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A novel extension of the exponential distribution with application in modeling complex lifetime and environmental data

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Probability distributions are widely utilized throughout several domains of life, particularly for studying data sets from environmental science, biology, medicine, economics, insurance, and many more. Standard probability distributions have been utilized in practice for an extended period. In this work, we proposed a continuous probability distribution based on the Ramos Louzada logic called the Ramos Louzada Exponential model with two parameters. The significance of the proposed model lies in its ability to effectively analyze the phenomena observed in nature. Its utility spans multiple disciplines. In particular, these distributions have demonstrated considerable efficacy in data modeling. The study presents some statistical and mathematical characteristics of the new distribution, such as the ordinary moment, the quantile function, the mean, the variance, and the moment generating function. To ensure precise parameter estimation, two estimation methods are evaluated, including maximum likelihood and Bayesian procedures under three suggested loss functions, accompanied by a simulation study that confirmed the reliability and consistency of the two proposed estimators. The performance of the estimators is evaluated through average estimate and mean square error. The utility of the model was demonstrated using three real-life data sets taken from the lifetime and environmental fields. Employing a meticulous comparative evaluation through an array of goodness-of-fit metrics, including Akaike Information Criterion, Correction Akaike Information Criterion ($\mathcal{CAIC}$), Hannan-Quin Information Criterion, Bayesian Information Criterion, Kolmogorov-Smirnov ($\mathcal{KS}$) statistics with its associated P-values, the proposed model consistently surpassed traditional competing models such as the generalized Rayleigh, truncated Poisson exponential, alpha power transformed exponential, extended exponential, gamma, Weibull, and two parameters Mira distributions. Based on (($\mathcal{CAIC}$), p-values) [(0.1206,0.5640), (0.1002, 0.62), and (0.1121, 0.2414)] for the three proposed datasets, we observe that the proposed distribution offers optimal fitting compared to other rival distributions.

Keywords Bayesian estimation, Lifetime, Loss functions, Maximum likelihood estimation, Ordinary moment, Simulation study

Statistical models have been extensively applied in industries such as survival, accounting, actuarial sciences, insurance, and engineering analysis to analyze data patterns. The success of statistical inference is strongly dependent on how well the chosen probability distribution matches the underlying data patterns. However, standard distributions frequently lack the versatility and accuracy required to effectively describe complex

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information. This constraint has culminated in the development of more adaptive and complex distributional models that better explain real-world data behavior.

Based on existing studies and findings, we can observe that engineering and environmental data sets most often possess a right-tailed distribution. For such types of data sets, probability distributions with heavy-tailed behavior are very competitive models. In this context, many authors have tried to find new statistical models, extending known ones, and better fitting various types of data sets. This flexibility is obtained by adding novel parameters or by some modifications to the classical distribution, increasing its ability to handle complicated scenarios. For more information, one may refer to some recently generated family of distributions, namely, Marshall and Olkin[1], progressive type-II censoring Marshall-Olkin distributions[2], α-power transformation[3], a new two-parameter mixture family of generalized distributions[4], inverse Weibull-G model[5], Kumaraswamy-G family[6], A new modified cubic transmuted-G family of distributions[7], a new power G-family of distributions[8], novel odd Type-X family[9], Tangent exponential-G family of distributions[10], transformed-transformer class of distributions[11], exponentiated generalized alpha power family[12], Topp-Leone Weibull generated family[13], Novel family of probability generating distributions[14], mixture Models of the New Topp-Leone-G family[15], generalized odd linear exponential family of distribution[16], Lomax tangent generalized family of distributions[17], new flexible power-X family[18], novel compound two components of exponentiated class[19], Power Lindley-G family[20], sine π-power odd-G family of distributions[21], new logarithmic tangent-U family[22], new family of upper-truncated distributions[23], transformation of the Chen distribution[24], modified Weibull distribution[25], modified tangent Weibull[26], arc cosine-G family[27], new logarithmic-G family[28], transformed Weibull distribution[29], the Fréchet–Power Rayleigh[30], and new Topp Leone-G family[31].

It is well documented that no particular probability distribution is available to handle every phenomenon, as real-world phenomena are complex, which is the main reason for developing more flexible probability distributions with greater distributional flexibility. Due to this,[32] suggested a new methodology for creating a new class of distributions known as the Ramos-Louzada (RL) class of distributions. The derivation of the distributions can be described as follows:

$$\begin{aligned} M(x) &= \int_0^{-\log[1-K(x)]} w(t)\, dt \\ &= \{W(t)\}_0^{-\log[1-K(x)]}. \end{aligned} \tag{1}$$

Here, $K(x)$ and $k(x)$ are respectively the cumulative distribution function (CDF) and the probability distribution function (PDF) of the standard distribution. Further, by taking respectively $w(t)$ and $W(t)$ as the PDF and CDF of the RL distribution with the forms

$$w(t) = \frac{(\beta^2 + t - 2\beta)}{\beta^2(\beta-1)}\, e^{-t/\beta}, \quad t > 0, \beta \geq 2 \tag{2}$$

and

$$W(t) = 1 - \frac{(\beta^2 + t - \beta)}{\beta(\beta-1)}\, e^{-t/\beta}, \quad t > 0, \beta \geq 2. \tag{3}$$

The CDF of the RL family of distributions is

$$M(x) = 1 - \left(1 - \frac{\log(1-K(x))}{\beta(\beta-1)}\right)[1-K(x)]^{\frac{1}{\beta}} \quad x \in \mathbb{R}, \quad \beta \geq 2. \tag{4}$$

Considering the computation of $\frac{d}{dx}M(x)$, brings us to the mathematical form of the PDF, denoted as $m(x)$, which is represented as follows:

$$m(x) = \frac{k(x)\ [1-K(x)]^{\frac{1}{\beta}-1}}{\beta^2(\beta-1)}\ \left[\beta^2 - 2\beta - \log(1-K(x))\right]. \tag{5}$$

The RL procedure is a novel tool for generalizing a given model, which adds an additional parameter to a distribution. The proposed method has many applications that extend to fitting, especially engineering and environmental areas, and it has an ability to model different kinds of data sets. However, recent works considering the RL technique include[33] who proposed the discrete RL distribution,[34] defined the generalization of the RL model using the exponentiated method, and[35] introduced the inverse power RL with medicine and geology applications.

To our knowledge, the exponential (Exp) distribution is a flexible one-parameter distribution. The Exp model can be utilized in various sectors, including survival, hydrology, insurance, and energy theory. Let Y follows the Exp distribution. The corresponding CDF and PDF are, respectively, given by

$$\Xi(y) = 1 - e^{-\alpha y}, \ 0 < y, \ \alpha > 0 \tag{6}$$

and

$$\xi(y) = \alpha e^{-\alpha y}. \tag{7}$$

In research, the Exp distribution has emerged as a well-known survival distribution. In engineering data modeling, this distribution has received a lot of attention, for example, see[45–47]. While there are several generators to attain adaptability, this can make estimation and interpretation difficult. To address this, we are motivated in this study to offer a new form of the Exp distribution based on the approach discussed in Eq. (1), which is intended to give increased flexibility by introducing an additional parameter. The resultant distribution is named the Ramos Louzada Exponential (RL-Exp) model. It is interesting because it offers a broad range of density shapes, including symmetric, skewed, and heavy-tailed forms while also allowing for a variety of failure rate behaviors, and it blends simplicity in definitions, enabling accurate mathematical understanding and effective data modeling. Such versatility renders it appropriate for a variety of statistical and reliability applications. Further, the n parameter offers increased versatility in analyzing the tail behavior of the specified density function. The following objectives of this study can be described as follows:

- Defined a novel two-parameter generalization of the Exp model by applying the RL tool, which can be enhanced in flexibility, robustness, and applicability to different data scenarios, and established its mathematical properties.
- Two estimation methods, namely, the maximum likelihood estimator (MLE) and Bayesian estimators based on:

  1. Square error (SE) loss function proposed by[38], and it measures the discrepancy between predicted and true values by taking the square of their difference.
  2. Linex (LI) loss function presented by[39], and it is an asymmetric function.
  3. General entropy (GE) defined by[40], and is also an asymmetric loss function

These loss functions can be apply to compute the cost of errors in parameter estimates.

- Three real data sets are applied from engineering and environmental fields to demonstrate the utility of the proposed RL-Exp model in application. For more reading see[36,37,48]

The further study is organized as follows: The recommended RL-Exp is defined in "Ramos Louzada-exponential distribution" with some distributional properties. In "Characterization of the RL-Exp model", various mathematical properties of the new model are established. Section "Statistical inference" demonstrates the estimation of the model parameters based on two different proposed methods. Section "Simulation study and algorithms" delineates the efficacy of the estimations using a Monte Carlo simulation, and three distinct real data sets taken from the engineering and environmental fields are applied to show the results of the application of the RL-Exp model. Section 7 offers final observations.

Ramos Louzada-exponential distribution

A random variable Z is said to follow the RL-Exp model with two parameters α and β if its CDF and PDF are of the form

$$\Psi(z) = 1 - \left(1 + \frac{\alpha z}{\beta(\beta-1)}\right) e^{-\frac{\alpha}{\beta}z}, \quad 0 < z, \quad \beta \geq 2, \alpha > 0 \tag{8}$$

and

$$\psi(z) = \frac{\alpha\, e^{-\frac{\alpha}{\beta}z}\,(\beta^2 - 2\beta + \alpha z)}{\beta^2(\beta-1)}, \; 0 < z, \quad \beta \geq 2, \alpha > 0. \tag{9}$$

The plots of the CDF and PDF functions under some selected parameter values are shown in Fig. 1, which might have a declining trend, unimodal, and right-skewed PDF shapes. Further, when $\alpha = 1$, we obtained the Ramos Lozuada (RL) distribution.

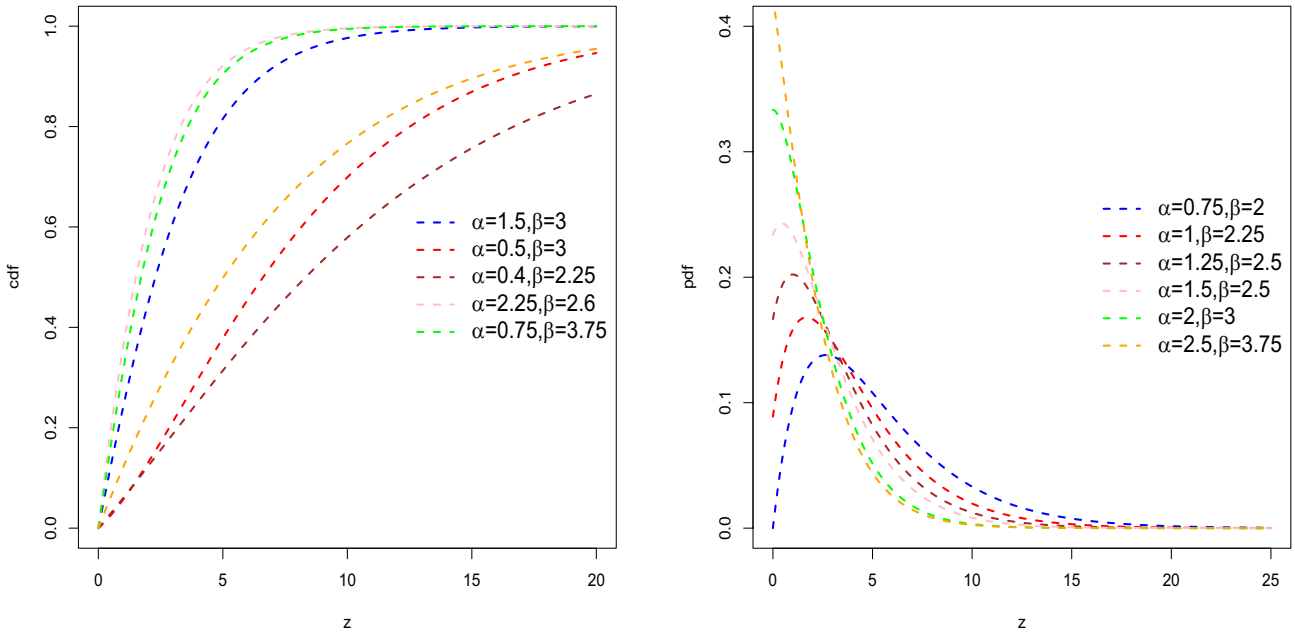
The survival function (SF) and hazard rate function (HRF) are formulated, respectively, as:

$$S(z) = \left(1 + \frac{\alpha z}{\beta(\beta-1)}\right) e^{-\frac{\alpha}{\beta}z}$$

and

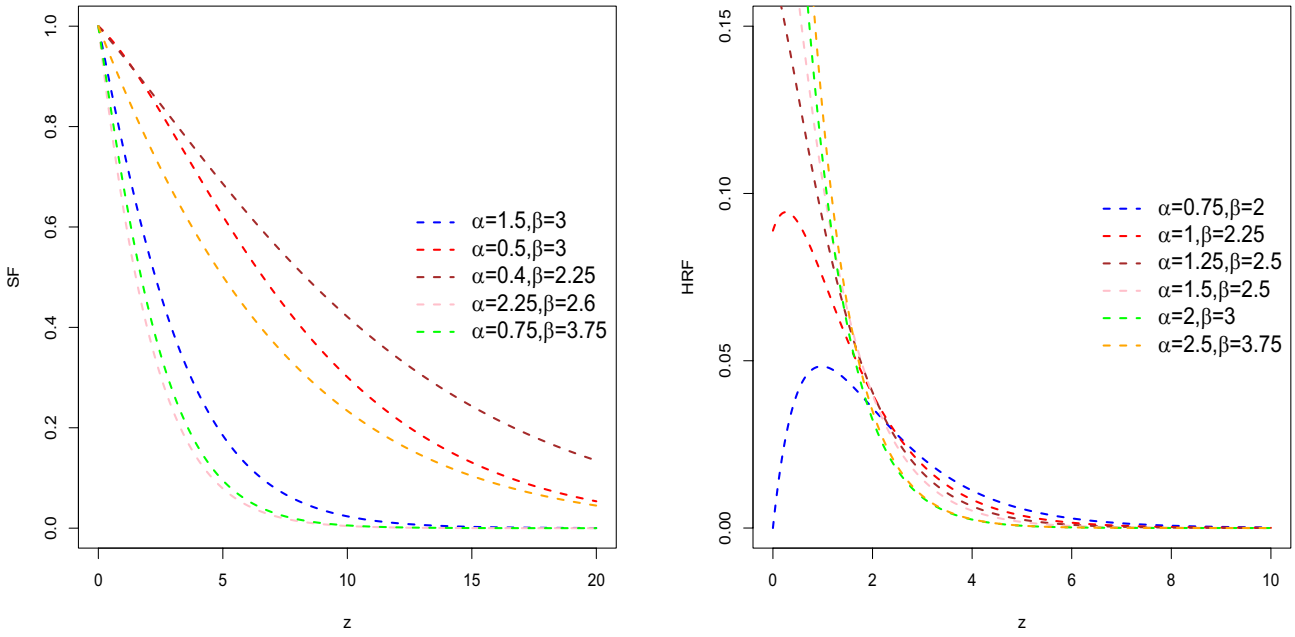
$$h(z) = \frac{\alpha\beta(\beta-1)(\beta^2 - 2\beta + \alpha z)}{\beta(\beta-1) + \alpha z}.$$

The graphical behavior of the SF and HRF for several selected parameters is sketched in Fig. 2 of the RL-Exp model. For the HRF, it has decreasing and unimodal shapes according to the parameter values.



(a) CDF (b) PDF

Figure 1. The CDF and PDF curves of the RL-Exp model are based on some choices of parameters.



(a) SF (b) HRF

Figure 2. The SF and HRF curves of RL-Exp model are based on some choices of parameters.

Characterization of the RL-Exp model

Quantile function and and random number generation

The quantile function serves a significant purpose in producing random numbers from a probability model, which are essential for executing a simulation study. The quantile function z_p associated with the RL-EXP MODEL is written in the following manner:

$$z_u = -\frac{1}{\alpha}\log\left\{\frac{W(-(\beta-1)(1-u)\ e^{-(\beta-1)})}{-(\beta-1)(1-u)}\right\}^{-\beta},\quad 0<u<1, \tag{10}$$

where $W(.)$ represents the Lambert function with the positive branch.

Proof: For $0 < u < 1$, we can define the quantile function of Z as follows:

$$\begin{aligned}
1-\left(1+\frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z} &= u\\
\left(1+\frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z} &= 1-u\\
\left(1+\frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha z}{\beta(\beta-1)}} &= (1-u)e^{-(\beta-1)}\\
-(\beta-1)(1-u)e^{-\frac{\alpha z}{\beta}} &= W(-(\beta-1)(1-u)e^{-(\beta-1)})\\
z &= -\frac{1}{\alpha}\log\left\{\frac{W(-(\beta-1)(1-u)\ e^{-(\beta-1)})}{-(\beta-1)(1-u)}\right\}^{-\beta}
\end{aligned}$$

From Eq. (10), we can be obtain the median value of the proposed RL-Exp model by setting $u = 0.5$, and it is given as

$$z_{0.5} = -\frac{1}{\alpha}\log\left\{\frac{W(-0.5(\beta-1)\ e^{-(\beta-1)})}{-0.5(\beta-1)}\right\}^{-\beta}.$$

Concerning the RL-Exp distribution, the skewness ($\mathcal{SKW}$) and kurtosis ($\mathcal{KUR}$) metrics are, receptively, given in the following way:

$$\mathcal{SKW} = \frac{z_{0.25}+z_{0.75}-2z_{0.5}}{z_{0.75}-z_{0.25}}$$

and

$$\mathcal{KUR} = \frac{z_{0.875}-z_{0.625}+z_{0.375}-z_{0.125}}{z_{0.75}-z_{0.25}}.$$

Ordinary moments

Assume that Z is a random variable following the RL-Exp distribution. The kth moment of Z can be obtained as

$$u'_k = \frac{\beta^k\ \Gamma(k+1)}{\alpha^k(\beta-1)}\ (k+\beta-1), \tag{11}$$

where $\Gamma(n+1) = n!, n = 0, 1, 2, \ldots$.

