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Convergence results for obstacle problems on metric spaces

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A R T I C L E I N F O

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A B S T R A C T

Let X be a complete metric space equipped with a doubling Borel measure supporting a p-Poincaré inequality. We obtain various convergence results for the single and double obstacle problems on open subsets of X. In particular, we consider single and double obstacle problems with fixed obstacles and boundary data on an increasing sequence of open sets.

1. Introduction

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a positive complete Borel measure μ which is *doubling*, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x, r) := \{y \in X\colon\ d(x, y) < r\}$ in X we have

$$0 < \mu(2B) \leqslant C\mu(B) < \infty,$$

where $\tau B = B(x, \tau r)$. The doubling property implies that X is complete if and only if X is proper, i.e., closed bounded sets are compact. We also require the space X to support a p-Poincaré inequality, see Section 2 for the definition.

In a metric space the gradient has no obvious meaning as in domains in $\mathbf{R}^n$. Therefore the concept of an upper gradient was introduced in Heinonen and Koskela [16] as a substitute for the modulus of the usual gradient. This makes it possible to define and study Sobolev spaces in metric spaces. There are many notions of Sobolev spaces in metric spaces; see for example Cheeger [9], Hajłasz [14] and Shanmugalingam [22,23]. We shall follow the definition of Shanmugalingam [22], where the Sobolev spaces $N^{1,p}(X)$ (called Newtonian spaces) were defined as the collection of p-integrable functions with p-integrable upper gradients.

Newtonian spaces $N^{1,p}(X)$ enable us to study variational integrals in metric spaces and to build a nonlinear potential theory for minimizers of the p-Dirichlet integral

$$\int g_u^p \, d\mu, \tag{1}$$

where g_u denotes the minimal p-weak upper gradient of u, whose existence and uniqueness were proved in Shanmugalingam [22]. Existence and uniqueness of minimizers of (1) were obtained in Shanmugalingam [23]. It was shown, in

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Kinnunen and Shanmugalingam [19], that under certain conditions on the space X, minimizers of (1) satisfy the Harnack inequality and the maximum principle, and are locally Hölder continuous.

Potential theory in metric spaces has been studied in the last fifteen years in many papers. The Dirichlet problem for minimizers was considered e.g. in Björn and Björn [1,2], Björn, Björn and Shanmugalingam [5,6] and Shanmugalingam [22,23].

Let Ω be a bounded open subset of X whose complement has positive capacity. We minimize the p-Dirichlet integral (1) on Ω among all functions which have prescribed boundary values f and lie between two given obstacles ψ_1 and ψ_2. The minimizer is called a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. This generalizes the Euclidean obstacle problem based on equations of p-Laplace type as e.g. in Kinderlehrer and Stampacchia [17] and Malý and Ziemer [21], see also Section 5.3 in [21] and the references therein. In particular existence and regularity for the solutions were shown. Further results about the obstacle problem in $\mathbf{R}^n$ can be found in Heinonen, Kilpeläinen and Martio [15], which concerns the single obstacle problem. In Dal Maso, Mosco and Vivaldi [10] the double obstacle problem in $\mathbf{R}^n$ was considered for $p = 2$ and $f \equiv 0$.

In the general setting of metric measure spaces, the single obstacle problem in metric spaces has been investigated in Kinnunen and Martio [18], where it was shown that there is a unique solution, up to equivalence in $N^{1,p}(X)$, of the single obstacle problem which satisfies the weak Harnack inequality and has a lower semicontinuous representative. In Farnana [11–13] the double obstacle problem on metric spaces was studied. In particular existence, uniqueness and regularity of the solutions were shown.

If Ω is not regular, then for some $f \in C(\partial\Omega)$, the solution of the Dirichlet problem for harmonic functions does not attain the boundary values at some points. This led Wiener [24] to his definition of generalized (Wiener) solutions of the Dirichlet problem which is based on approximating Ω by regular sets (e.g. polyhedra) and showing that the solutions of the Dirichlet problem in these sets converge, in some sense, to a unique harmonic function. In metric measure spaces it has been shown that any open set Ω can be approximated by regular sets and moreover there exists a unique Wiener solution of the Dirichlet problem for $f \in C(\partial\Omega)$, see Björn and Björn [2], Theorems 1.1 and 4.2.

In this paper we study various convergence properties of the obstacle problem. In particular, we consider an increasing sequence of open sets Ω_j whose union is Ω. We analyze the convergence of the solutions u_j of the obstacle problems corresponding to the sets Ω_j. In this work we give several generalizations of Theorem 4.3 in Björn and Björn [2]. Our purpose here is to give sufficient conditions on the obstacles and the boundary values which imply that the sequence of solutions u_j converges to the solution of the obstacle problem corresponding to the set Ω, in some sense.

In particular we have the following result as a special case of Theorem 4.1.

Theorem 1.1. *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets and $f \in N^{1,p}(\Omega)$. Then*

$$H_{\Omega_j} f \to Hf \quad \text{as } j \to \infty,$$

locally uniformly in Ω, where $H_{\Omega_j} f$ is the p-harmonic extension of f to Ω_j (the continuous solution of the $\mathcal{K}_{-\infty,\infty,f}(\Omega_j)$-problem) and Hf is the p-harmonic extension of f to Ω.

This extends Theorem 4.3 from Björn and Björn [2] where a similar result was proved for $f \in C(\overline{\Omega})$.

Moreover, we extend the previous result to double obstacle problems i.e. for $\psi_1 \geqslant -\infty$ and $\psi_2 \leqslant \infty$. In order to obtain convergence of solutions we impose some additional assumptions on the boundary values, the obstacles and the set Ω. In particular we obtain the following result which is essentially a special case of Theorem 4.2, see also Theorem 4.3.

