradient of u, whose existence was proved in Shanmugalingam [14] and [15]. Indeed, in Kinnunen–Shanmugalingam [10] it was shown that under certain conditions the minimizers of (2) satisfy the Harnack inequality and the maximum principle, and are locally Hölder continuous. The standard assumptions for the theory and for this thesis are that X is doubling and supports a p-Poincaré inequality, which means that the local mean oscillation of every function is controlled by the L^p-norm of its upper gradient.

The Dirichlet boundary value problem on a domain Ω is to find a function u satisfying $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω (or minimizing (2)), so that $u = f$ on $\partial\Omega$, where $f : \partial\Omega \to \mathbf{R}$ is a given function. Several results concerning solubility of the Dirichlet problem for p-harmonic functions have been extended to metric spaces in e.g. Cheeger [5], Shanmugalingam [15] and Björn–Björn–Shanmugalingam [4], [3]. Furthermore the single obstacle problem has been extended to the setting of metric spaces, in Kinnunen–Martio [9].

In this thesis, we study the double obstacle problem in metric spaces. For domains in $\mathbf{R}^n$, the double obstacle problem was defined and studied in e.g. Dal Maso–Mosco–Vivaldi [6], Kilpeläinen–Ziemer[8], Li–Martio [11],[12] and Olek–Szczepaniak [13]. The definitions therein deal with partial differential equations. Due to the notion of the upper gradient, it is not clear how to employ partial differential equations in this setting. Our approach is based only on the variational integrals.

Since the single obstacle problem is a special case of the double obstacle problem, one cannot expect better results in the latter case. One significant difference between the single and double obstacle problems is that the solution of the single obstacle problem turns out to be a superminimizer whereas this is no longer true in the double obstacle situation. This does not allow for the use of the weak Harnack inequality for superminimizers, which was a main tool in the analysis of the single obstacle problem, and therefore new arguments are needed. However we are still able to obtain many useful results for the double obstacle problem.

Let Ω be a bounded open subset of X. We study the double obstacle problem with boundary data $f \in N^{1,p}(\Omega)$ and obstacles $\psi_j : \Omega \to \overline{\mathbf{R}}$, $j = 1, 2$. Let

$$\mathcal{K}_{\psi_1,\psi_2,f}(\Omega) = \{v \in N^{1,p}(\Omega) : v - f \in N_0^{1,p}(\Omega) \text{ and } \psi_1 \le v \le \psi_2 \text{ q.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$ is a *solution of the* $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$*-obstacle problem* if

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_{\psi_1,\psi_2,f}(\Omega),$$

where g_u is the minimal p-weak upper gradient of u.

This thesis is organized in two papers. In Paper 1, we define the double obstacle problem, and prove that there exists a unique solution (up to sets of capacity zero) of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-obstacle problem. We also show that there is a continuous solution of the double obstacle problem provided the two obstacles are continuous, in this case we also prove that the solution is a minimizer in the open set where the continuous solution does not touch the two obstacles. Furthermore we study the boundary regularity for the double obstacle problem, and prove that under certain conditions the solution of the obstacle problem is continuous up to the boundary. We also give two new characterizations of regular boundary points. Our work in this paper extends some results from [9] and [1] in which a similar investigations was undertaken for the case of a single obstacle problem.

In Paper 2 we study various convergence properties of the obstacle problem. First we consider two sequences of obstacles $\{\psi_j\}_{j=1}^\infty$, $\{\varphi_j\}_{j=1}^\infty$ converging to ψ, φ, respectively. We assume that the sequence $\{\psi_j\}_{j=1}^\infty$ converges to ψ q.e. from below while the sequence $\{\varphi_j\}_{j=1}^\infty$ converges to φ q.e. from above. We prove that the solutions of the $\mathcal{K}_{\psi_j,\varphi_j,f}$-problem, with $f \in N^{1,p}(\Omega)$, converge to the solution of the $\mathcal{K}_{\psi,\varphi,f}$-problem. In the Euclidean case this result was proved in Olek–Szczepaniak [13], by a completely different method.

Second, we consider an increasing sequence of open sets Ω_j whose union is Ω. We analyze the convergence of the solutions of the obstacle problems corresponding to the sets Ω_j. We show that the p-harmonic extensions of $f \in N^{1,p}(\Omega)$ to Ω_j (the solution of the obstacle problem with $\psi_1 \equiv -\infty$, $\psi_2 \equiv \infty$ and boundary values f) converge to the p-harmonic extension of f to Ω. We also prove that if Ω is regular, $f \in N^{1,p}(\overline{\Omega}) \cap C(\overline{\Omega})$ and the obstacles are continuous then the solutions of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega_j)$-problem converge to the solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. Finally when Ω is not regular we prove that the solutions of the single obstacle problem in Ω_j with a continuous obstacle and boundary values $f \in N^{1,p}(\overline{\Omega}) \cap C(\overline{\Omega})$ converge to the solution of the corresponding problem in Ω. Our work in this part is an extension of Theorem 4.3 in Björn–Björn [2] which proves the existence and uniqueness of Wiener solutions.

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