The mean and variance of Z can be obtained in special case from Eq. (11), and they are in the following form:

$$\mu'_1 = \frac{\beta^2}{\alpha(\beta-1)}$$

and

$$V_Z = \frac{\beta^2(\beta^2-4\beta+2)}{\alpha^2(\beta-1)^2}.$$

The index of dispersion of Z is derived as bellow

$$I_Z = \frac{(\beta^2-4\beta+2)^{\frac{1}{2}}}{\beta}.$$

Moment generating function

Let the random variable Z has a RL-Exp model. Hence, the associated moment generating function (MGF) can be provided in the following form:

$$M_Z(t) = \sum_{k=0}^{\infty}\frac{t^k\beta^k\ \Gamma(k+1)}{\alpha^k(\beta-1)k!}\ (k+\beta-1). \tag{12}$$

Table 1 demonstrates different mathematical properties of the recommended RL-Exp model using various selected parameter values, and Fig. 3 is the 3-dimensional plots of the proposed statistical properties of the new distribution. It can be seen that as β increase and for fixed values of α, the values of μ_1', V_Z, I_Z, $\mathcal{SKW}$, and $\mathcal{KUR}$ increase. Further, for fixed values of β and as α increase the values of μ_1' and V_Z decrease, where the values of I_Z, $\mathcal{SKW}$ and $\mathcal{KUR}$ are constant. In addition, multiple non-monotonic forms are seen, which confirms that the explored RL-Exp model is an adaptable model that accommodates larger data sets.

Order statistics

Let a random sample $z_1, z_2 \ldots z_m$ represent the RL-Exp model. The PDF of the jth order statistic can be expressed as follows

$$f_{(j:m)}(z) = \frac{m!}{(j-1)!(m-j)!}\ \psi(z)\ [\Psi(z)]^{j-1}\ [1-\Psi(z)]^{m-j}$$

$$= \frac{\alpha\ m!\ e^{-\frac{\alpha}{\beta}z}\ (\beta^2 - 2\beta + \alpha z)}{(j-1)!\beta^2(\beta-1)(m-j)!}\ \left[1 - \left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{j-1}$$

$$\times \left[\left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{m-j}.$$

Specially, the PDF of the highest order statistics $f_{(m:m)}(z)$ is obtained as setting $j = m$ in Eq. () to get:

$$f_{(m:m)}(z) = \frac{\alpha\ m\ e^{-\frac{\alpha}{\beta}z}\ (\beta^2 - 2\beta + \alpha z)}{\beta^2(\beta-1)}\ \left[1 - \left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{m-1}.$$

Similarly, the PDF of the lowest order statistics $f_{(1:m)}(z)$ is obtained as setting $j = 1$ in Eq. ()

$$f_{(1:m)}(z) = \frac{\alpha\ m\ e^{-\frac{\alpha}{\beta}z}\ (\beta^2 - 2\beta + \alpha z)}{\beta^2(\beta-1)}\ \left[\left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{m-1}.$$

Furthermore, the jth CDF of the order statistic of the RL-Exp model can be given by

$$F_{(j:m)}(z) = \sum_{i=j}^{m} \Psi^i(z)\ [1-\Psi(z)]^{m-i}$$

$$= \sum_{i=j}^{m} \left[1 - \left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{i}\ \left[\left(1 + \frac{\alpha z}{\beta(\beta-1)}\right)\ e^{-\frac{\alpha}{\beta}z}\right]^{m-i}.$$

α	μ_1'	V_Z	I_Z	$\mathcal{SKW}$	$\mathcal{KUR}$
β = 2 1.5	2.6679	3.5750	0.7087	1.4006	2.8185
3	1.3340	0.8937	0.7087	1.4006	2.8185
4.5	0.8893	0.3972	0.7087	1.4006	2.8185
6	0.6670	0.2234	0.7087	1.4006	2.8185
β = 2.5 1.5	2.7796	5.2747	0.8262	1.4979	3.1436
3	1.3898	1.3187	0.8262	1.4979	3.1436
4.5	0.9265	0.5861	0.8262	1.4979	3.1436
6	0.6949	0.3297	0.8262	1.4979	3.1436
β = 3 1.5	3.0025	7.0366	0.8835	1.6040	3.5708
3	1.5012	1.7592	0.8835	1.6040	3.5708
4.5	1.0008	0.7818	0.8835	1.6040	3.5708
6	0.7506	0.4398	0.8835	1.6040	3.5708
β = 3.5 1.5	3.2697	8.9755	0.9163	1.6817	3.9232
3	1.6348	2.2439	0.9163	1.6817	3.9232
4.5	1.0899	0.9973	0.9163	1.6817	3.9232
6	0.8174	0.5610	0.9163	1.6817	3.9232

Table 1. Various computational metrics of the RL-exp model.

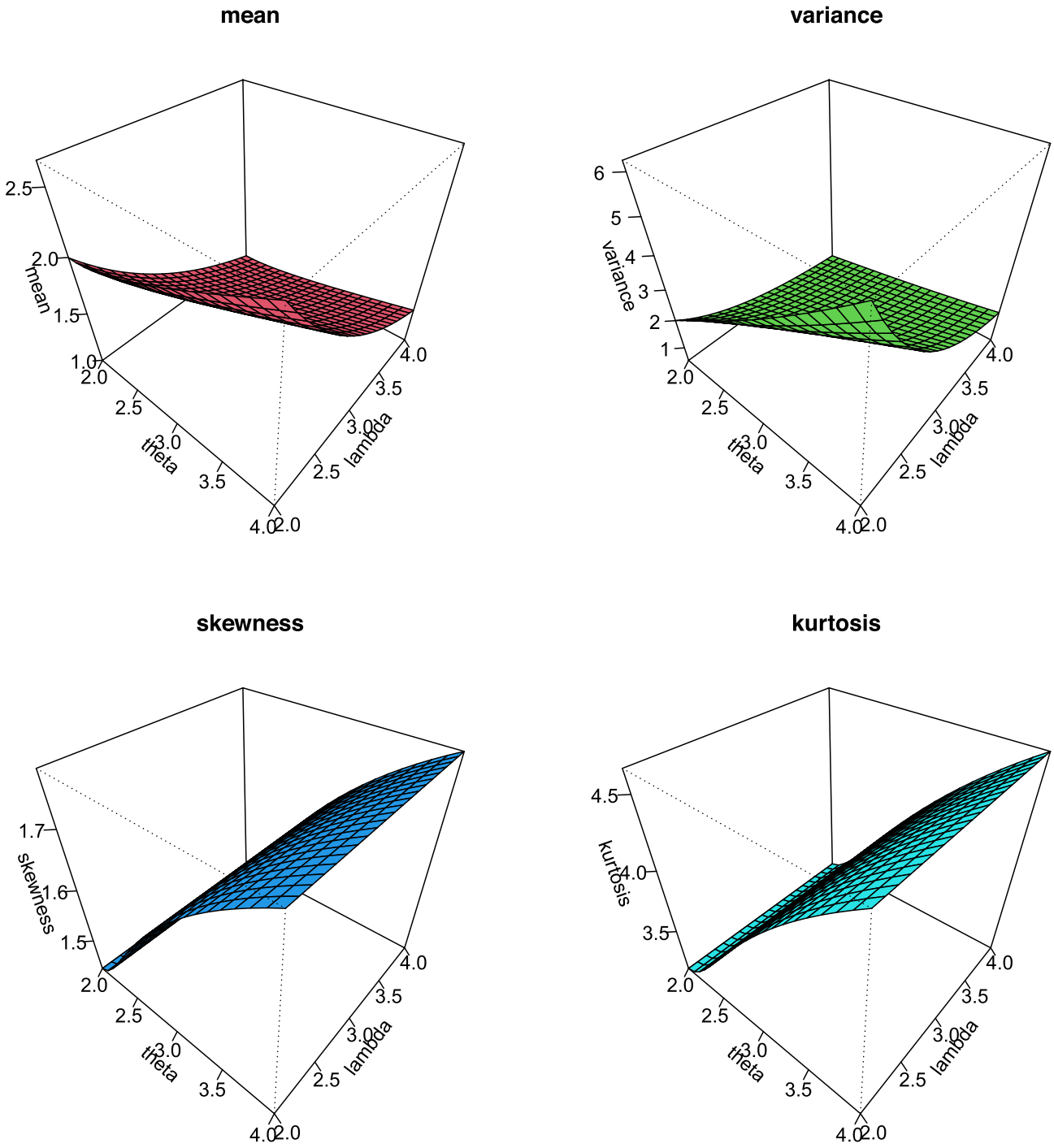


Figure 3. The plots of mean, variance, skewness and kurtosis for the RL-Exp.

Statistical inference

Maximum likelihood estimation

In this section, we estimate model parameters $\rho = (\alpha, \beta)$ using MLE technique based on complete data. Assume that $\{z_1, z_2, \ldots, z_m\}$ is a random sample follows the equation given in Eq. (9). The likelihood function can be represented as

$$L(\rho) = \prod_{i=1}^{m} \psi(z_i). \tag{13}$$

Using Eqs. (9) in (13), we get that:

$$L(\rho) = \frac{1}{\beta^{2m}(\beta-1)^m} \prod_{i=1}^{m} \alpha \; e^{-\frac{\alpha}{\beta} z_i} \; (\beta^2 - 2\beta + \alpha z_i). \tag{14}$$

The corresponding log-likelihood function is given by

$$\log L(\rho) = -2m \log(\beta) - m \log(\beta - 1) + m \log \alpha - \frac{\alpha}{\beta} \sum_{i=1}^{m} z_i + \sum_{i=1}^{m} \log(\beta^2 - 2\beta + \alpha z_i). \tag{15}$$

The first partial derivatives of log-likelihood function in Eq. (15) with respect to α and β provide likelihood equations given as

$$\frac{\partial \log L}{\partial \alpha} = \frac{m}{\alpha} - \frac{1}{\beta} \sum_{i=1}^{m} z_i + \sum_{i=1}^{m} \frac{z_i}{\beta^2 - 2\beta + \alpha z_i} = 0, \tag{16}$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{2m}{\beta} - \frac{m}{\beta - 1} + \frac{\alpha}{\beta^2} \sum_{i=1}^{m} z_i + (2\beta - 2) \sum_{i=1}^{m} \frac{1}{\beta^2 - 2\beta + \alpha z_i} = 0. \tag{17}$$

The maximum likelihood estimates of unknown parameters $\rho = (\alpha, \beta)$ can now be obtained by solving the above two equations. We adopt a numerical method like the Newton-Raphson or secant methods to evaluate these estimates, as the above equations are nonlinear in nature. The well-known statistical programs for maximizing the log-likelihood function are the R package (Adequacy- Model), the Ox program (subroutine-Max-BFGS), and SAS (PROC-NLMIXED). In this study, we applied the R script with the function (optim), employing the method = L-BFGS-B.

Bayesian estimation

In contrast to the maximum likelihood estimation approach, Bayesian estimation is a more efficient approximation. Taking into consideration past data and samples, it calculates the interest parameters that have not been stated. From a Bayesian standpoint, there is no hard and fast method for choosing priors because one cannot clearly assert that one prior is superior to another. When there is insufficient interpretive information about the unknown parameter, a non-informative prior is frequently adopted. In contrast, if considerable information about the parameter(s) is available, an informative prior is generally more effective. Let the parameters α and β have, respectively, the prior gamma distributions (Gamma(a_1, b_1) and Gamma(a_2, b_2). The main motivation for the Gamma prior is usually to constrain the random variables to positive values. This is especially useful when dealing with random variables that represent the variance of another distribution. Next, let define the joint priors for the parameters $\rho = (\alpha, \beta)$ as follows:

$$\Pi(\rho) \propto \alpha^{a_1 - 1} \; \beta^{a_2 - 1} e^{-\frac{\alpha}{b_1} - \frac{\beta}{b_2}}, \quad \alpha, \; \beta > 0. \tag{18}$$

The posterior density of ρ has the following form

$$\Pi^*(\alpha, \beta) = L(\alpha, \beta) \propto \Pi(\alpha, \beta) = \frac{\alpha^{m + a_1 - 1} \; \beta^{a_2 - 1 - 2m}}{(\beta - 1)^m} \; e^{-\frac{\alpha}{b_1} - \frac{\beta}{b_2}} \; e^{-\frac{\alpha}{\beta} \sum_{i=1}^{m} z_i} \prod_{i=1}^{m} (\beta^2 - 2\beta + \alpha z_i). \tag{19}$$

The joint posterior distribution of ρ is given by

$$\begin{aligned} \Pi(\alpha, \beta \mid z) &= \frac{L(\alpha, \beta) \propto \Pi(\alpha, \beta)}{\int_0^\infty \int_0^\infty L(\alpha, \beta) \propto \Pi(\alpha, \beta) \; d\alpha \; d\beta} \\ &= \frac{\frac{\alpha^{m + a_1 - 1} \; \beta^{a_2 - 1 - 2m}}{(\beta - 1)^m} \; e^{-\frac{\alpha}{b_1} - \frac{\beta}{b_2}} \; e^{-\frac{\alpha}{\beta} \sum_{i=1}^{m} z_i} \prod_{i=1}^{m} (\beta^2 - 2\beta + \alpha z_i)}{\int_0^\infty \int_0^\infty \frac{\alpha^{m + a_1 - 1} \; \beta^{a_2 - 1 - 2m}}{(\beta - 1)^m} \; e^{-\frac{\alpha}{b_1} - \frac{\beta}{b_2}} \; e^{-\frac{\alpha}{\beta} \sum_{i=1}^{m} z_i} \prod_{i=1}^{m} (\beta^2 - 2\beta + \alpha z_i) \; d\alpha \; d\beta}. \end{aligned} \tag{20}$$

The marginal posterior of the α and β are obtained to be

$$\Pi(\alpha \mid \beta, z) = \alpha^{m + a_1 - 1} \; e^{-\alpha \left(\frac{1}{b_1} + \frac{1}{\beta} \right) \sum_{i=1}^{m} z_i} \prod_{i=1}^{m} (\beta^2 - 2\beta + \alpha z_i), \tag{21}$$

and

$$\Pi(\beta \mid \alpha, z) = \beta^{a_2-2m-1}\ (\beta-1)^{-m}\ e^{-\beta\left(\frac{1}{b_2}+\frac{\alpha}{\beta^2}\right)\sum_{i=1}^{m} z_i}\ \prod_{i=1}^{m}(\beta^2 - 2\beta + \alpha z_i) \tag{22}$$

Hence, the Bayes $\phi(\rho)$ based on SE loss function is

$$\tilde{\phi}_{SE}(\rho) = \int_{\rho}(\rho - \hat{\rho})^2\ \Pi(\rho \mid z)\ d\rho. \tag{23}$$

Next, we provide other loss functions, including LI and GE functions. The Bayes estimators based on these loss functions can be formulated, respectively, as $\tilde{\phi}_{LI}(\rho) = -\frac{1}{q}\log\left(E_\rho\left(e^{-q\Pi(\rho)} \mid z\right)\right)$ and $\tilde{\phi}_{GE}(\rho) = \left(E_\rho\left((\Pi(\rho))^{-q} \mid z\right)\right)^{-\frac{1}{q}}$.

By obtaining the joint prior, the posterior function, which is not from a well-known distribution may be determined. Due to this, we are unable to create samples from it; hence, we will employ the Metropolis-Hastening method.

Metropolis hasting method

1. Start with initial values $\alpha^{(0)} = \hat{\alpha}, \beta^{(0)} = \hat{\beta}$.
2. Let $m = 1$ define the first iteration.
3. obtain α^*, β^*, respectively, from a proposal distribution.
4. Compute $H_\alpha = \min\left[1, \frac{\pi^*(\alpha^* \mid \beta^{m-1})}{\pi^*(\alpha^{m-1} \mid \beta^{m-1})}\right], H_\beta = \min\left[1, \frac{\pi^*(\beta^* \mid \alpha^{m-1})}{\pi^*(\beta^{m-1} \mid \alpha^{m-1})}\right]$.
5. For $1 \geq j \geq 2$, generate random samples Q_j, where $Q_j \sim U(0,1)$.
6. If $Q_1 \leq H_\alpha,\ Q_2 \leq H_\beta$, then $\alpha^{(m)} = \alpha^*,\ \beta^{(m)} = \beta^*$, else $\alpha^{(m)} = \alpha^{m-1},\ \beta^{(m)} = \beta^{m-1}$.
7. $m = m + 1$.
8. For *M* repeat of process, the steps (3) to (7) are repeated, and finally, we obtain $\alpha^{(m)}, \beta^{(m)}$ for $m = 1, \ldots, B$.