Theorem 1.2. *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets and $f \in N^{1,p}(\Omega) \cap C(\overline{\Omega})$. Let also $\psi_1 : \Omega \to [-\infty, \infty)$ and $\psi_2 : \Omega \to (-\infty, \infty]$ be continuous (as $\overline{\mathbf{R}}$-valued functions) and such that $\psi_1 \leqslant f \leqslant \psi_2$ in Ω. Then the continuous solutions u_j of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega_j)$-problems converge q.e. in Ω to the continuous solution u of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem.*

2. Notation and preliminaries

A nonnegative Borel function g is said to be an *upper gradient* of an extended real-valued function f on X if for all rectifiable curves $\gamma : [0, l_\gamma] \to X$ parameterized by arc length ds, we have

$$\left|f\big(\gamma(0)\big) - f\big(\gamma(l_\gamma)\big)\right| \leqslant \int_\gamma g\,ds \tag{2}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2) holds for p-almost every curve, then g is a *p-weak upper gradient* of f.

By saying that (2) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [22]. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \leqslant g$ a.e., see Corollary 3.7 in Shanmugalingam [23].

The operation of taking the upper gradient is not linear. However, we have the following useful property. If $a, b \in \mathbf{R}$ and g_1 and g_2 are upper gradients of u_1 and u_2 respectively, then $|a|g_1 + |b|g_2$ is an upper gradient of $au_1 + bu_2$.

In Shanmugalingam [22], upper gradients have been used to define Sobolev type spaces on metric spaces. We will use the following equivalent definition.

Definition 2.1. Let $u \in L^p(X)$. Then we define

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p},$$

where g_u is the minimal p-weak upper gradient of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u\colon \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p. 249 in Shanmugalingam [22].

Definition 2.2. The *capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geqslant 1$ on E.

We say that a property holds *quasieverywhere* (q.e.) in X, if it holds everywhere except on a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u, v \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [22] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

We shall need the following result. For a proof see Corollary 3.3 in Björn, Björn and Parviainen [4].

Lemma 2.3. *Assume that $\{u_j\}_{j=1}^\infty$ is a bounded sequence in $N^{1,p}(X)$ and that $u_j \to u$ q.e. in X. Then $u \in N^{1,p}(X)$ and*

$$\int_X g_u^p \, d\mu \leqslant \liminf_{j\to\infty} \int_X g_{u_j}^p \, d\mu.$$

From now on we assume that X supports a *p-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \geqslant 1$ such that for all balls $B(z,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \leqslant Cr \left(\fint_{B(z,\lambda r)} g^p \, d\mu \right)^{1/p},$$

where $u_{B(z,r)} := \fint_{B(z,r)} u \, d\mu$.

Under the above assumptions, every function $u \in N^{1,p}(X)$ is quasicontinuous, i.e. for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{X\setminus G}$ is continuous, see Theorem 1.1 in Björn, Björn and Shanmugalingam [7]. Moreover, when restricted to $\mathbf{R}^n$ the Newtonian space $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space $W^{1,p}(\mathbf{R}^n)$, as defined in Chapter 4 in Heinonen, Kilpeläinen and Martio [15].

For $\Omega \subset X$ open we define the space $N^{1,p}(\Omega)$ with respect to the restrictions of the metric d and the measure μ to Ω. It is well known in the field that the restriction to Ω of a minimal p-weak upper gradient in X remains minimal with respect to Ω, see Björn and Björn [3].

A function u is said to belong to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(G)$ for every open $G \Subset \Omega$, where by $G \Subset \Omega$ we mean that the closure of G is a compact subset of Ω.

To be able to compare the boundary values of Newtonian functions we need to define a Newtonian space with zero boundary values outside of Ω as follows

$$N_0^{1,p}(\Omega) = \{f|_\Omega\colon f \in N^{1,p}(X) \text{ and } f = 0 \text{ q.e. in } X \setminus \Omega\}.$$

Under our assumptions, Newtonian functions with compact support are dense in $N_0^{1,p}(\Omega)$, see Shanmugalingam [23]. Moreover the proof of this result in Björn and Björn [3] shows that if $0 \leqslant u \in N_0^{1,p}(\Omega)$, then we can choose the Newtonian approximations to be nonnegative and pointwise smaller than the function u.

We assume throughout the rest of this paper that $\Omega \subset X$ is a nonempty bounded open set such that $C_p(X \setminus \Omega) > 0$. Also, the letter C represents various positive constants whose values can change even within the same line of a calculation.

We shall need the following Poincaré type inequality. For a proof see e.g. Kinnunen and Shanmugalingam [19], Lemma 2.1.

Lemma 2.4. *There exists a constant $C > 0$ such that for all $u \in N_0^{1,p}(\Omega)$ we have*

$$\int_\Omega |u|^p \, d\mu \leqslant C \int_\Omega g_u^p \, d\mu.$$

The following definition is slightly different from the notation used in Kinnunen and Martio [18]. We use q.e. inequalities rather than a.e. inequalities as in [18], for more discussion see p. 265 in Farnana [11].

Definition 2.5. Let $V \subset X$ be a nonempty bounded open set such that $C_p(X \setminus V) > 0$, let $f \in N^{1,p}(V)$ and $\psi_i : V \to \overline{\mathbf{R}}$, $i = 1, 2$. Then we define the obstacle problem with obstacles ψ_1, ψ_2 and boundary values f by

$$\mathcal{K}_{\psi_1,\psi_2,f}(V) = \bigl\{v \in N^{1,p}(V)\colon\ v - f \in N_0^{1,p}(V) \text{ and } \psi_1 \leqslant v \leqslant \psi_2 \text{ q.e. in } V\bigr\}.$$

Furthermore, a function $u \in \mathcal{K}_{\psi_1,\psi_2,f}(V)$ is a *solution of the* $\mathcal{K}_{\psi_1,\psi_2,f}(V)$*-problem* if

$$\int_V g_u^p \, d\mu \leqslant \int_V g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_{\psi_1,\psi_2,f}(V).$$

We also let $\mathcal{K}_{\psi_1,\psi_2,f} = \mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$, $\mathcal{K}_{\psi,f}(V) = \mathcal{K}_{\psi,\infty,f}(V)$ and $\mathcal{K}_{\psi,f} = \mathcal{K}_{\psi,f}(\Omega)$.

A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *minimizer* in Ω if it is a solution of the $\mathcal{K}_{-\infty,u}(\Omega')$-problem for every open $\Omega' \Subset \Omega$. Similarly, a function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *superminimizer* in Ω if it is a solution of the $\mathcal{K}_{u,u}(\Omega')$-problem for every open $\Omega' \Subset \Omega$. A solution of the $\mathcal{K}_{\psi,f}$-obstacle problem is a superminimizer in Ω, but the converse is not true in general. However, if $u \in N^{1,p}(\Omega)$ and u is a superminimizer in Ω, then u is a solution of the $\mathcal{K}_{u,u}(\Omega)$-obstacle problem. We also say that u is a *subminimizer* if $-u$ is a superminimizer.

The following result, which we will need later, is a combination of Theorem 4.2 and Remark 4.4 in Kinnunen and Shanmugalingam [19] and Lemma 4.1, Theorem 4.4 and Remark 4.5 in Kinnunen and Martio [18].

Proposition 2.6. *Let $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ and $B(x,r) \subset \Omega$. Assume that either*

(a) *u is a solution of the $\mathcal{K}_{\psi,f}(\Omega)$-problem and $k \geqslant \psi$ q.e. in $B(x,r)$; or*
(b) *u is a subminimizer in Ω and $k \in \mathbf{R}$.*

Then for all $q > 0$ and $r > 0$,

$$\operatorname*{ess\,sup}_{B(x,r/2)} u \leqslant k + C\biggl(\fint_{B(x,r)} (u-k)_+^q \, d\mu\biggr)^{1/q}.$$

We shall need the following results from Farnana [11].

Theorem 2.7. *Let $f \in N^{1,p}(\Omega)$ and $\psi_i : \Omega \to \overline{\mathbf{R}}$, $i = 1, 2$. If $\mathcal{K}_{\psi_1,\psi_2,f}$ is nonempty, then there is a unique solution (up to equivalence in $N^{1,p}(\Omega)$) of the $\mathcal{K}_{\psi_1,\psi_2,f}$-problem.*

Lemma 2.8. *Let $\psi, \psi', \varphi, \varphi' : \Omega \to \overline{\mathbf{R}}$ and $f, f' \in N^{1,p}(\Omega)$. Let u be a solution of the $\mathcal{K}_{\psi,\varphi,f}$-problem and u' be a solution of the $\mathcal{K}_{\psi',\varphi',f'}$-problem. Assume that $(f - f')_+ \in N_0^{1,p}(\Omega)$ and that $\psi \leqslant \psi'$ and $\varphi \leqslant \varphi'$ q.e. in Ω. Then $u \leqslant u'$ q.e. in Ω.*

Moreover if u and u' are continuous then $u \leqslant u'$ everywhere in Ω.

The following result shows that we can obtain a continuous solution of the obstacle problem if the obstacles are continuous. It extends Theorem 3.9 in Farnana [11] to the $\overline{\mathbf{R}}$-valued continuous obstacles. For a proof see Corollary 3.4 in Farnana [12] and Theorem 3.10 in [11].

Theorem 2.9. *Let $\psi_1 : \Omega \to [-\infty, \infty)$ and $\psi_2 : \Omega \to (-\infty, \infty]$ be continuous (as $\overline{\mathbf{R}}$-valued functions). Let also $f \in N^{1,p}(\Omega)$ and assume that $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$ is nonempty. Then there exists a continuous solution u in Ω of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. Moreover, u is a minimizer in the open set $\{x \in \Omega\colon\ \psi_1(x) < u(x) < \psi_2(x)\}$.*

A function $v : \Omega \to \mathbf{R}$ is p-*harmonic* in Ω if it is a continuous minimizer.

If Ω is bounded and $C_p(X \setminus \Omega) > 0$, then for every $f \in C(\partial\Omega)$ there exists a unique bounded p-harmonic function $H_\Omega f = Hf$ in Ω such that

$$\lim_{\Omega \ni x \to x_0} Hf(x) = f(x_0) \quad \text{for q.e. } x_0 \in \partial\Omega, \tag{3}$$

see Theorem 6.1 and Corollary 6.2 in Björn, Björn and Shanmugalingam [6] together with Theorem 3.9 in Björn, Björn and Shanmugalingam [5].

Definition 2.10. Let Ω be bounded with $C_p(X \setminus \Omega) > 0$. A point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni x \to x_0} Hf(x) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

If all $x_0 \in \partial\Omega$ are regular, then Ω is *regular*. We also say that $x_0 \in \partial\Omega$ is *irregular* if it is not regular.

Regularity can be characterized in many different ways, see Björn and Björn [1], Theorem 6.1 and Farnana [11], Theorem 4.7.

The following property is called the *Kellogg property*: If I_p denotes the set of all irregular points in $\partial\Omega$, then $C_p(I_p) = 0$, see Björn, Björn and Shanmugalingam [5], Theorem 3.9.

We recall the following result from Farnana [11].

Proposition 2.11. *Let $\psi_i : \Omega \to \bar{\mathbf{R}}$, $i = 1, 2$, and $f \in N^{1,p}(\Omega)$. Let $x_0 \in \partial\Omega$ be a regular boundary point and such that $f(x_0) := \lim_{\Omega \ni y \to x_0} f(y)$ exists and that*

$$C_p\text{-}\limsup_{\Omega \ni y \to x_0} \psi_1(y) \leqslant f(x_0) \leqslant C_p\text{-}\liminf_{\Omega \ni y \to x_0} \psi_2(y).$$

If u is a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}$-problem, then

$$C_p\text{-}\lim_{\Omega \ni y \to x_0} u(y) = f(x_0).$$

Here C_p-$\limsup$, C_p-$\liminf$ and C_p-$\lim$ stand for essential limits up to sets of capacity zero.