At the end, the Bayes estimators of α and β using the Metropolis-Hastening algorithm under the proposed loss error functions can be given as

$$\hat{\rho}_{SE} = \frac{1}{D-B}\sum_{l=B+1}^{D}\rho(\alpha^{(l)}, \beta^{(l)}), \tag{24}$$

$$\hat{\rho}_{LI} = -\frac{1}{q}\log\left(\frac{1}{D-B}\sum_{l=B+1}^{D} e^{-q\rho(\alpha^{(l)},\beta^{(l)})}\right) \tag{25}$$

and

$$\hat{\rho}_{GE} = \left(\frac{1}{D-B}\sum_{l=B+1}^{D}(\rho(\alpha^{(l)}, \beta^{(l)}))^{-q}\right)^{-1/q}, \tag{26}$$

where *B* denotes the burn-in period.

Interval estimation

Credible interval

In this subsection, we construct the credible confidence intervals of parameters of $\rho = (\alpha, \beta)$ using the Metropolis-Hastening technique. To cancel the issue of the initial guess, we delete *B* initial samples (burn-in period). Furthermore, applying the idea of Chen and Shao[41], and generating posterior samples, we can compute the credible intervals. Consequently, the $100(1-p)\%$ Cred Int of $\Xi = (\delta, \eta)$ is defined by

$$\frac{1}{M^*}\sum_{j=i_1^*}^{M^*}\rho_{(j)} \leq 1 - p \leq \frac{1}{M^*}\sum_{j=i_1^*}^{M^*}\rho_{(j)}, \tag{27}$$

where, $i_1^* < i_2^*,\ i_1^*, i_2^* \in \{1, 2, \ldots, D\}$ and $M^* = D - B$.

Bootstrap confidence intervals

In this section, we discuss the construction of bootstrap-percentile (boot-p) and bootstrap-t (boot-t) confidence intervals. Step-by-step algorithms are provided below for the two methods:

(A) Boot-p

1. Obtain the estimate $\tilde{\rho}$ based on complete data generated by considering the true parameter as ρ.

2. Obtain the updated estimate $\tilde{\rho}^*$ using $\tilde{\rho}$ as the true parameter.
3. Repeat step 2, desired N_B number of times.
4. Let $\tilde{F}_1(z) = P(\tilde{\rho}^* \leq z)$ is the CDF of $\tilde{\rho}^*$, then obtain $\tilde{\rho}_{B_p}(z) = \tilde{F}_1^{-1}(z)$. An approximate $100(1-p)\%$ confidence interval for η is now given by

$$\left(\tilde{\rho}^*_{B_p}\left(\frac{p}{2}\right), \tilde{\rho}^*_{B_p}\left(1-\frac{p}{2}\right)\right).$$

(B) Boot-t

1. Obtain $\tilde{\rho}^*$ using the boot-p method as discussed in the above algorithm.
2. Compute the statistic $T = \frac{\tilde{\rho}^* - \tilde{\rho}}{\sqrt{\tilde{V}(\tilde{\rho}^*)}}$.
3. Repeat step 2, N_B number of times.
4. Let $\tilde{F}_2(z) = P(\tilde{T} \leq z)$ is the CDF of $\tilde{T}$, then obtain $\tilde{\rho}_{B_t}(z) = \tilde{\rho} + \sqrt{\tilde{V}(\tilde{\rho})}\tilde{F}_2^{-1}(z)$. Further, the approximate $100(1-p)\%$ confidence interval for η is evaluated as

$$\left(\tilde{\rho}^*_{B_p}\left(\frac{p}{2}\right), \tilde{\rho}^*_{B_p}\left(1-\frac{p}{2}\right)\right).$$

Simulation study and algorithms

In this subsection, we examine the performance of MLE and Bayes under the three loss functions estimators considering a finite number of samples. In the context of the RL-Exp distribution, sampling is performed by selecting random samples such as $n = 20, 40, 70, 100$. The samples are obtained through the application of the quantile function, given by

$$z_u = -\frac{1}{\alpha}\log\left\{\frac{W(-(\beta-1)(1-u)\ e^{-(\beta-1)})}{-(\beta-1)(1-u)}\right\}^{-\beta}, \quad 0 < u < 1. \tag{28}$$

The simulation study is conducted for two different sets of $\rho = (\alpha, \beta)$. The values pertaining to these combinations are indicated as follows: $\rho = (0.5, 2)$, and Set2: $\rho = (1.75, 2.3)$. multiple repetitions were performed and yielded consistent results. $N = 1000$ random samples from the RL-Exp distribution are generated and different criteria including average estimates (Mean) and mean square error (MSE) are applied to asses the simulation results. Multiple repetitions were performedsuch as ($\rho = (2.5, 3.75)$ and $\rho = (4, 4.5)$) and yielded consistent results. The proposed criteria can be defined as below

$$\text{Mean} = \frac{1}{N}\sum_{i=1}^{N}\hat{\rho}_i,$$

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\rho}_i - \rho)^2.$$

The simulation process is conducted by using the R language software with the optim function. It is well noted that for the Bayes estimator using the MCMC procedure, the obtained estimators are computed with conditions $q = 0.5, D = 10000$, the burn-in period $B = 2000$, and the values of prior distribution $a_1 = b_1 = a_2 = b_2 = 1$. Figs. 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 and 19 plotted the histogram, trace, the auto-correlation function (ACF), cumuplots function, and Gelman-Rubin diagnostic curves of the unknown parameters α and β to assess MCMC convergence.

On the other hand, the CIs are constructed using the three proposed techniques with 95% confidence level. We computed the average interval length (ALs) and coverage probabilities (CPs) for each method. The results are presented in numerical form in Tables 4 and 5.

After that, to compare the performance of the two estimation approaches, we have assigned the values of the number of iterations (NI) and MSEs on the basis of their value in decreasing order for the two estimation methods corresponding to the same sample size. From these results obtained in Tables 2 and 3, we can observe that for all recommended estimators, the average parameter estimations approach the actual parameter values. Further, the MSEs decrease in size as m increases, which ensures that the obtained estimated parameter values are consistent and asymptotically unbiased. Moreover, we observed that Bayes estimation under the SE loss function appears as the best estimation method for the RL-Exp model with minimum MSE as compared to other methods mentioned. Figures 4, 5, 6 and 7 confirm this result. In addition, as the value of m continues to rise, the ALs of ρ ultimately decrease for all suggested techniques. Also, the Boot-t method has greater CPs among credible and Boot-p techniques.

Application of the RL-Exp model to life events

This section demonstrates the RL-Exp distribution as a practical tool, demonstrating its efficiency through the application of real data sets, with a specific focus on case studies in engineering and environmental fields.

m		MLE		Bayes(SE)		Bayes(LI)		Bayes(GE)	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	α	0.5403	0.0026	0.4835	0.0015	0.4832	0.0018	0.4816	0.0021
	β	1.7637	0.0558	1.9643	0.0019	1.9641	0.0021	1.9643	0.0023
	NI	12		9		11		10	
40	α	0.5256	0.0022	0.5161	0.0013	0.5158	0.0015	0.5146	0.0018
	β	1.8127	0.0350	1.9546	0.0017	1.9541	0.0019	1.9539	0.0022
	NI	11		7		10		10	
70	α	0.5238	0.0018	0.4885	0.0009	0.4883	0.0012	0.4873	0.0014
	β	1.9100	0.0080	1.9715	0.0014	1.9714	0.0016	1.9713	0.0018
	NI	9		7		8		9	
100	α	0.5151	0.0014	0.4867	0.0006	0.4866	0.0008	0.4860	0.0009
	β	1.9184	0.0066	1.9887	0.0007	1.9886	0.0009	1.9885	0.0011
	NI	10		6		7		8	

Table 2. Numerical values of the RL-Exp model simulation for Set 1.

m		MLE		Bayes (SE)		Bayes (LI)		Bayes (GE)	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
20	α	2.1679	0.1532	1.6999	0.0034	1.6997	0.0037	1.6996	0.0039
	β	3.0649	0.5479	2.3979	0.0128	2.3971	0.0131	2.3969	0.0133
	NI	13		10		11		11	
40	α	2.0867	0.1133	1.7251	0.0013	1.7249	0.0016	1.7248	0.0018
	β	2.9957	0.4840	2.2633	0.0028	2.2629	0.0031	2.2628	0.0033
	NI	14		11		12		13	
70	α	1.8376	0.0076	1.7360	0.0008	1.7359	0.0010	1.7359	0.0012
	β	2.4753	0.0307	2.2804	0.0025	2.2799	0.0027	2.2797	0.0028
	NI	12		9		11		12	
100	α	1.7976	0.0022	1.7696	0.0005	1.7694	0.0008	1.7693	0.0009
	β	2.3744	0.0055	2.3046	0.0020	2.3041	0.0022	2.3040	0.0024
	NI	10		8		8		9	

Table 3. Numerical values of the RL-Exp model simulation for Set 2.

m		Credible		Boot-t		Boot-p	
		AL	CP	AL	CP	AL	CP
20	α	0.1131	0.840	0.0935	0.880	0.1149	0.801
	β	0.1242	0.844	0.0695	0.925	0.1256	0.836
40	α	0.0986	0.880	0.0823	0.942	0.0995	0.853
	β	0.1232	0.860	0.0739	0.932	0.1238	0.914
70	α	0.0946	0.920	0.0696	0.987	0.0972	0.937
	β	0.1219	0.880	0.0940	0.950	0.1221	0.928
100	α	0.0925	0.945	0.0628	0.975	0.0929	0.962
	β	0.1203	0.925	0.0870	0.951	0.1242	0.947

Table 4. The ALs and the CPs for the RL-Exp model for Set 1.

First data

The first data set defined the failure times of product reliability during the design phase for 50 elements. It is studied by[42] and the values are presented in Table 6.

m		Credible		Boot-t		Boot-p	
		AL	CP	AL	CP	AL	CP
20	α	0.1225	0.852	0.1064	0.880	0.1253	0.801
	β	0.1412	0.859	0.0835	0.936	0.1416	0.861
40	α	0.1166	0.900	0.1003	0.954	0.1185	0.8867
	β	0.1392	0.880	0.0819	0.955	0.1408	0.929
70	α	0.1086	0.934	0.0986	0.989	0.1092	0.948
	β	0.1359	0.911	0.0790	0.961	0.1381	0.938
100	α	0.1003	0.954	0.0828	0.991	0.1026	0.974
	β	0.1306	0.934	0.0770	0.966	0.1311	0.959

Table 5. The ALs and the CPs for the RL-Exp model for Set 2.

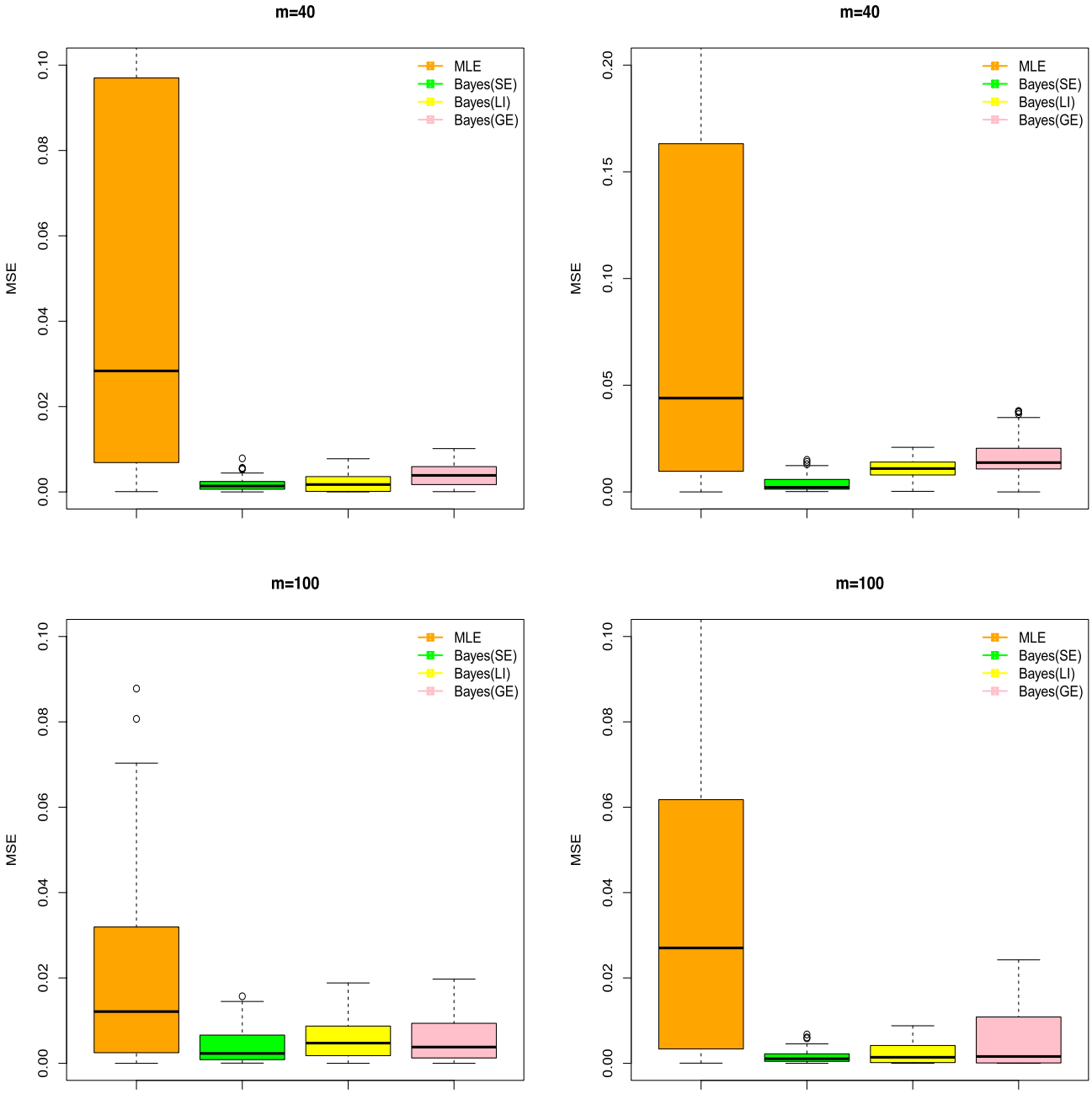


Figure 4. Box plots for the MSEs based on proposed estimators using Set 1.

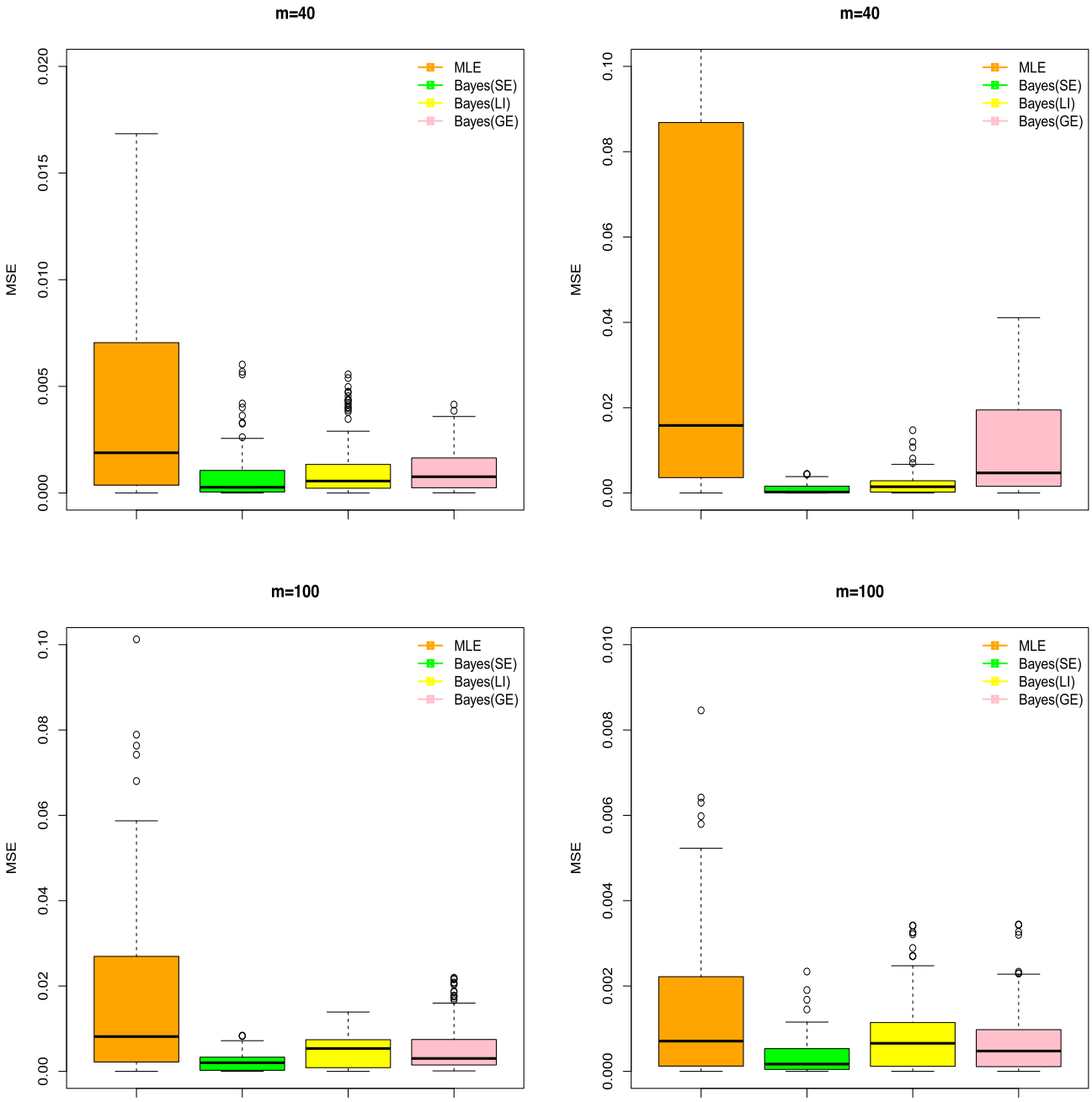


Figure 5. Box plots for the MSEs based on proposed estimators using Set 2.