We shall need the following result from Farnana [13].

Proposition 2.12. *Let $\{f_j\}_{j=1}^{\infty}$, $\{\psi_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ be q.e. decreasing sequences converging to f, ψ and φ, respectively, such that $\psi_j \to \psi$ in $N^{1,p}(\Omega)$. Assume that $\{f_j\}_{j=1}^{\infty}$ is bounded in $N^{1,p}(\Omega)$ and $\psi_j - f_j \in N_0^{1,p}(\Omega)$, $j = 1, 2, \ldots$. Let also u_j be a solution of the $\mathcal{K}_{\psi_j,\varphi_j,f_j}(\Omega)$-problem, $j = 1, 2, \ldots$, and u be a solution of the $\mathcal{K}_{\psi,\varphi,f}(\Omega)$-problem. Then u_j decreases to u q.e. in Ω.*

3. Auxiliary results

Definition 3.1. Let $\{f_n\}_{n \in \mathbf{N}}$ be a family of continuous functions. Then $\{f_n\}_{n \in \mathbf{N}}$ is said to be *equicontinuous* at x_0 if for each $\varepsilon > 0$ there exists a neighborhood $U \ni x_0$ such that

$$\sup_U f_n - \inf_U f_n \leqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

The following lemma is Theorem 1.6 in Li and Martio [20]. As the proof therein is for domains in $\mathbf{R}^n$ we include the proof here for completeness.

Lemma 3.2. *Let $\mathcal{S}$ be a family of continuous solutions of the single obstacle problem in Ω, i.e. for each $u \in \mathcal{S}$ there are $\psi_u : \Omega \to \mathbf{R}$ and $f_u \in N^{1,p}(\Omega)$ such that u is the continuous solution of the $\mathcal{K}_{\psi_u,f_u}$-problem. Let $\mathcal{Q} = \{\psi_u \colon u \in \mathcal{S}\}$. Suppose that for $x_0 \in \Omega$ there are $M < \infty$ and a neighborhood $V \ni x_0$ such that*

$$\sup_V u - \inf_V u \leqslant M \quad \textit{for all } u \in \mathcal{S}.$$

If the family $\mathcal{Q}$ is equicontinuous at x_0 then the family $\mathcal{S}$ is equicontinuous at x_0.

Proof. Fix $\varepsilon > 0$ and $B = B(x_0, r)$ with $20\lambda B = B(x_0, 20\lambda r) \Subset \Omega$, such that

$$\sup_B u - \inf_B u \leqslant M \tag{4}$$

and

$$\sup_{20\lambda B} \psi_u - \inf_{20\lambda B} \psi_u \leqslant \varepsilon \quad \text{for all } u \in \mathcal{S}. \tag{5}$$

Fix $u \in \mathcal{S}$. We consider two cases. First assume that

$$\inf_{\frac{1}{2}B} u > \sup_{\frac{1}{2}B} \psi_u,$$

this implies that $\frac{1}{2}B \subset \{x \in \Omega\colon\ u(x) > \psi_u(x)\}$ and hence by Theorem 2.9 u is a minimizer in B. Theorem 5.2 in Kinnunen and Shanmugalingam [19] implies that for $0 < s < r/2$ we have

$$\sup_{B(x_0,s)} u - \inf_{B(x_0,s)} u \leqslant 8^\alpha \left(\frac{s}{r}\right)^\alpha \left(\sup_{\frac{1}{2}B} u - \inf_{\frac{1}{2}B} u\right) \leqslant 8^\alpha \left(\frac{s}{r}\right)^\alpha M,$$

for some α with $0 < \alpha \leqslant 1$. Choosing $s \in (0, r/2]$ with

$$8^\alpha \left(\frac{s}{r}\right)^\alpha M < \varepsilon$$

we see that

$$\sup_{B(x_0,s)} u - \inf_{B(x_0,s)} u \leqslant \varepsilon. \tag{6}$$

Next assume that

$$\inf_{\frac{1}{2}B} u \leqslant \sup_{\frac{1}{2}B} \psi_u. \tag{7}$$

Let $m = \sup_B \psi_u$ and note that $m = -\infty$ is not possible here, since it would imply that

$$\sup_B u \leqslant \inf_B u + M \leqslant \sup_B \psi_u + M = -\infty,$$

i.e. $u \equiv -\infty$ in B which contradicts the fact that $u \in N^{1,p}(B)$. Also by (5), $m = \infty$ is always impossible. Therefore we may assume that $m = 0$. Now u is a solution of the $\mathcal{K}_{\psi_u,u}(B)$-problem and $\psi_u \leqslant m = 0$ in B. Proposition 2.6 implies that

$$\sup_{\frac{1}{2}B} u \leqslant C\left(\fint_B u_+^q \, d\mu\right)^{1/q} \quad \text{for all } q > 0, \tag{8}$$

where $u_+ = \max\{u, 0\}$. Next from (5) we have

$$u + \varepsilon \geqslant \psi_u + \varepsilon \geqslant \inf_{20\lambda B} \psi_u + \varepsilon \geqslant \sup_{20\lambda B} \psi_u \geqslant 0,$$

hence $u + \varepsilon$ is a nonnegative superminimizer in $20\lambda B$. Theorem 9.2 in Björn and Marola [8] provides us with $q > 0$ and $C > 0$, only depending on p, the doubling constant and the constants in the Poincaré inequality, such that

$$\left(\fint_B (u+\varepsilon)^q \, d\mu\right)^{1/q} \leqslant C \inf_{\frac{1}{2}B} (u + \varepsilon). \tag{9}$$

Now combining (7), (8) and (9) and using that $u_+ \leqslant u + \varepsilon$, yields

$$\begin{aligned}\sup_{\frac{1}{2}B} u &\leqslant C\left(\fint_B u_+^q \, d\mu\right)^{1/q} \leqslant C\left(\fint_B (u+\varepsilon)^q \, d\mu\right)^{1/q} \\ &\leqslant C \inf_{\frac{1}{2}B} (u+\varepsilon) \leqslant C \sup_{\frac{1}{2}B} (\psi_u + \varepsilon) \leqslant C\varepsilon.\end{aligned} \tag{10}$$

Using that $u \geqslant \psi_u$ and (5) we see that

$$\inf_{\frac{1}{2}B} u \geqslant \inf_{\frac{1}{2}B} \psi_u \geqslant \inf_{20\lambda B} \psi_u \geqslant -\varepsilon.$$

This and (10) yield

$$\sup_{\frac{1}{2}B} u - \inf_{\frac{1}{2}B} u \leqslant (C+1)\varepsilon. \tag{11}$$

Since $u \in \mathcal{S}$ was arbitrary, inequalities (6) and (11) show that $\mathcal{S}$ is equicontinuous at x_0. $\square$

Next, we prove the following proposition which we will need later. It shows that it is enough to test a function $u \in \mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$ locally in order to show that it is a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem.

Proposition 3.3. *Let $\psi_i : \Omega \to \overline{\mathbf{R}}$, $i = 1, 2$, and $f \in N^{1,p}(\Omega)$. Let also $u \in \mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$. Then u is a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem if and only if u is a solution of the $\mathcal{K}_{\psi_1,\psi_2,u}(\Omega')$-problem for all $\Omega' \Subset \Omega$.*

Proof. The only if direction follows directly from Lemma 3.6 in Farnana [11]. For the other direction assume that $u \in \mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$ is a solution of the $\mathcal{K}_{\psi_1,\psi_2,u}(\Omega')$-problem for all $\Omega' \Subset \Omega$ and let $\tilde{u}$ be a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. We will show below that u is a solution of the $\mathcal{K}_{\psi_1,u,u}(\Omega)$-problem and also a solution of the $\mathcal{K}_{u,\psi_2,u}(\Omega)$-problem. The comparison Lemma 2.8 then implies that $u \leqslant \tilde{u} \leqslant u$ q.e. in Ω. Hence $u = \tilde{u}$ q.e. in Ω and u is also a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem.

To show that u is a solution of the $\mathcal{K}_{\psi_1,u,u}(\Omega)$-problem, let $v \in \mathcal{K}_{\psi_1,u,u}(\Omega)$. Then $v \leqslant u$ q.e. in Ω and $u - v \in N_0^{1,p}(\Omega)$. Using that Newtonian functions with compact support are dense in $N_0^{1,p}(\Omega)$ we see that, for every $\varepsilon > 0$ there exists $\varphi \in N_0^{1,p}(\Omega')$, for some $\Omega' \Subset \Omega$, such that $0 \leqslant \varphi \leqslant u - v$ q.e. and

$$\begin{aligned}\left(\int_\Omega g_{u-\varphi}^p \, d\mu\right)^{1/p} &\leqslant \left(\int_\Omega g_v^p \, d\mu\right)^{1/p} + \left(\int_\Omega g_{u-v-\varphi}^p \, d\mu\right)^{1/p} \\ &\leqslant \left(\int_\Omega g_v^p \, d\mu\right)^{1/p} + \varepsilon. \end{aligned} \tag{12}$$

Further we have $\psi_1 \leqslant v \leqslant u - \varphi \leqslant u \leqslant \psi_2$ q.e. in Ω. Hence $u - \varphi \in \mathcal{K}_{\psi_1,\psi_2,u}(\Omega')$. As u is a solution of the $\mathcal{K}_{\psi_1,\psi_2,u}(\Omega')$-problem we get

$$\int_{\Omega'} g_u^p \, d\mu \leqslant \int_{\Omega'} g_{u-\varphi}^p \, d\mu.$$

This and the fact that $g_u = g_{u-\varphi}$ a.e. in $\Omega \setminus \Omega'$ together with (12) imply that

$$\left(\int_\Omega g_u^p \, d\mu\right)^{1/p} \leqslant \left(\int_\Omega g_{u-\varphi}^p \, d\mu\right)^{1/p} \leqslant \left(\int_\Omega g_v^p \, d\mu\right)^{1/p} + \varepsilon.$$

Letting $\varepsilon \to 0$ shows that u is a solution of the $\mathcal{K}_{\psi_1,u,u}(\Omega)$-problem. Similarly, we conclude that u is a solution of the $\mathcal{K}_{u,\psi_2,u}(\Omega)$-problem, by using the above argument for the solution $-u$ of the $\mathcal{K}_{-\psi_2,-\psi_1,-f}(\Omega)$-problem. $\square$

We shall need the following lemma which can be proved easily using the comparison Lemma 2.8.

Lemma 3.4. *Let $\psi, \psi_j, \varphi, \varphi_j : \Omega \to \overline{\mathbf{R}}$ and $f, f_j \in N^{1,p}(\Omega)$, $j = 1, 2, \ldots$, be such that $\psi_j \to \psi$, $\varphi_j \to \varphi$ and $f_j \to f$ q.e. uniformly in Ω. Let also u_j be a solution of the $\mathcal{K}_{\psi_j,\varphi_j,f_j}$-problem and u be a solution of the $\mathcal{K}_{\psi,\varphi,f}$-problem. Then $u_j \to u$ q.e. uniformly in Ω.*

Here we say that $w_j \to w$ q.e. uniformly in Ω if there exists a set $E \subset \Omega$ such that $C_p(E) = 0$ and $w_j \to w$ uniformly in $\Omega \setminus E$.