Second data

The second data set provided 56 yearly records of the highest amounts of rainfall in Karachi, Pakistan, from 1953 to 2009, which offers a comprehensive view of the city's precipitation patterns. This data set was previously considered[43]. The values of the data set are summarized in Table 7.

Third data

The third data consisted of the length of the intervals between the periods in which cars pass a spot on a road. The proposed data was sourced from Jorgensen[44], and its values were reported in Table 8.

Table 9 summarizes the statistical description of all the proposed data sets, and visual comparisons, including scaled total time on the test (TTT), probability-probability (P-P), and box plots are sketched in Fig. 20 drawn from the three recommended data sets. These representations help in studying the historical performance of the engineering and environmental sectors.

Note that the RL-Exp model is applied for the three data sets, with the following initial parameters:

1. Data I: $\alpha = 0.2, \quad \beta = 3.8$.

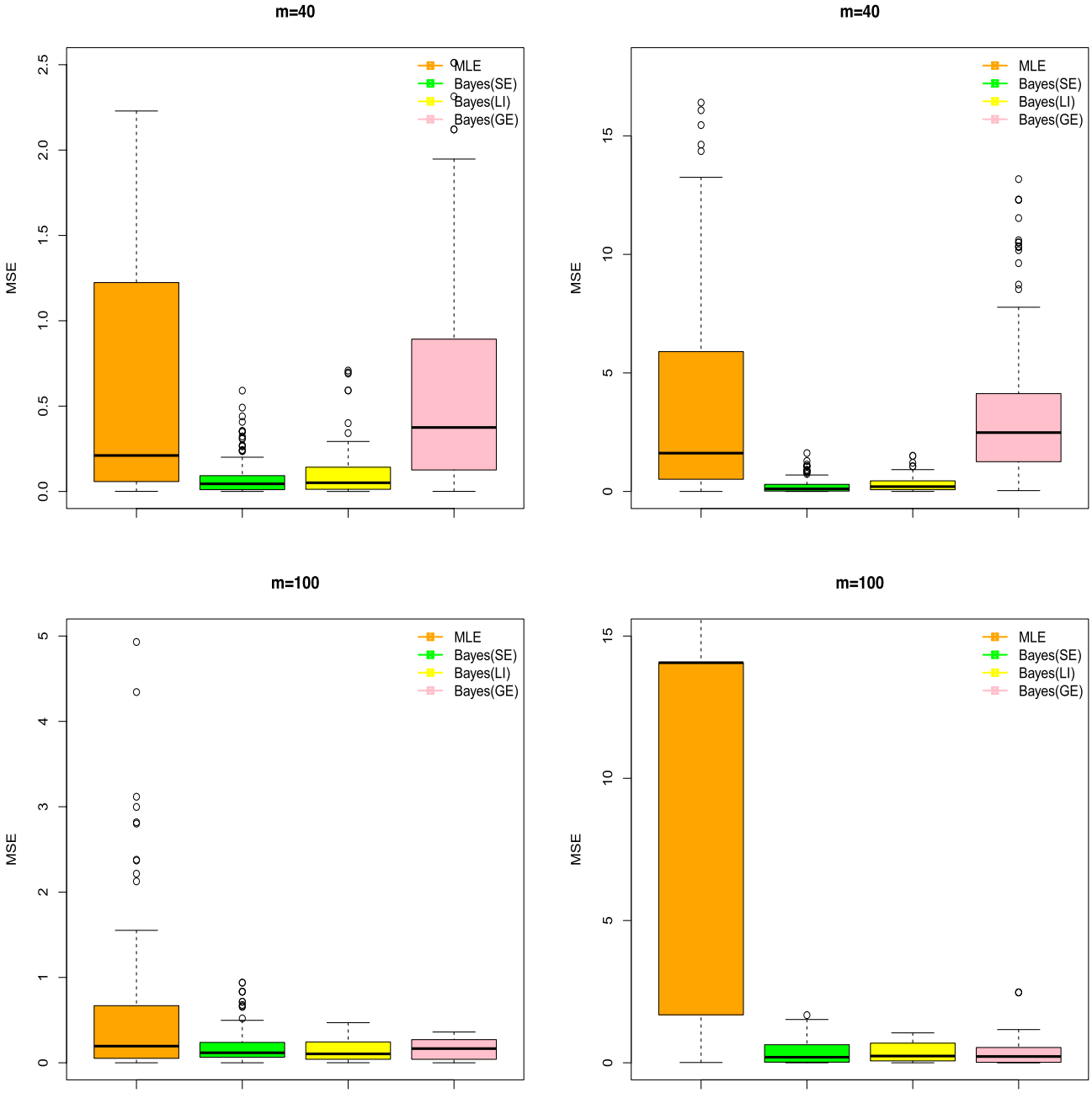


Figure 6. Box plots for the MSEs based on proposed estimators at $\rho = (2.5, 3.75)$.

2. Data II: $\alpha = 0.05, \quad \beta = 2.1$.
3. Data III: $\alpha = 0.95, \quad \beta = 18$.

Now, the proposed model is compared with some renowned distributions, generalized Rayleigh (GR), zero-truncated Poisson exponential (ZTP-Exp), alpha power transformed exponential (APT-Exp), Weibull (Wei), extended exponential (Ex-Exp), moment exponential (M-Exp), Gamma, XLindley, and two parameters Mira (TPM) distributions for fitting the three data sets.

Hence, several goodness-of-fit metrics like log likelihood function ($\mathcal{LLF}$), Akaike Information Criterion ($\mathcal{AIC}$), Correction Akaike Information Criterion ($\mathcal{CAIC}$), Hannan-Quin Information Criterion($\mathcal{HQIC}$), Bayesian Information Criterion ($\mathcal{BIC}$), Kolmogorov-Smirnov ($\mathcal{KS}$) statistics with its associated P-values, Anderson Darling ($\mathcal{A}$), and Cramer-Von Mise ($\mathcal{W}$) to select the best appropriate model for the two data sets. For a particular probability model, the distribution with the least value of these measures and a greater P-value offers the best fit for modeling a given real-life data set. The parameter estimates of all distributions with goodness-of-fit metrics for the proposed data are given in Tables 10 and 11. To verify whether the obtained estimate actually converges to the MLEs, we have tried the grid search algorithm with some other initial estimates also, and it converges to the same set of estimates. Moreover, we have conducted an extensive

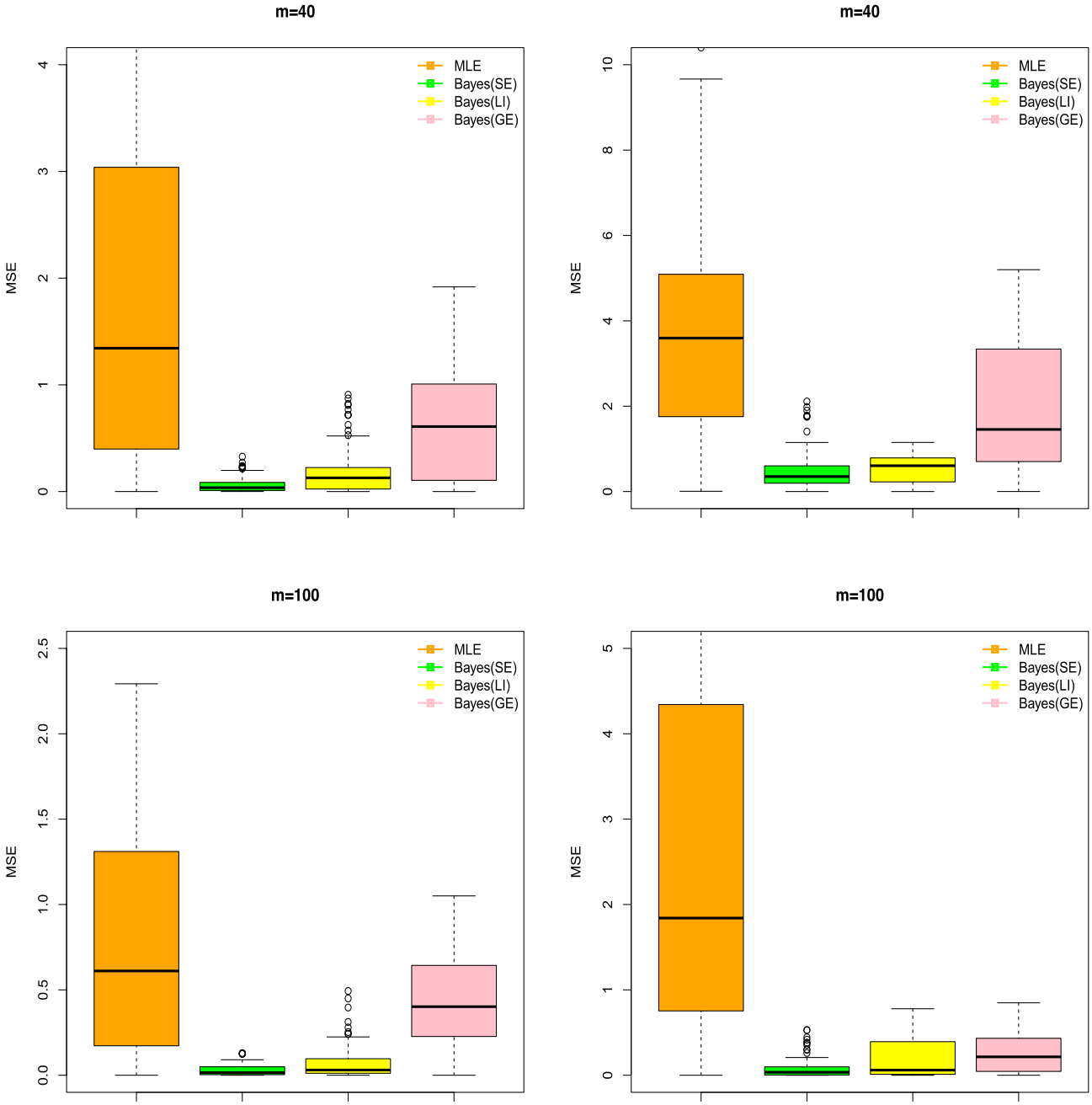


Figure 7. Box plots for the MSEs based on proposed estimators at $\rho = (4, 4.5)$.

grid search method with a grid size of 0.001, and it is noticed that they coincide, which guarantees that the final estimates converge to the MLEs. The statistical software utilizes the R package (AdquacyModel) and the "BFGS" algorithm to perform all of these calculations. Obviously, the numerical values of Tables 10 and 11 demonstrate that the RL-Exp model fits the three data sets better than other fitted distributions, as indicated by the p value. Figures 21, 22, 23, 24, 25 and 26 present the fitted functions for the RL-Exp with the fitting models, including the PDF, CDF, SF, and P-P plots for the applied data sets. These plots further support the findings from Tables 10 and 11, offering a very close fit to each data set as compared to the other contender distributions.

Finally, the three recommended data sets are evaluated by applying the Bayes estimators under SE, LI, and GE loss functions. The obtained results are summarized in Table 12. In addition, the histogram, trace, and ACF plots of the RL-Exp distribution for the MCMC technique are illustrated in Figs. 27, 28 and 29. All the codes used exist in Sect. 8.

Conclusion

The article introduces a new probabilistic model, the Ramos Louzada exponential model, specifically designed for engineering and environmental analysis applications. The proposed model can fit real-world data with

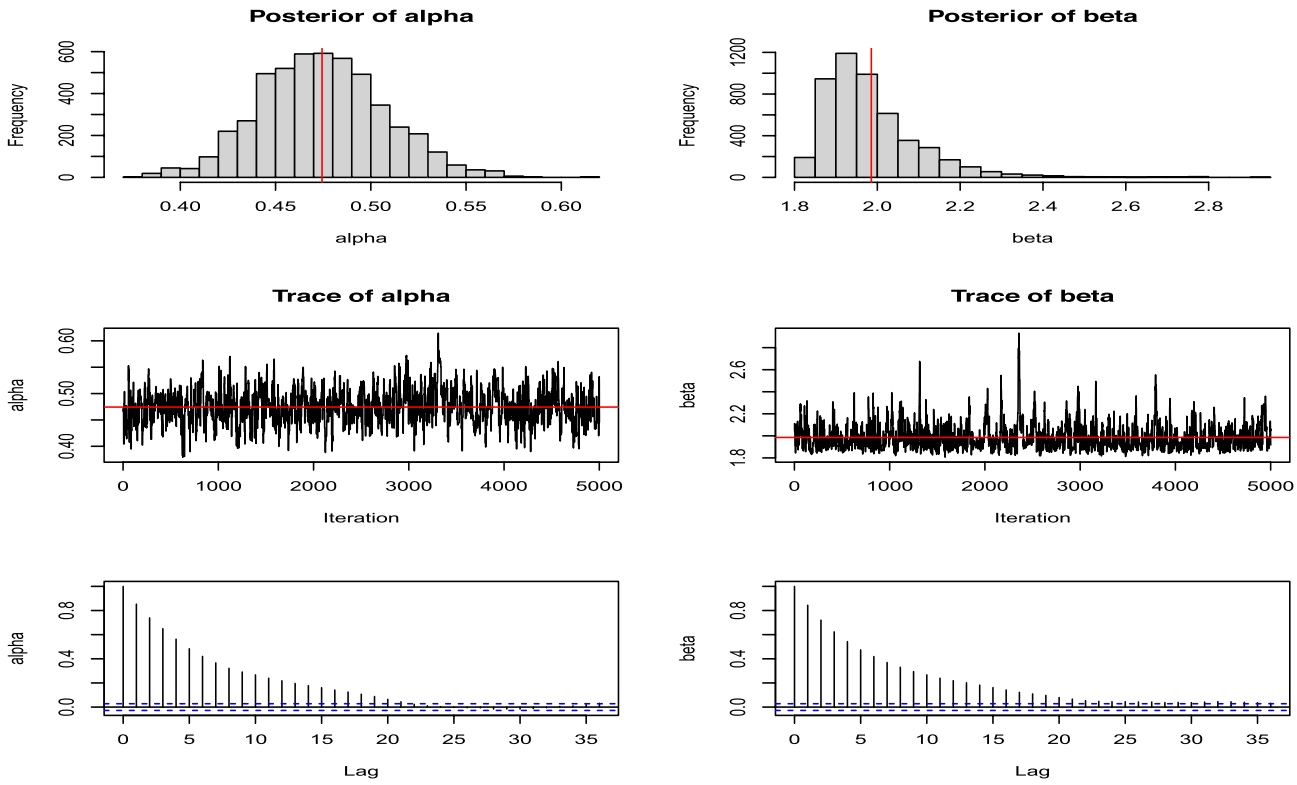


Figure 8. MCMC convergence plots of RL-Exp model for Table 2 at $m = 100$.

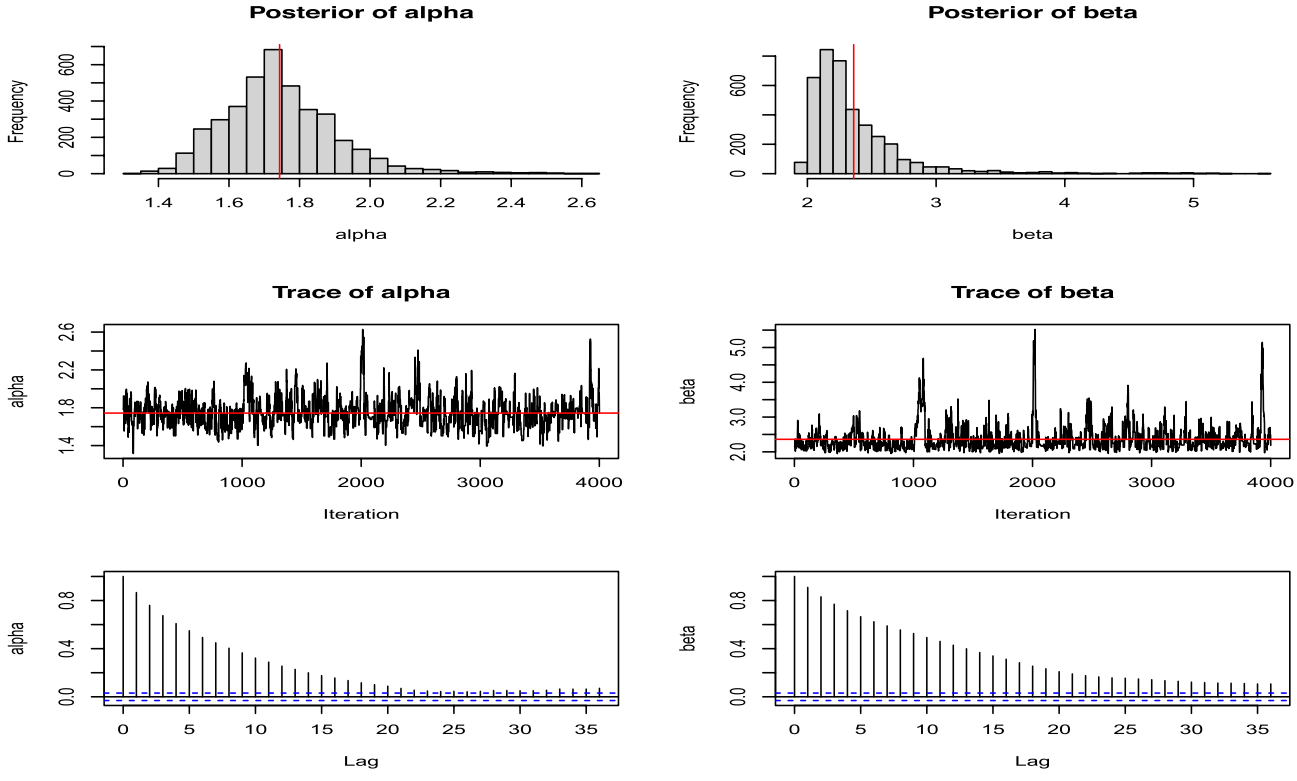


Figure 9. MCMC convergence plots of RL-Exp model for Table 3 at $m = 100$.

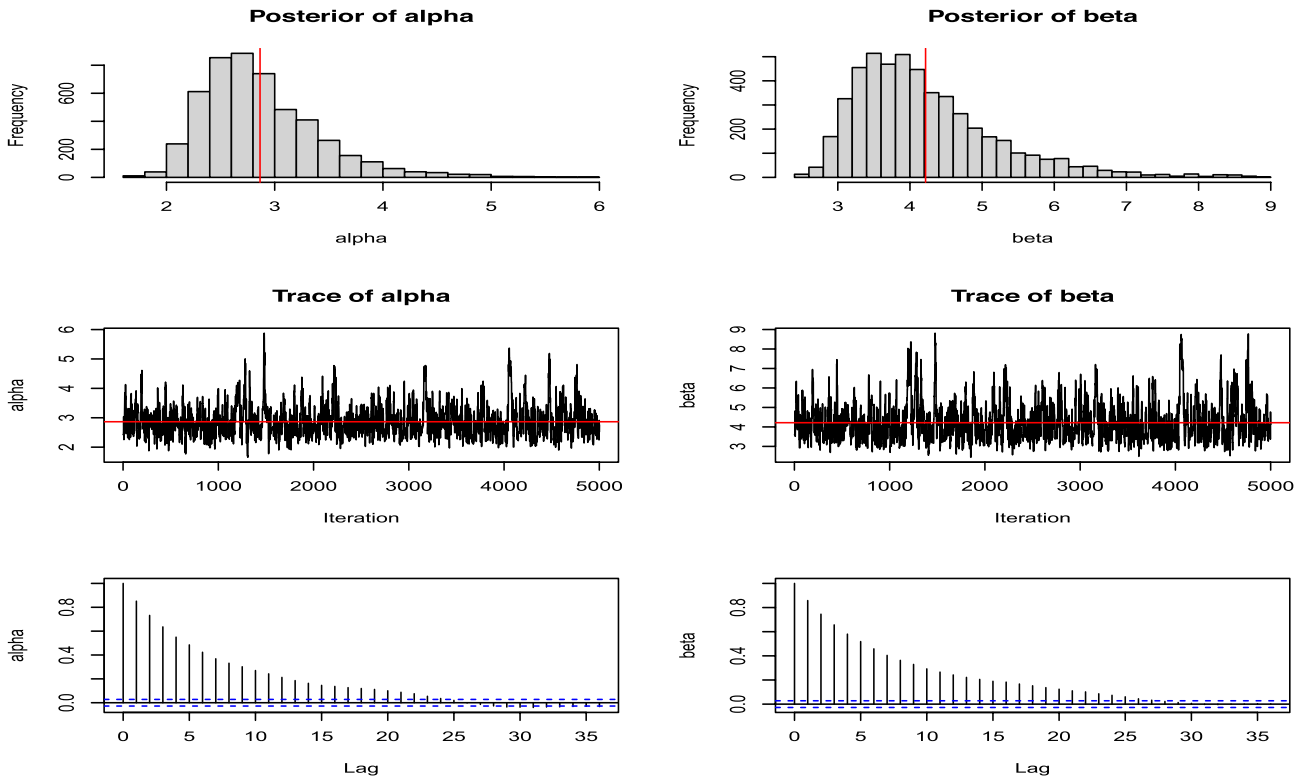


Figure 10. MCMC convergence plots of RL-Exp model for $\rho = (2.5, 3.75)$.

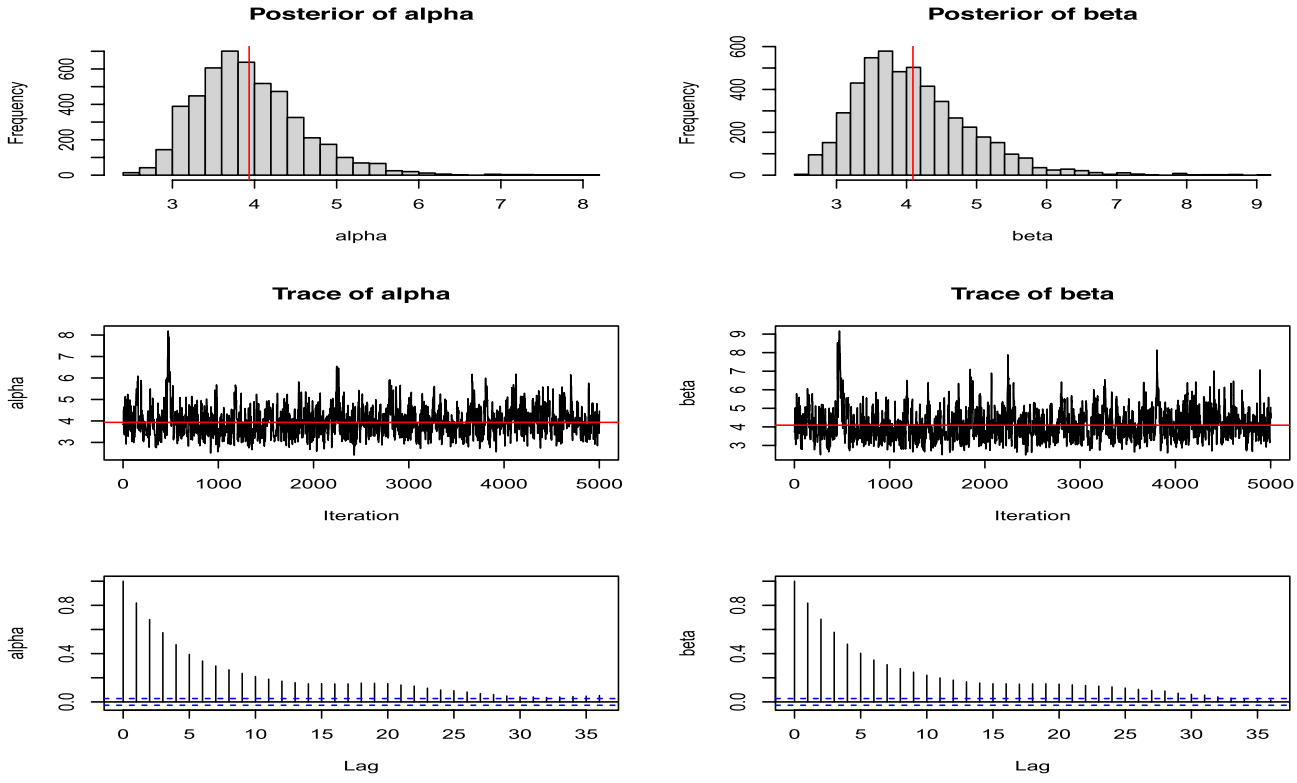


Figure 11. MCMC convergence plots of RL-Exp model for $\rho = (4, 4.5)$.

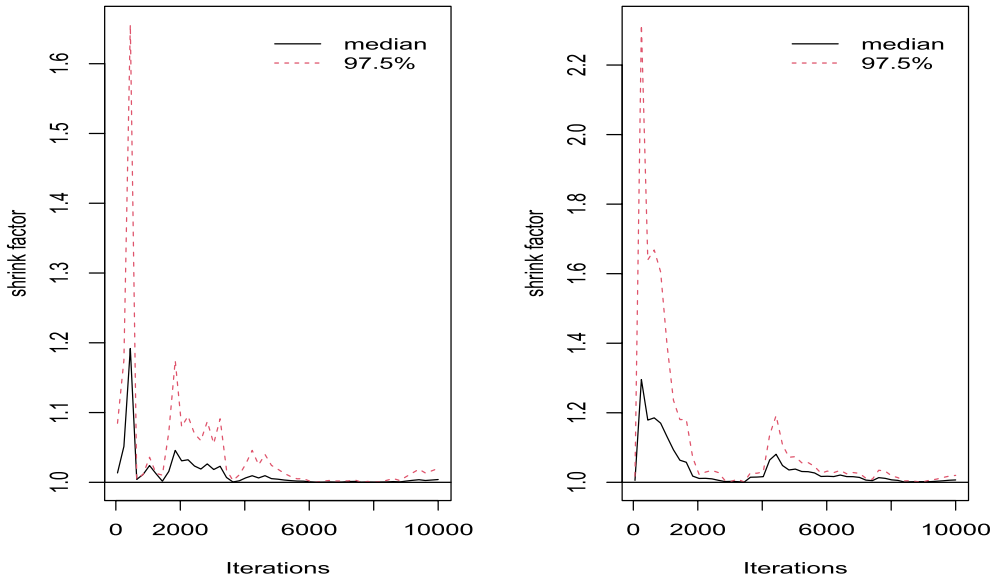


Figure 12. Gelman-Rubin diagnostic plots of α and β for Table 2 at $m = 100$.

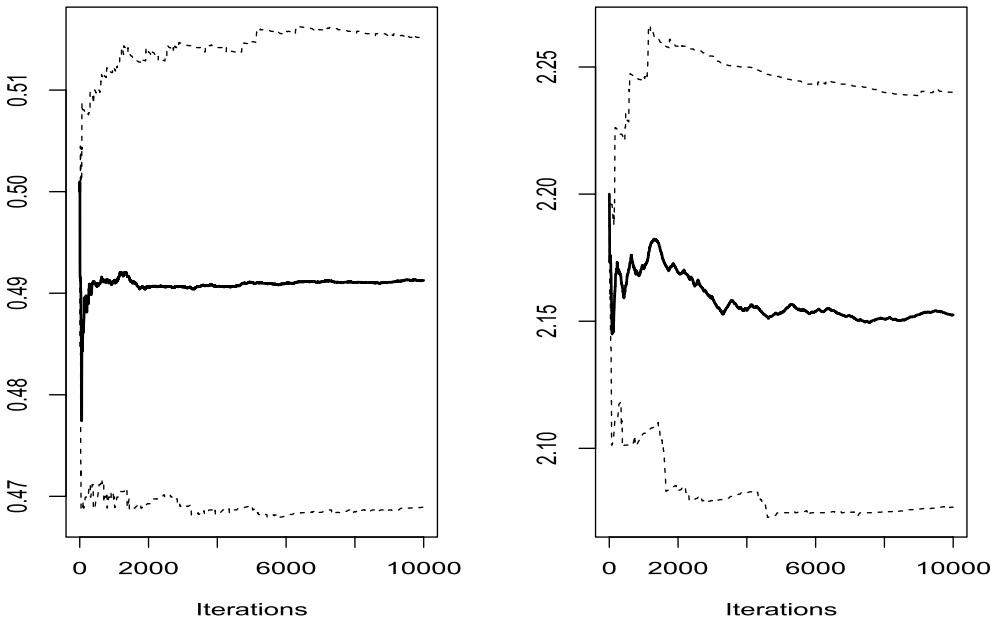


Figure 13. Cumuplot function of α and β for Table 2 at $m = 100$.

ambiguity; our exploration of the RL-Exp model has demonstrated its ability to handle ambiguity and data ambiguity. The proposed model provides insight into reliability properties, moment functions, and order statistics and demonstrates its versatility as a powerful tool for decoding complex, skewed, and symmetric data sets. Using a wide range of methods, we can handle statistical analysis of the RL-Exp model and its unknown parameters. Therefore, we concluded that the Bayesian approach under the SE loss function is superior to the conventional estimating method since it consistently produces lower values for the MSE and number of iterations. We applied the proposed distribution to real-world data sets to verify its practical importance, comparing it directly with well-known classical distributions. The analysis results showed unequivocally that the RL-Exp model outperformed its classical counterpart in capturing the complex nuances of the underlying data, indicating its potential as a superior modeling tool in various fields. Henceforth, to best-fitting capability, the RL-Exp mode has certain disadvantages. For example, it has a non-closed form for the quantile function, and the cost of Bayesian estimates is generally high to compute. For future studies, we can apply the RL-Exp model in other cases, such as bivariate and multivariate versions, regression frameworks, and other statistical inferences,

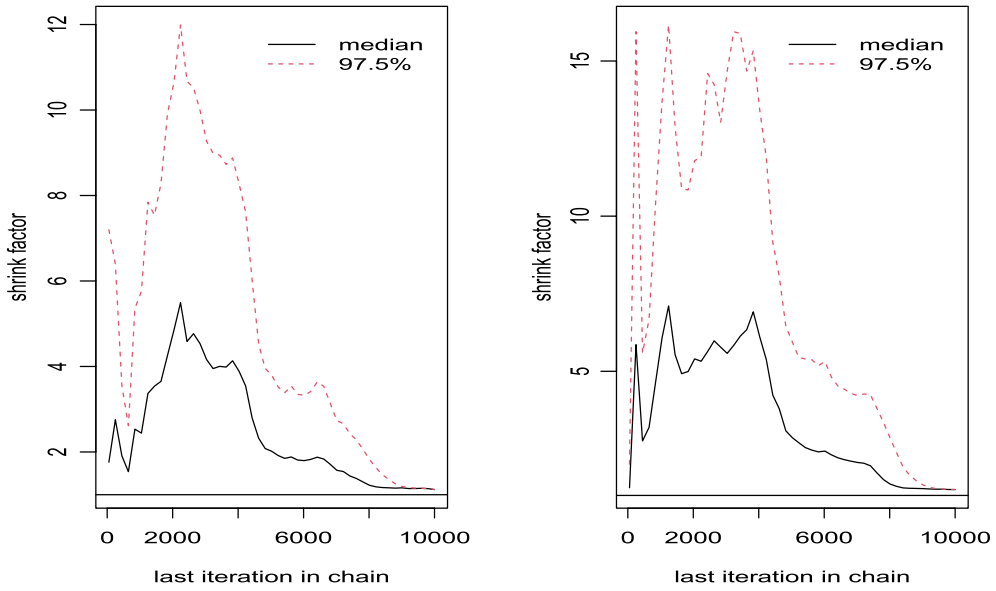


Figure 14. Gelman-Rubin diagnostic plots of α and β for Table 3 at $m = 100$.

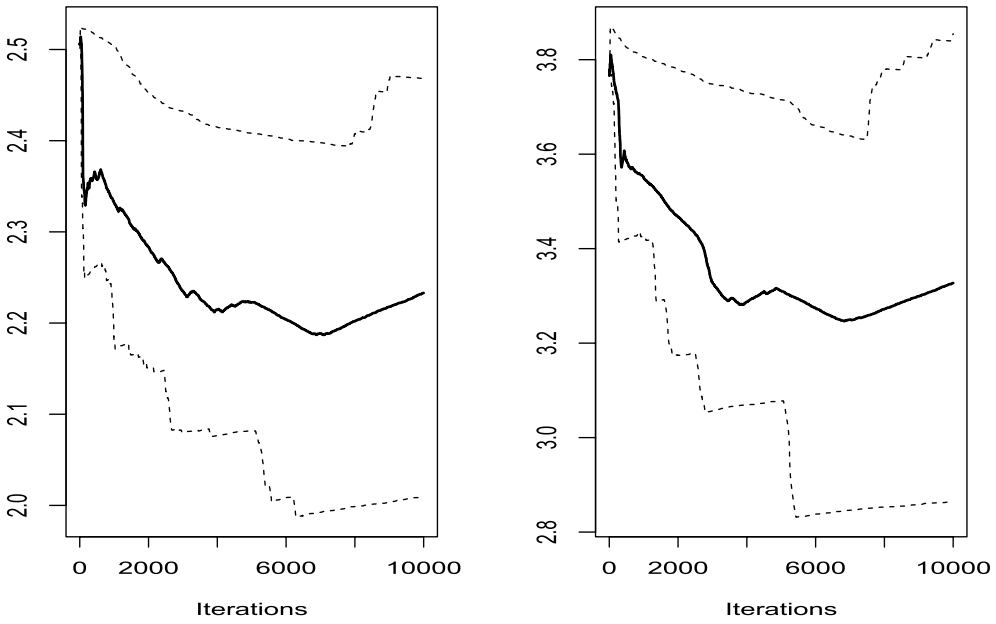


Figure 15. Cumuplot function of α and β for Table 3 at $m = 100$.