4. Convergence of the obstacle problems in varying domains

If Ω is not regular, then for some $f \in C(\partial\Omega)$, the solution of the Dirichlet problem for p-harmonic functions does not attain the prescribed boundary values at some points. This led Wiener [24] to his definition of generalized (Wiener) solutions of the Dirichlet problem for harmonic functions which is based on approximating Ω by regular sets. In metric measure spaces it was shown that any open set Ω can be approximated by regular sets and moreover there exists a unique Wiener solution of the Dirichlet problem for $f \in C(\partial\Omega)$, see Björn and Björn [2], Theorems 1.1, 4.2 and 4.3. In this section we give several generalizations of Theorem 4.3 in [2].

Theorem 4.1. *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \bigcup_{j=1}^\infty \Omega_j$ be open sets, $\psi : \Omega \to [-\infty, \infty)$ be continuous (as an $\overline{\mathbf{R}}$-valued function) and $f \in N^{1,p}(\Omega)$ be such that $f \geqslant \psi$ in Ω. Let also u_j be the continuous solution of the $\mathcal{K}_{\psi,f}(\Omega_j)$-problem provided by Theorem 2.9 and u be the continuous solution of the $\mathcal{K}_{\psi,f}(\Omega)$-problem. Then $u_j \to u$ locally uniformly in Ω, where we define $u_j = f$ in $\Omega \setminus \Omega_j$.*

Proof. As $f \in \mathcal{K}_{\psi,f}(\Omega_j)$ and u_j is a solution of the $\mathcal{K}_{\psi,f}(\Omega_j)$-problem, $j = 1, 2, \dots,$ we have

$$\int_{\Omega_j} g_{u_j}^p \, d\mu \leqslant \int_{\Omega_j} g_f^p \, d\mu. \tag{13}$$

Since $u_j - f \in N_0^{1,p}(\Omega_j)$ it follows from Lemma 2.4 and (13) that

$$\begin{aligned} \|u_j - f\|^p_{N^{1,p}(\Omega)} = \|u_j - f\|^p_{N^{1,p}(\Omega_j)} &\leqslant C \int_{\Omega_j} g^p_{u_j - f} \, d\mu \\ &\leqslant C\biggl(\int_{\Omega_j} g^p_{u_j} \, d\mu + \int_{\Omega_j} g^p_f \, d\mu \biggr) \leqslant C \int_{\Omega} g^p_f \, d\mu. \end{aligned} \tag{14}$$

Hence $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(\Omega)$.

Next, fix $k \in \mathbf{N}$. For $x \in \Omega_k$ choose $B = B(x,r) \Subset \Omega_k$. As ψ is continuous and $B \Subset \Omega_k$ we can find $M \in \mathbf{R}$ such that $\psi \leqslant M$ in B. It follows from Proposition 2.6 and the boundedness of $\{u_j\}_{j=1}^\infty$ in $N^{1,p}(\Omega)$ that

$$\begin{aligned} \sup_{\frac{1}{2}B} u_j &\leqslant M + C\biggl(\fint_B |u_j - M|^p \, d\mu \biggr)^{1/p} \\ &\leqslant M + \frac{C}{\mu(B)^{1/p}} \biggl(\int_\Omega |u_j - M|^p \, d\mu \biggr)^{1/p} \leqslant C_x \end{aligned} \tag{15}$$

where C_x is independent of j, since $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(\Omega)$. On the other hand we have that $-u_j$ is a subminimizer in B, for $j \geqslant k$, and Proposition 2.6, with $k = 0$, then implies that

$$\begin{aligned} \sup_{\frac{1}{2}B}(-u_j) &\leqslant C\biggl(\fint_B |-u_j|^p \, d\mu \biggr)^{1/p} \\ &\leqslant \frac{C}{\mu(B)^{1/p}} \biggl(\int_\Omega |u_j|^p \, d\mu \biggr)^{1/p} \leqslant C'_x. \end{aligned} \tag{16}$$

Hence we conclude from (15) and (16) that u_j, $j \geqslant k$, are locally equibounded in Ω_k. Moreover, as ψ is continuous and u_j is the continuous solution of the $\mathcal{K}_{\psi,u_j}(\Omega_k)$-problem, for all $j \geqslant k$, Lemma 3.2 shows that $\{u_j\}_{j=k}^\infty$ is locally equicontinuous in Ω_k. By the Arzelà–Ascoli theorem we conclude that there is a subsequence $\{u'_j\}_{j=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ which converges locally uniformly in Ω_1 and the limit function is a continuous superminimizer, by Remark 6.7 in Kinnunen and Martio [18]. Now $\{u'_j\}_{j=2}^\infty$ is an equicontinuous and equibounded sequence of superminimizers in Ω_2. Another application of the Arzelà–Ascoli theorem provides us with a subsequence which converges locally uniformly to a continuous superminimizer in Ω_2. Proceeding in this way and taking a diagonal sequence we obtain a subsequence of our original sequence (also denoted $\{u_j\}_{j=1}^\infty$) that converges locally uniformly to a continuous superminimizer in Ω. Let $\tilde{u} = \lim_{j\to\infty} u_j$.

Next we show that $\tilde{u} \in \mathcal{K}_{\psi,f}(\Omega)$. For this let $v_j = u_j - f$ extended by zero outside Ω_j and

$$v = \begin{cases} \tilde{u} - f & \text{in } \Omega, \\ 0 & \text{in } X \setminus \Omega. \end{cases}$$

Thus $v_j \to v$ in X. As $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(\Omega)$, we conclude that $\{v_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(X)$ and Lemma 2.3 then shows that $v \in N^{1,p}(X)$ and hence $\tilde{u} - f \in N_0^{1,p}(\Omega)$. Since $u_j \geqslant \psi$ q.e. in Ω, we have $\tilde{u} \geqslant \psi$ q.e. in Ω and hence $\tilde{u} \in \mathcal{K}_{\psi,f}(\Omega)$.