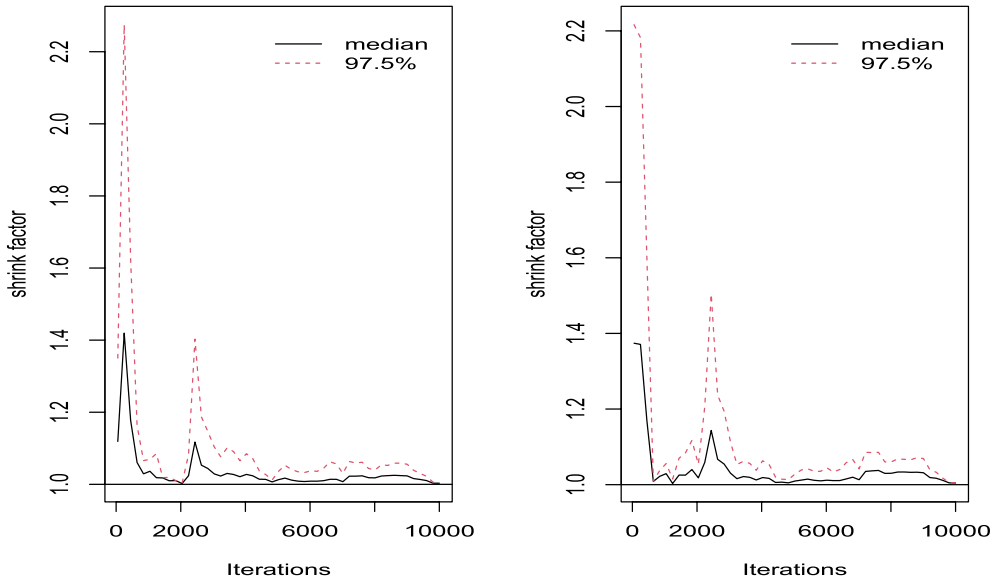


Figure 16. Gelman-Rubin diagnostic plots of α and β for $\rho = (2.5, 3.75)$.

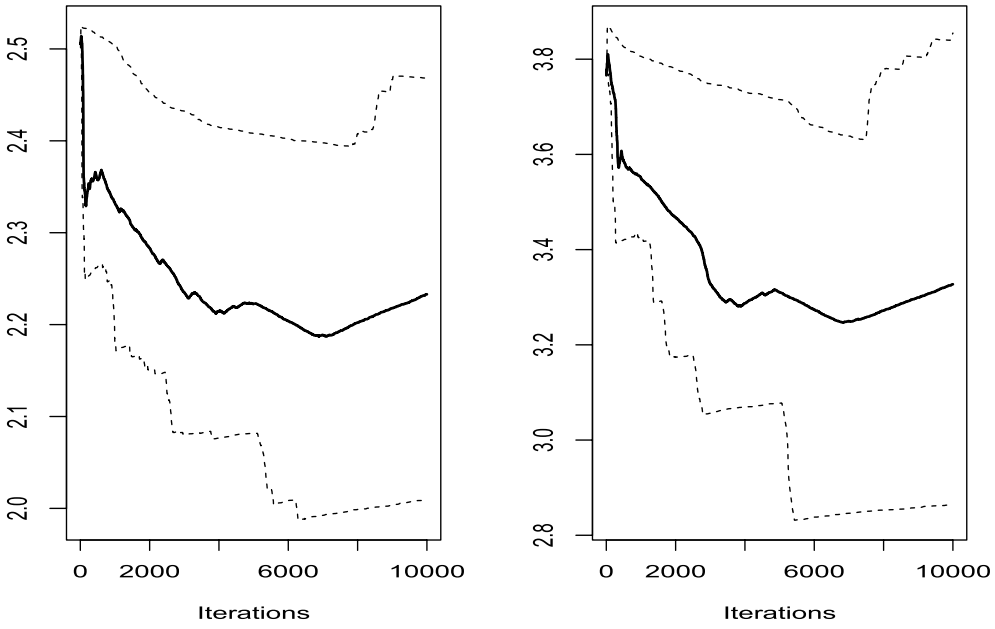


Figure 17. Cumuplot function of α and β for $\rho = (2.5, 3.75)$.

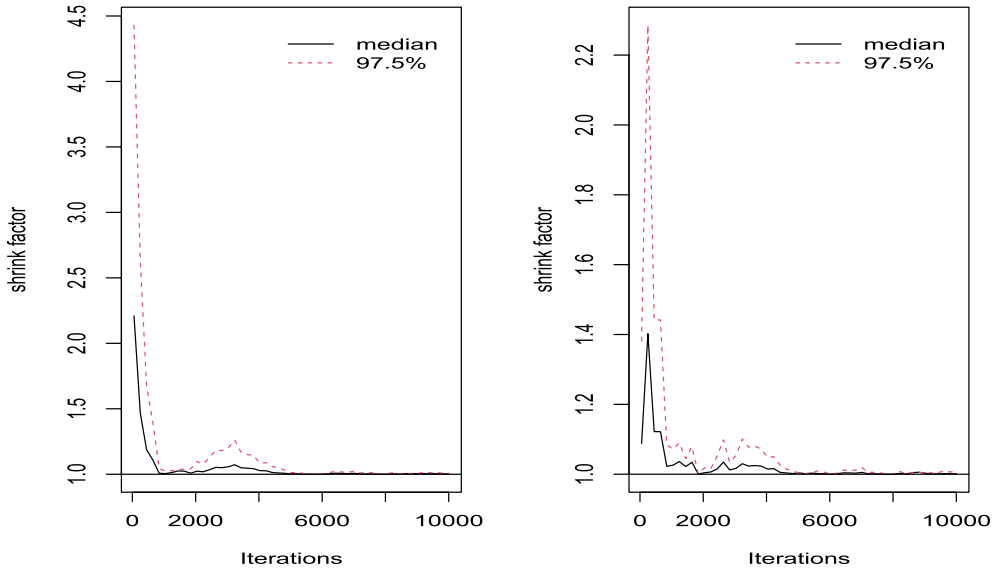


Figure 18. Gelman-Rubin diagnostic plots of α and β for $\rho = (4, 4.5)$.

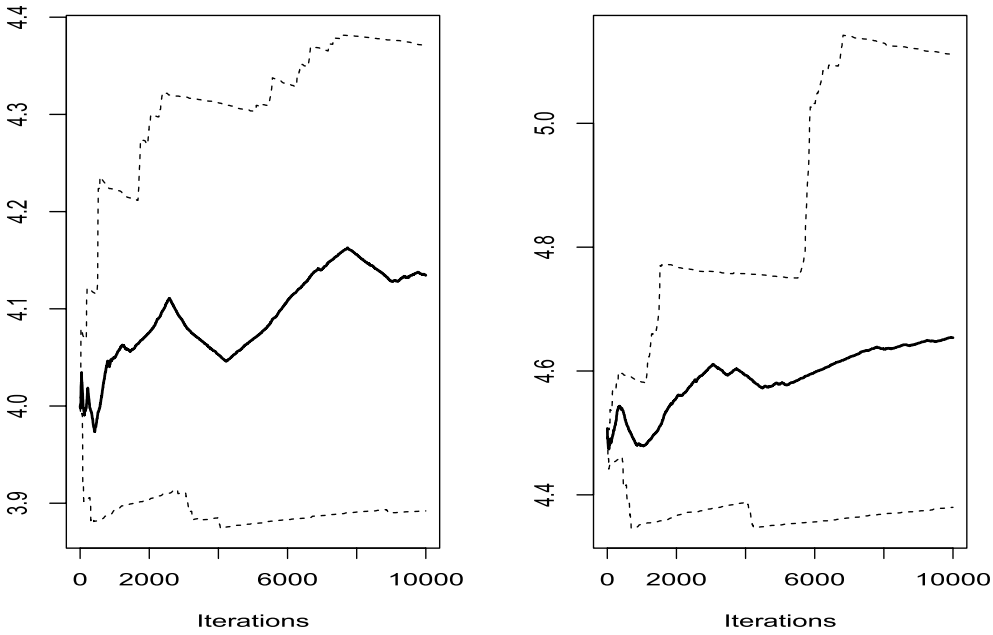


Figure 19. Cumuplot function of α and β for $\rho = (4, 4.5)$.

0.12	0.43	0.92	1.14	1.24	1.61	1.93	2.38	4.51	5.09
6.79	7.64	8.45	11.9	11.94	13.01	13.25	14.32	17.47	18.1
18.66	19.23	24.39	25.01	26.41	26.8	27.75	29.69	29.84	31.65
32.64	35	40.7	42.34	43.05	43.40	44.36	45.4	48.14	49.1
49.44	51.17	58.62	60.29	72.13	72.22	72.25	72.29	85.2	89.52

Table 6. The values of data set I.

117.6	157.7	148.6	63.6	62.4	11.8	54.9	39.9	16.8	30.2	38.4	76.9
73.4	85	256.3	24.9	148.6	160.5	131.3	77	155.2	217.2	105.5	166.8
157.9	73.6	291.4	210.3	315.7	107.7	33.3	302.6	159.1	78.7	33.2	52.2
92.7	150.4	43.7	68.3	20.8	179.4	245.7	19.5	30	270.4	160	96.3
185.7	429.3	184.9	262.5	80.6	138.2	28	39.3				

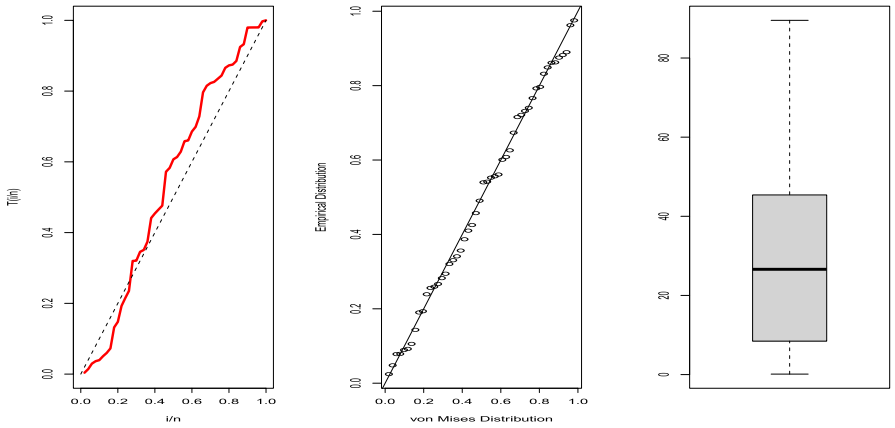
Table 7. The values of data set II.

2.50	2.60	2.60	2.70	2.80	2.80	2.90	3.00	3.00	3.10	3.20	3.40	3.70
3.90	3.90	3.90	4.60	4.70	5.00	5.60	5.70	6.00	6.00	6.10	6.60	6.90
6.90	7.30	7.60	7.90	8.00	8.30	8.80	8.80	9.30	9.40	9.50	10.1	11.
11.3	11.9	11.9	12.3	12.9	12.9	13.0	13.8	14.5	14.9	15.3	15.4	15.9
16.2	17.6	20.1	20.3	20.6	21.4	22.8	23.7	23.7	24.7	29.7	30.6	31.0
34.1	34.7	36.8	40.1	40.2	41.3	42.0	44.8	49.8	51.7	55.7	56.5	58.1
70.5	72.6	87.1	88.6	91.7	119.8							

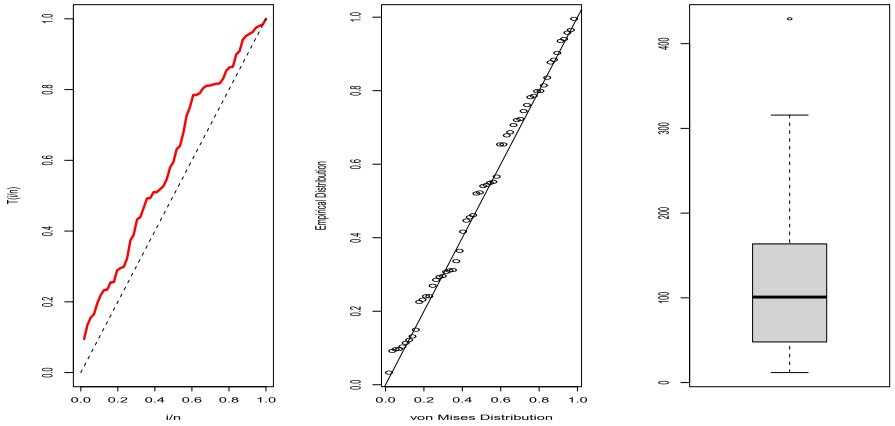
Table 8. Values of the data set III.

Data	Q_1	Q_2	μ_1'	Q_3	$\mathcal{SKW}$	$\mathcal{KUR}$
I	9.312	26.605	30.179	45.14	0.6354	– 0.5790
II	50.08	100.90	124.32	162.07	0.9978	0.6510
III	5.925	12.10	21.60	29.92	1.881	3.425

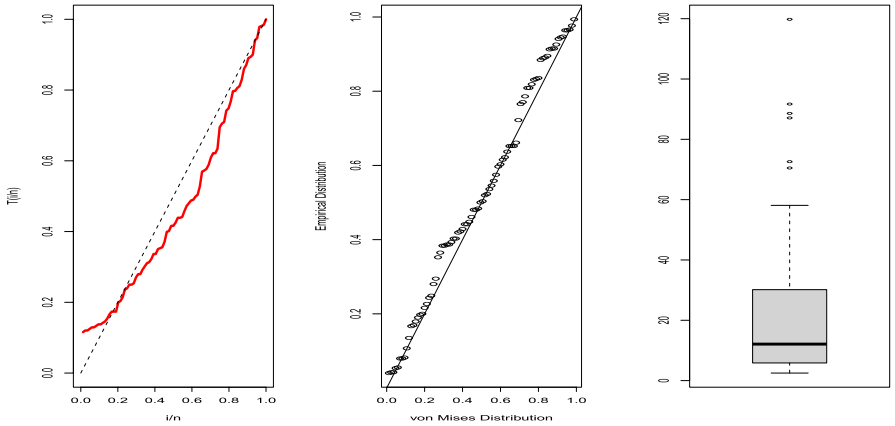
Table 9. Descriptive statistics for the recommended data sets.



(a) Data I



(b) Data II



(c) DataIII

Figure 20. Several descriptive plots of the proposed data set.

Data	Model	$\hat{\alpha}$	$\hat{\beta}$	$\mathcal{KS}$	P-value	$\mathcal{A}$	$\mathcal{W}$
	RL-Exp	0.1596	3.4018	0.0950	0.7110	0.6234	0.0969
	GR	0.3934	35.683	0.1080	0.5665	0.6662	0.1496
	ZTP-Exp	0.0391	0.7505	0.0981	0.6840	0.6241	0.1015
I	APT-Exp	0.0423	3.1670	0.1060	0.5897	0.6576	0.1004
	Wei	1.0151	30.344	0.1111	0.5304	0.6706	0.1020
	Ex-Exp	0.0311	0.9075	0.1245	0.3881	0.7566	0.1131
	M-Exp		15.087	0.1544	0.1658	0.7704	0.1219
	Gamma	0.9262	0.0306	0.1227	0.4062	0.7452	0.1125
	XL		0.0625	0.1353	0.2922	0.6937	0.1097
	TPM	0.0985	0.7318	0.2010	0.0301	0.8071	0.1475
	RL-Exp	0.0321	1.9985	0.1002	0.6272	0.3714	0.0617
	GR	0.7756	97.403	0.1586	0.1193	0.4995	0.0745
	ZTP-Exp	0.0085	0.2031	0.1074	0.5376	0.3724	0.0622
II	APT-Exp	0.0084	1.2016	0.1088	0.5202	0.3774	0.0674
	Wei	1.4289	149.701	0.1082	0.5285	0.3756	0.0663
	Ex-Exp	0.0126	2.1699	0.1083	0.5263	0.3765	0.0668
	M-Exp		69.037	0.1208	0.3869	0.3783	0.0674
	Gamma	2.2261	0.0178	0.1145	0.4542	0.3781	0.0672
	XL		0.0158	0.1651	0.0452	0.9825	0.2632
	TPM	0.0240	0.1067	0.1570	0.4262	0.3913	0.0730
	RL-Exp	0.3436	6.2784	0.1121	0.2414	1.5194	0.2344
	GR	0.3983	27.840	0.1542	0.0368	2.9796	0.4966
	ZTP-Exp	0.0596	1.1776	0.1526	0.0398	1.9827	0.3200
III	APT-Exp	0.0551	2.1908	0.1358	0.0900	1.8419	0.2946
	Wei	0.9733	19.739	0.1252	0.1433	1.5530	0.2425
	Ex-Exp	0.0632	1.6257	0.1659	0.0196	2.9915	0.6261
	M-Exp		10.802	0.2204	0.0005	3.5632	0.8443
	Gamma	1.0876	0.0503	0.1138	0.2267	1.5486	0.2417
	XL		0.0857	0.1894	0.0048	1.9198	0.33074
	TPM	0.1381	2.0442	0.2894	1.54×10^{-06}	3.7736	1.2800

Table 10. Estimation and goodness-of-fit comparison measures for the fitting models by applying the three recommended data sets.

Data	Model	$\mathcal{LLF}$	$\mathcal{AIC}$	$\mathcal{CAIC}$	$\mathcal{HQIC}$	$\mathcal{BIC}$
	RL-Exp	− 219.815	443.630	443.885	445.086	447.454
	GR	− 220.200	444.440	444.695	445.896	448.264
	ZTP-Exp	− 219.945	443.891	444.146	445.347	447.715
I	APT-Exp	− 220.079	444.161	444.416	445.617	447.985
	Wei	− 220.349	444.698	444.953	446.154	448.522
	Ex-Exp	− 220.208	444.417	444.673	445.874	448.241
	M-Exp	− 232.469	466.938	467.021	467.666	468.850
	Gamma	− 220.260	444.520	444.775	445.976	448.344
	XL	− 224.698	451.396	451.480	452.124	453.308
	TPM	− 239.930	483.876	484.131	485.332	487.700
	RL-Exp	− 321.817	647.633	647.860	649.204	651.684
	GR	− 328.177	660.408	660.634	661.978	664.458
	ZTP-Exp	− 325.567	655.134	655.360	656.704	659.184
II	APT-Exp	− 325.616	655.232	655.458	656.802	659.282
	Wei	− 325.572	655.158	655.384	656.728	659.208
	Ex-Exp	− 326.781	657.562	657.789	659.133	661.613
	M-Exp	− 329.395	661.105	661.179	661.890	663.130
	Gamma	− 326.056	656.118	656.344	657.688	660.1691
	XL	− 331.425	664.970	665.044	665.755	666.996
	TPM	− 327.057	658.114	658.340	659.684	662.164
	RL-Exp	− 342.464	688.928	689.076	690.882	693.789
	GR	− 351.897	707.845	707.993	709.799	712.706
	ZTP-Exp	− 350.940	705.899	706.047	707.853	710.760
III	APT-Exp	− 346.299	696.600	696.748	698.554	701.462
	Wei	− 345.791	695.631	695.779	697.585	700.493
	Ex-Exp	− 353.268	710.637	710.785	712.591	715.498
	M-Exp	− 356.085	734.218	734.267	735.195	736.649
	Gamma	− 342.605	689.255	689.404	691.210	694.117
	XL	− 355.937	715.935	716.083	717.889	720.796
	TPM	− 378.454	760.908	761.056	762.862	765.7697

Table 11. Various statistical metrics for the fitting using the suggested data sets.

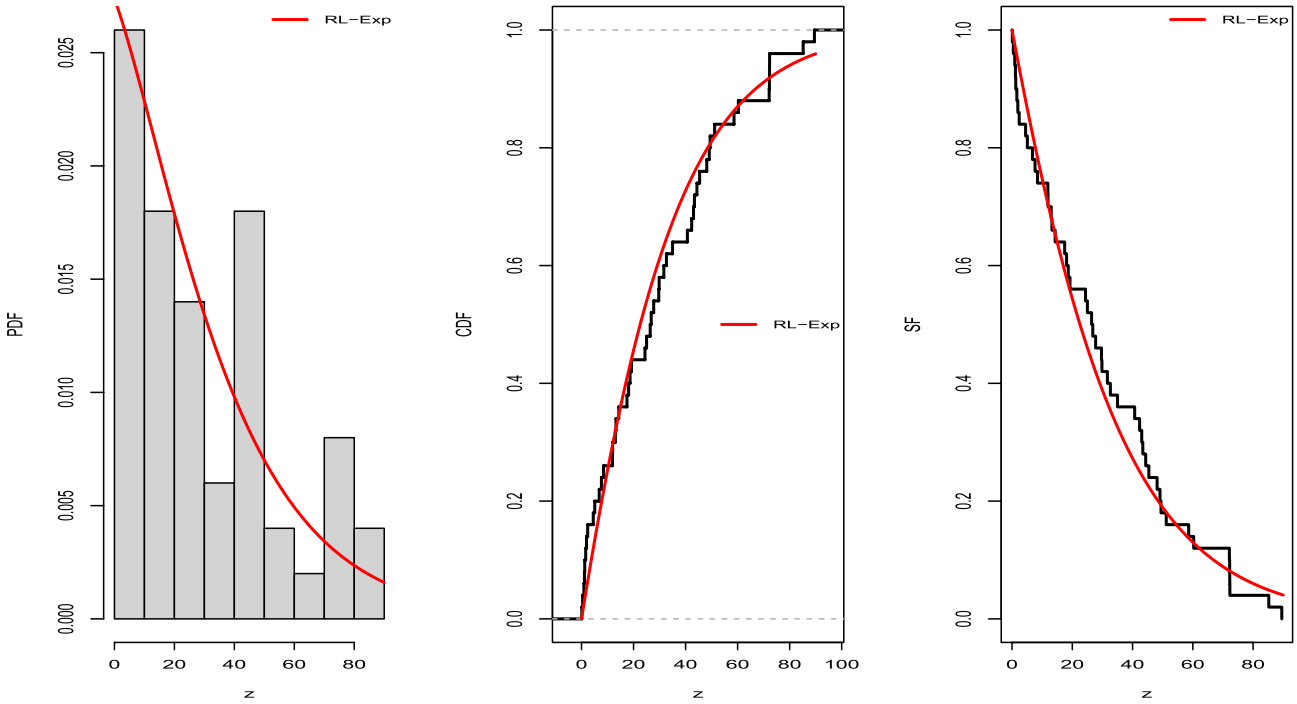


Figure 21. Fitted PDF, CDF, and SF plots of RL-Exp model for the first data set.

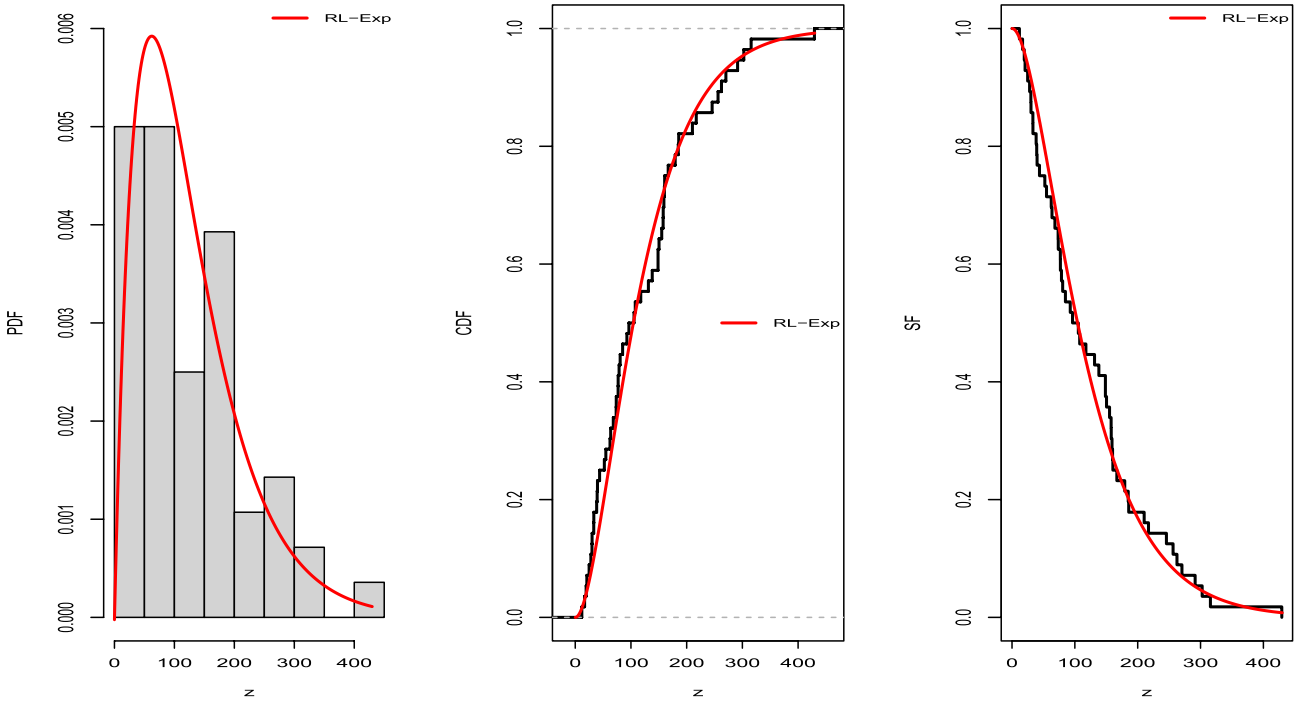


Figure 22. Fitted PDF, CDF, and SF plots of RL-Exp model for the second data set.

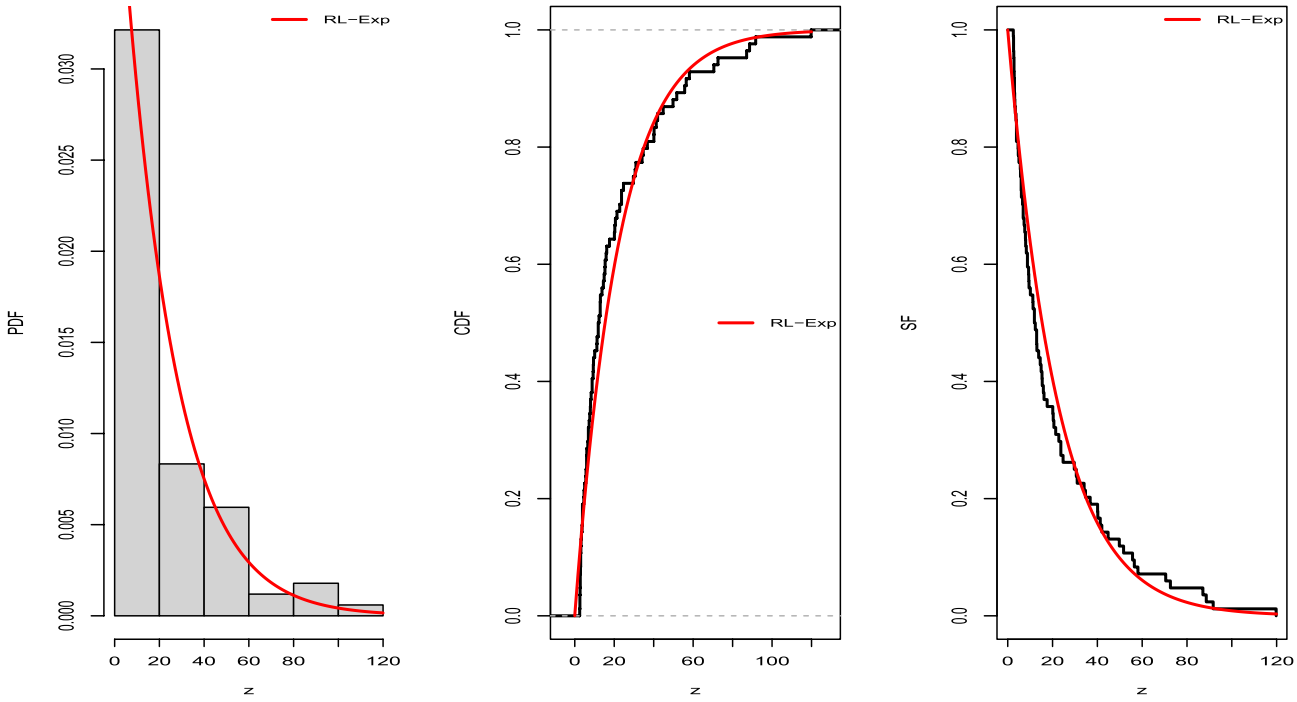


Figure 23. Fitted PDF, CDF, and SF plots of RL-Exp model for the third data set.

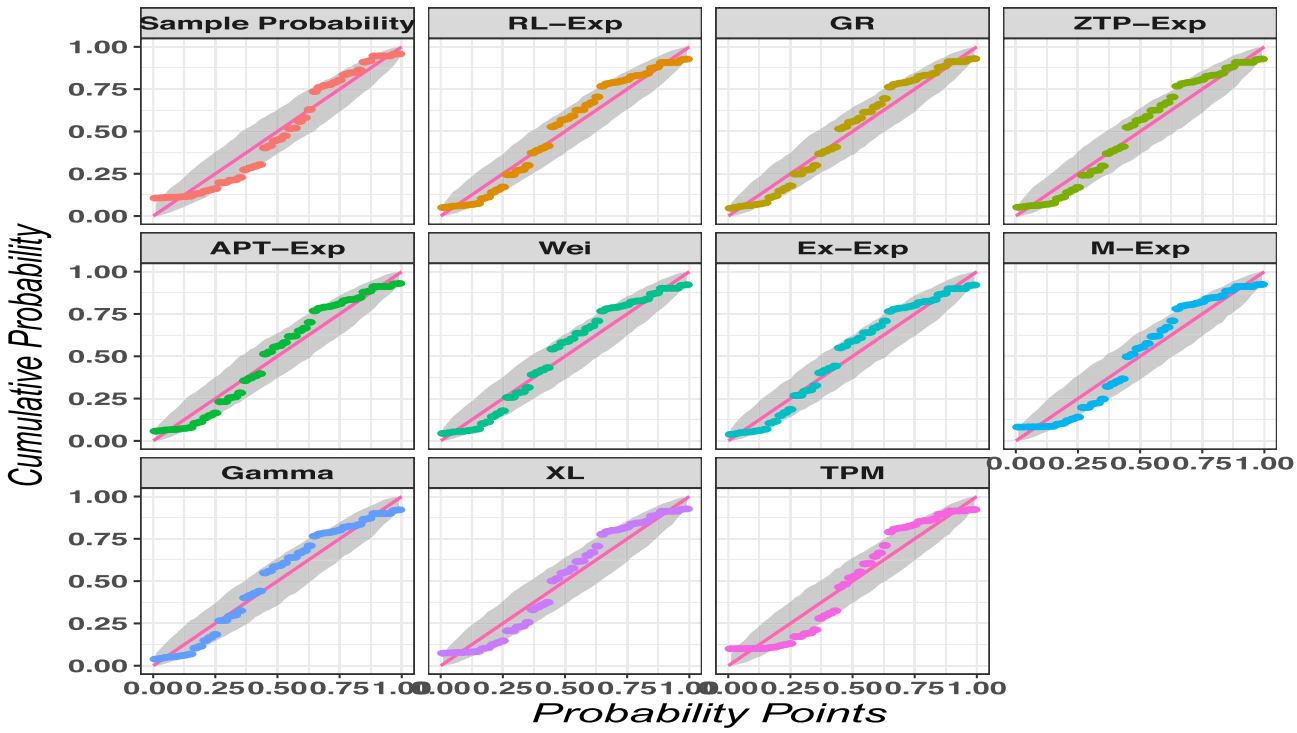


Figure 24. P–P plots of the fitting distributions using data I.

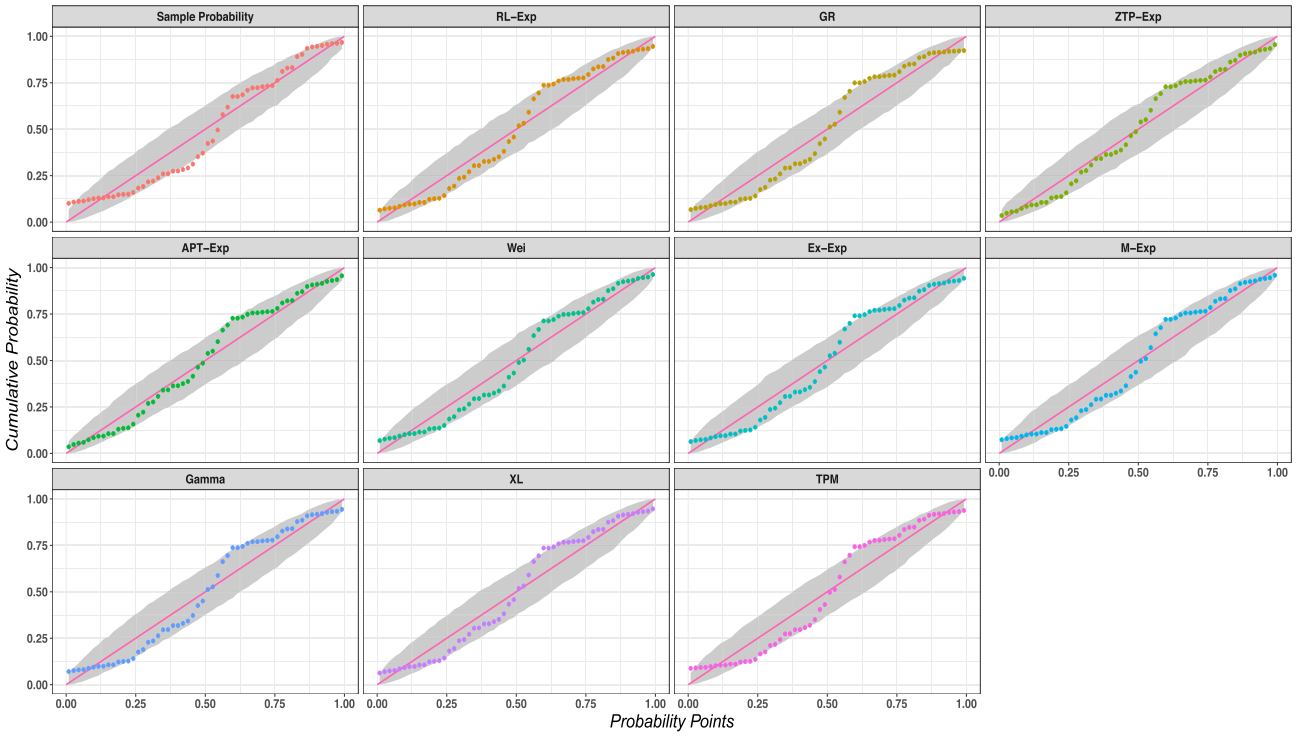


Figure 25. P–P plots of the fitting distributions using data II.