Now, it remains to show that $\tilde{u}$ is a solution of the $\mathcal{K}_{\psi,f}(\Omega)$-problem. To this end let $\Omega' \Subset \Omega$. By compactness we have $\Omega' \Subset \Omega_k$ for some $k \in \mathbf{N}$. As u_j is a solution of the $\mathcal{K}_{\psi,u_j}(\Omega')$-problem, for all $j \geqslant k$, and $u_j \to \tilde{u}$ uniformly in Ω', Lemma 3.4 implies that $\tilde{u}$ is a solution of the $\mathcal{K}_{\psi,\tilde{u}}(\Omega')$-problem. As $\Omega' \Subset \Omega$ was arbitrary it follows from Proposition 3.3 that $\tilde{u}$ is a solution of the $\mathcal{K}_{\psi,f}(\Omega)$-problem and hence $u = \tilde{u}$ in Ω, by the uniqueness of the solution of the obstacle problem.

We also conclude that the original sequence $\{u_j\}_{j=1}^\infty$ converges to u locally uniformly in Ω, since otherwise there would be an $\Omega' \Subset \Omega$, $\varepsilon > 0$, a subsequence $\{u_{j_k}\}_{k=1}^\infty$ and $x_k \in \overline{\Omega'}$, $k = 1, 2, \dots,$ such that

$$\bigl| u_{j_k}(x_k) - u(x_k) \bigr| > \varepsilon. \tag{17}$$

The compactness of $\overline{\Omega'}$ implies that there exists a subsequence of $\{x_k\}_{k=1}^\infty$ (also denoted $\{x_k\}_{k=1}^\infty$) such that $x_k \to x_0 \in \overline{\Omega'}$. By what we have shown above $\{u_{j_k}\}_{k=1}^\infty$ has a subsequence $\{u'_{j_k}\}_{k=1}^\infty$ which converges locally uniformly to u, i.e. uniformly in $\overline{\Omega'}$. Thus, for sufficiently large k, we have

$$\left|u'_{j_k}(x_0) - u(x_0)\right| \leqslant \varepsilon/4. \tag{18}$$

Next, as u is continuous in Ω and $\{u'_{j_k}\}_{k=1}^\infty$ is equicontinuous on $\overline{\Omega'}$ we obtain that, for sufficiently large k,

$$\left|u'_{j_k}(x_k) - u'_{j_k}(x_0)\right| \leqslant \varepsilon/4 \quad \text{and} \quad \left|u(x_k) - u(x_0)\right| \leqslant \varepsilon/4. \tag{19}$$

The triangle inequality together with (17), (18) and (19) then imply that

$$\begin{aligned}\varepsilon &\leqslant \left|u'_{j_k}(x_k) - u(x_k)\right| \\ &\leqslant \left|u'_{j_k}(x_k) - u'_{j_k}(x_0)\right| + \left|u'_{j_k}(x_0) - u(x_0)\right| + \left|u(x_0) - u(x_k)\right| \\ &\leqslant 3\varepsilon/4,\end{aligned}$$

a contradiction. Hence $u_j \to u$ locally uniformly in Ω. $\square$

Now we concentrate on double obstacle problems. In order to get convergence of the solutions we impose some additional assumptions on the obstacles and the boundary values.

Theorem 4.2. *Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \bigcup_{j=1}^\infty \Omega_j$ be open sets and $f \in N^{1,p}(\Omega)$ be bounded and such that $f(x_0) := \lim_{\Omega \ni x \to x_0} f(x)$ exists for all regular boundary points $x_0 \in \partial\Omega$. Assume also that $\psi_j : \Omega \to \bar{\mathbf{R}}$, $j = 1, 2$, are such that ψ_1 is bounded from above and ψ_2 is bounded from below in Ω, and*

$$C_p\text{-}\limsup_{\Omega \ni x \to x_0} \psi_1(x) \leqslant f(x_0) \leqslant C_p\text{-}\liminf_{\Omega \ni x \to x_0} \psi_2(x) \tag{20}$$

for all regular boundary points $x_0 \in \partial\Omega$. Let u_j be a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega_j)$-problem, $j = 1, 2, \ldots$, and u be a solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem. Then $u_j \to u$ q.e. in Ω, where we define $u_j = f$ q.e. in $\Omega \setminus \Omega_j$.

In particular Theorem 4.2 applies if $f \in N^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $\psi_1 \leqslant f \leqslant \psi_2$ in Ω. Note that it is possible to have a soluble obstacle problem without (20), see Example 5.2 in Björn and Björn [1].

Note also that if $f \in N^{1,p}(\Omega)$ in Theorem 4.1 is bounded and $\lim_{x \to x_0} f(x)$ exists for all regular boundary points $x_0 \in \partial\Omega$, then Theorem 4.1 is a special case of Theorem 4.2, if we only consider q.e.-convergence in Theorem 4.1. Otherwise they are unrelated.

Proof. As f is bounded and ψ_2 is bounded from below in Ω we may assume without loss of generality that $0 \leqslant f \leqslant 1$ and $\psi_2 \geqslant 0$ in Ω.

Let $I \subset \partial\Omega$ be the set of all irregular points. Then $C_p(I) = 0$ by the Kellogg property and hence there is a decreasing sequence of open sets $\{V_i\}_{i=1}^\infty$ such that $V_i \supset I$ and $C_p(V_i) < 1/2^{ip}$. Lemma 5.3 in Björn, Björn and Shanmugalingam [6] provides us with a decreasing sequence of nonnegative functions $\{\eta_i\}_{i=1}^\infty$ such that $\|\eta_i\|_{N^{1,p}(X)} < 1/2^i$ and $\eta_i \geqslant 1$ in V_{i+1}.