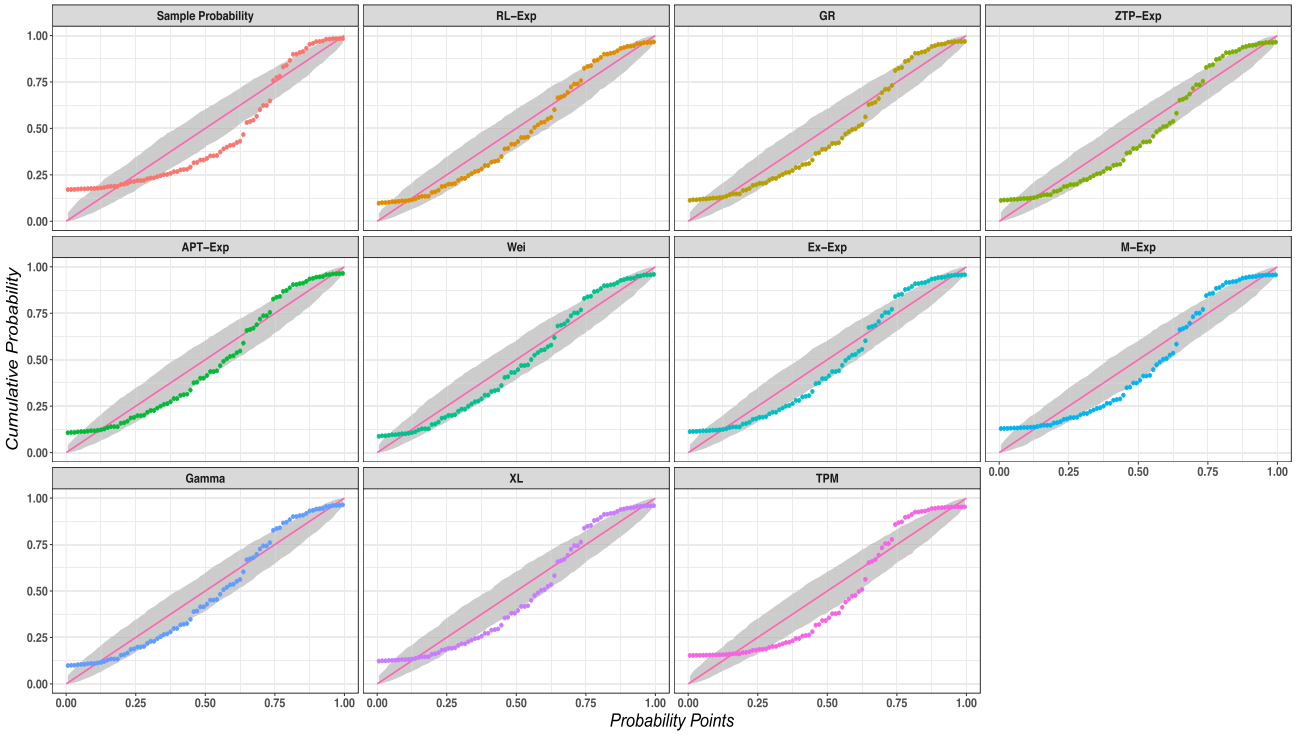


Figure 26. P–P plots of the fitting distributions using data III.

Data			SE	LI	GE
I	α		0.1581	0.1580	0.1564
	β		3.2333	3.2312	3.2314
II	α		0.0323	0.0321	0.0322
	β		1.9536	1.9535	0.9535
III	α		0.8983	0.8982	0.8978
	β		18.5268	18.5266	18.5268

Table 12. Result of Bayesian estimate of the RL-Exp model using SE, LI and GE loss functions for the three data sets.

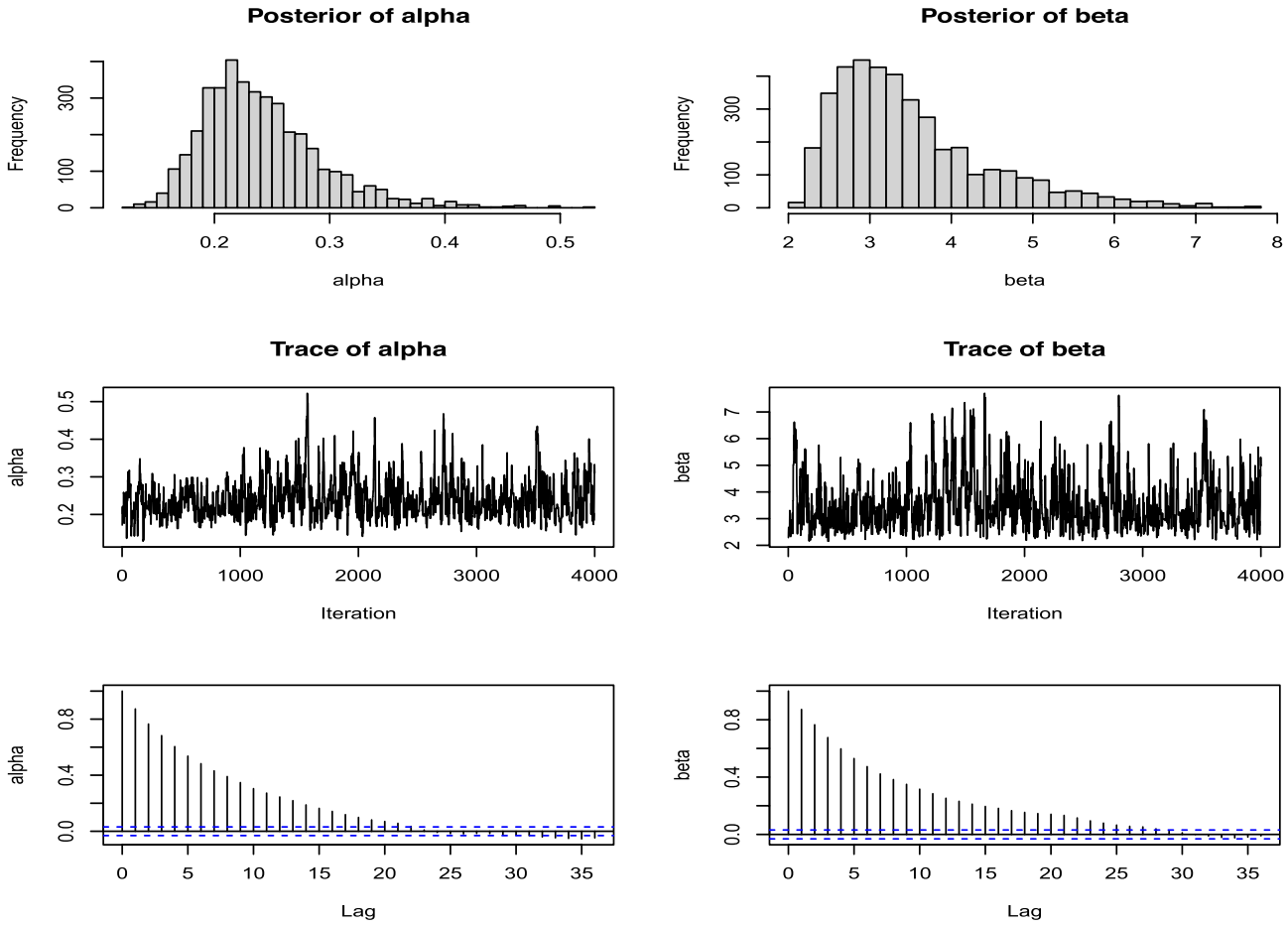


Figure 27. MCMC Convergence results using data I.

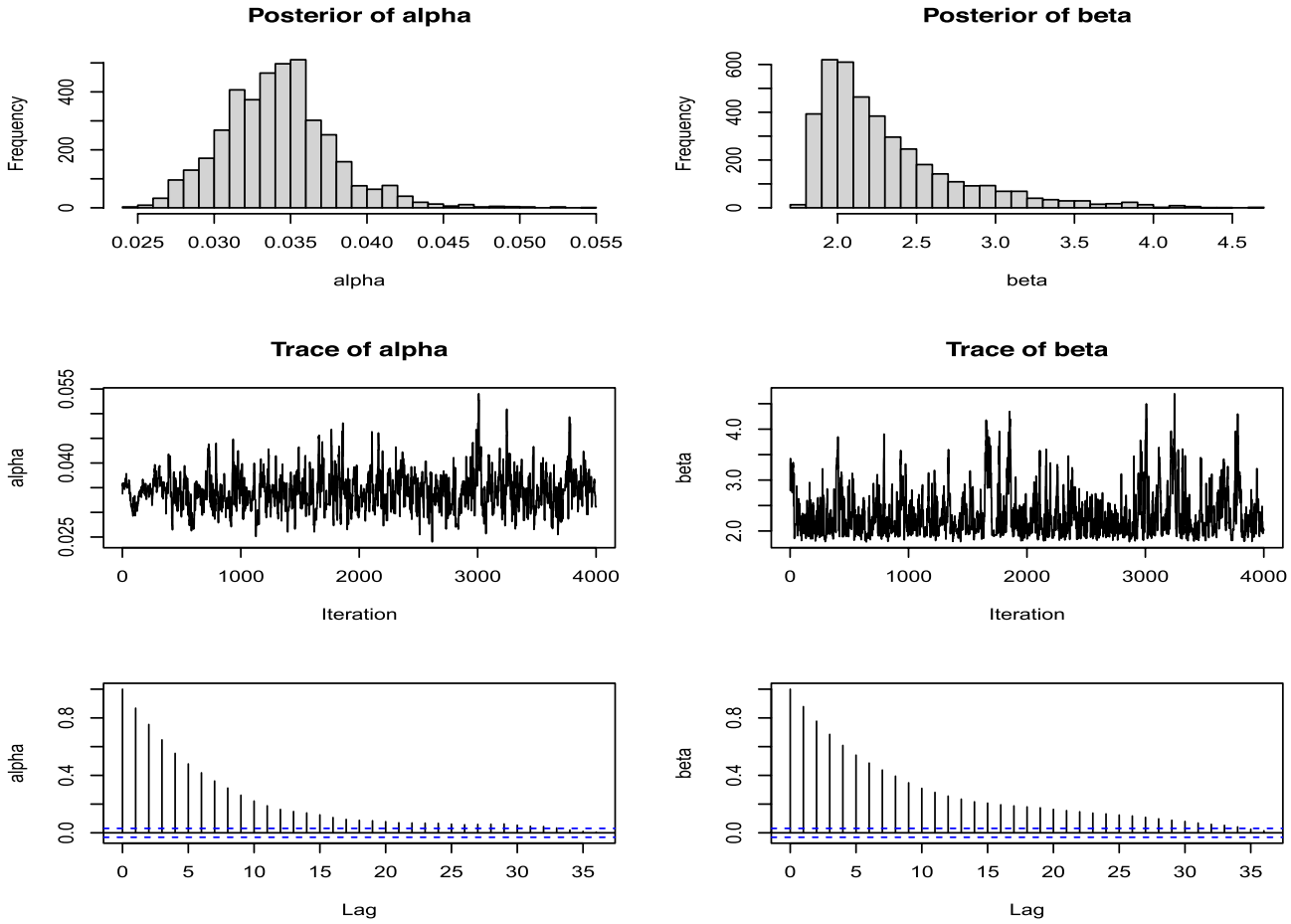


Figure 28. MCMC Convergence results using data II.

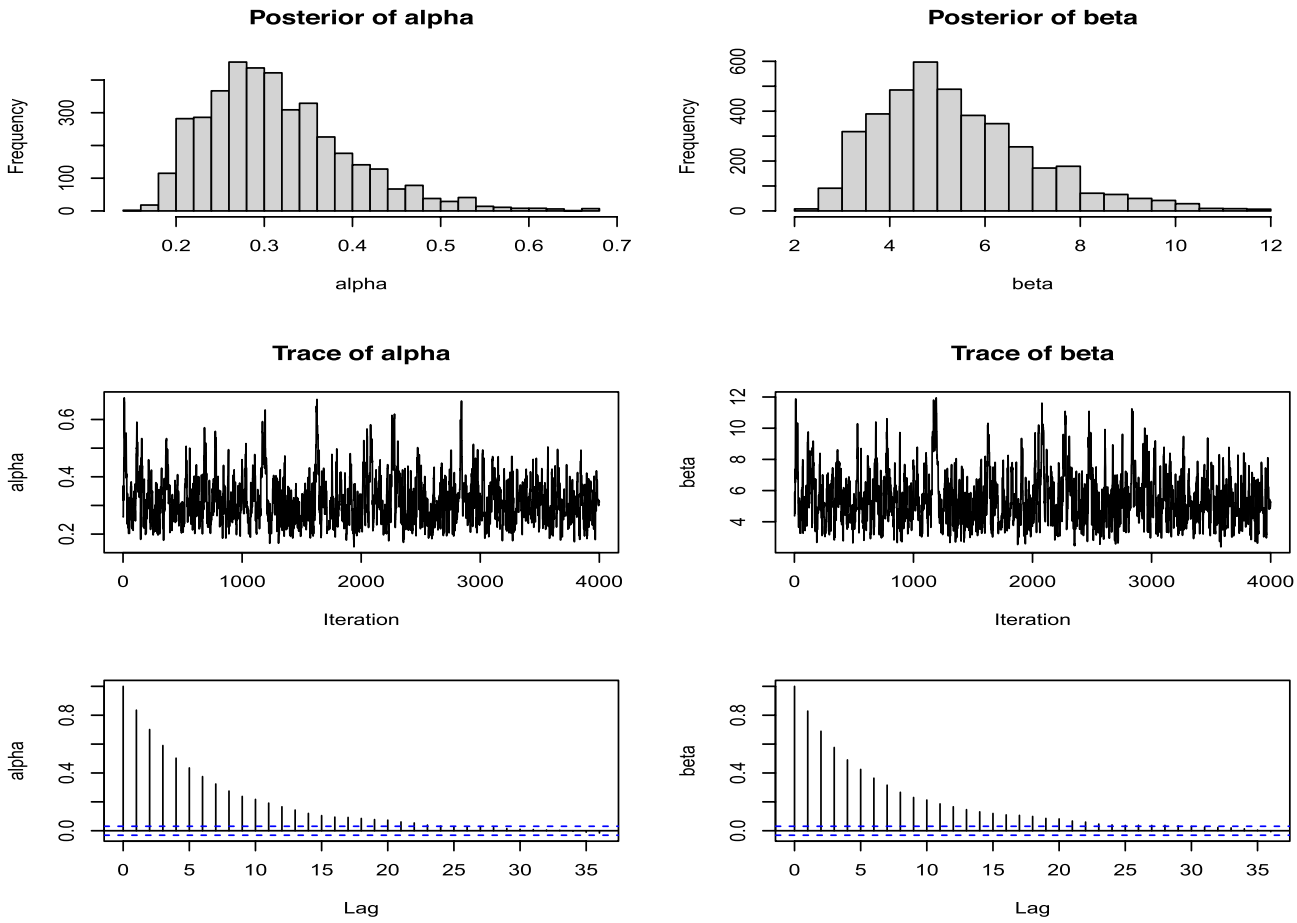


Figure 29. MCMC Convergence results using data III.

like censored data. Also, the proposed model may be utilized in different areas that require flexible and tractable probabilistic modeling.

Codes used in the paper

```
 ## PDF of RL-Exp
 pdf=function(par,x){
lambda=par[1]
theta=par[2]
lambda*exp(-lambda*x/theta)*(theta^2-2*theta+lambda*x)/(theta^2*(theta-1))
}
t=seq(0,25,len=1000)
plot(t,pdf(c(2.5,5),t),type="l",lwd=2,lty=2,col="blue",xlab="z",ylab="PDF")

## PDF of RL-Exp
cdf<- function(par,x){
lambda=par[1]
theta=par[2]
1-(1+lambda*x/(theta*(theta-1)))*exp(-lambda/theta*x)
}
t=seq(0,25,len=1000)
plot(t,cdf(c(2.5,5),t),type="l",lwd=2,lty=2,col="blue",xlab="z",ylab="CDF")

##Random Generation
lambda=1; theta=2.5; nb=1e5
u=runif(nb,0,1)
Z=-1/lambda*log((lambertWm1(-(theta-1)*(1-u)*
exp(-(theta-1)))/(-(theta-1)*(1-u)) )**(-theta) )

## Estimation of Parameters under MLE
MLE<-function(par,x){
lambda=par[1]
theta=par[2]
 - sum(log(lambda*exp(-lambda*x/theta)*(theta^2-2*theta+lambda*x)/(theta^2*(theta-1))
))
}

NB=100;nb=20;res.theta=numeric(NB);res.lambda=numeric(NB)
for(k in 1:NB){
lambda=1.75; theta=2.3
u=runif(nb,0,1)
X=-1/lambda*log((lambertWm1(-(theta-1)*(1-u)*
exp(-(theta-1)))/(-(theta-1)*(1-u)) )**(-theta) )
res.lambda[k]=optim(c(lambda,theta),MLE,method="N",x=X)$par[1]
res.theta[k]=optim(c(lambda,theta),MLE,method="N",x=X)$par[2]
}
 MLE.lambda=mean(res.lambda);MLE.theta=mean(res.theta)
 MSE.theta=mean(res.theta-theta)**2
 MSE.lambda=mean(res.lambda-lambda)**2
cbind(MLE.lambda,MSE.lambda);cbind(MLE.theta,MSE.theta)
## Application to real Data Set
result=goodness.fit(pdf = pdf, cdf =cdf, method = "N",
starts = c(lambda,theta), data = data,domain = c(0,Inf),mle = NULL)
```

Data availability
All data exists in the paper with its related references in the Application section.

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Author contributions
Fatma and Amirah did the writing, Meraou, Eslam , Wafa, Zakiah, Abdisalam did the mathematics and formal analysis, and Abeer, Hassan did the programming, and editing the language

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Declarations

Competing interests
The authors declare no competing interests.

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