Next, as $f \geqslant 0$ and $\psi_2 \geqslant 0$, the comparison Lemma 2.8 implies that $u \geqslant 0$ q.e. in Ω. It then follows that $u - f + \eta_i \geqslant 0$ q.e. in $V_{i+1} \cap \Omega$. On the other hand we have $u - f + \eta_i \geqslant 0$ q.e. in $V_{i+1} \setminus \Omega$, where we extend $u - f$ by zero outside Ω. Hence we obtain $u - f + \eta_i \geqslant 0$ q.e. in V_{i+1}. Fix $\varepsilon > 0$, $i \in \mathbf{N}$ and let $V := V_{i+1}$. It then follows that

$$u - f + \eta_i + \varepsilon > u - f + \eta_i \geqslant 0 \quad \text{q.e. in } V. \tag{21}$$

For $x \in \partial\Omega \setminus I$ we can find a ball B_x such that

$$u - f + \eta_i + \varepsilon \geqslant u - f + \varepsilon > 0 \quad \text{q.e. in } B_x, \tag{22}$$

by Proposition 2.11. The compactness of $\overline{\Omega}$ implies that

$$\overline{\Omega} \subset \Omega_n \cup V \cup \bigcup_{k=1}^m B_{x_k} \quad \text{for some } n, m \in \mathbf{N}$$

and hence $\Omega \setminus \Omega_j \subset \bigcup_{k=1}^m B_{x_k} \cup V$ for all $j \geqslant n$. It follows from (21) and (22) that $u + \eta_i \geqslant f - \varepsilon$ q.e. in $\Omega \setminus \Omega_j$ for all $j \geqslant n$.

Now, let v_i be a solution of the $\mathcal{K}_{u+\eta_i,\psi_2+\eta_i,u+\eta_i}(\Omega)$-problem and fix $j \geqslant n$. As v_i is a solution of the $\mathcal{K}_{u+\eta_i,\psi_2+\eta_i,v_i}(\Omega_j)$-problem, $u + \eta_i \geqslant \psi_1 - \varepsilon$, $\psi_2 + \eta_i \geqslant \psi_2 - \varepsilon$ q.e. in Ω_j and $v_i \geqslant u + \eta_i \geqslant f - \varepsilon$ q.e. on $\partial\Omega_j$, the comparison Lemma 2.8 and the fact that $u_j - \varepsilon$ is a solution of the $\mathcal{K}_{\psi_1-\varepsilon,\psi_2-\varepsilon,f-\varepsilon}(\Omega_j)$-problem imply that $v_i \geqslant u_j - \varepsilon$ q.e. in Ω_j. On the other hand we have

$$v_i \geqslant u + \eta_i \geqslant f - \varepsilon = u_j - \varepsilon \quad \text{q.e. in } \Omega \setminus \Omega_j.$$

Hence $v_i \geqslant u_j - \varepsilon$ q.e. in Ω. Letting $j \to \infty$ we obtain $v_i \geqslant \limsup_{j\to\infty} u_j - \varepsilon$ q.e. in Ω. By Proposition 2.12 we have $v_i \to u$ q.e. in Ω. Thus letting $i \to \infty$ and then $\varepsilon \to 0$ we conclude that

$$u \geqslant \limsup_{j\to\infty} u_j \quad \text{q.e. in } \Omega.$$

Applying this to $-u$, a solution of the $\mathcal{K}_{-\psi_2,-\psi_1,-f}(\Omega)$-problem, we obtain

$$u = -(-u) \leqslant -\limsup_{j\to\infty}(-u_j) = \liminf_{j\to\infty} u_j \leqslant \limsup_{j\to\infty} u_j \leqslant u \quad \text{q.e. in } \Omega.$$

Hence $u = \lim_{j\to\infty} u_j$ q.e. in Ω. $\square$

Proof of Theorem 1.2. This follows directly from Theorem 4.2. $\square$

Finally, when Ω is regular and f is continuous, we show that Theorem 4.1 can be extended to double obstacle problems by a rather short proof.

Theorem 4.3. *Let Ω be regular and $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets. Let also $\psi_1 : \Omega \to [-\infty, \infty)$ and $\psi_2 : \Omega \to (-\infty, \infty]$ be continuous (as $\bar{\mathbf{R}}$-valued functions) and $f \in N^{1,p}(\Omega) \cap C(\overline{\Omega})$ be such that $\psi_1 \leqslant f \leqslant \psi_2$ in Ω. Then*

$$u_j \to u, \quad \text{as } j \to \infty,$$

locally uniformly in Ω, where u is the continuous solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega)$-problem and u_j is the continuous solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega_j)$-problem, $j = 1, 2, \ldots$.

Proof. Let $u = f$ on $\partial\Omega$. It follows from Proposition 2.11 that $u \in C(\overline{\Omega})$. Let $\varepsilon > 0$ and

$$G = \{x \in \overline{\Omega}\colon\ u(x) + \varepsilon > f(x)\}.$$

Then G is an open set in the relative topology on $\overline{\Omega}$, by the continuity of $u - f$. The set G contains $\partial\Omega$ by assumption. The compactness of $\overline{\Omega}$ implies that $\overline{\Omega} = G \cup \Omega_k$, for some $k \in \mathbf{N}$, and hence $u + \varepsilon > f$ on $\partial\Omega_j$ for all $j \geqslant k$. Since $u + \varepsilon$ is the continuous solution of the $\mathcal{K}_{\psi_1+\varepsilon,\psi_2+\varepsilon,u+\varepsilon}(\Omega_j)$-problem and u_j is the continuous solution of the $\mathcal{K}_{\psi_1,\psi_2,f}(\Omega_j)$-problem, it follows from the comparison Lemma 2.8 that, for all $j \geqslant k$,

$$u_j \leqslant u + \varepsilon \quad \text{in } \Omega_j.$$

Applying this to $-u_j$ the continuous solution of the $\mathcal{K}_{-\psi_2,-\psi_1,-f}(\Omega_j)$-problem and $-u$ the continuous solution of the $\mathcal{K}_{-\psi_2,-\psi_1,-f}(\Omega)$-problem we obtain

$$-u_j \leqslant -u + \varepsilon \quad \text{in } \Omega_j.$$

Hence we have $|u_j - u| \leqslant \varepsilon$ in Ω_j for all $j \geqslant k$. Letting $\varepsilon \to 0$ implies that $u_j \to u$ locally uniformly in Ω. $\square$

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