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The Marshall-Olkin Gull Alpha Power family of distributions: development, statistical properties, and comparative applications

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ABSTRACT

The Marshall-Olkin Gull alpha power (MOGAP) family of probability distributions is an innovative distributional technique that we establish in this article. The introduced family approach is incorporated by joining the Marshall-Olkin and Gull alpha power probability approaches. The quantile function, ordinary moments, moment generating function, residual life, reverse residual life, and order statistics are among the standard statistical features that are investigated. A specific member of the proposed approach, utilizing the two-parameter Weibull distribution as a baseline distribution, is introduced. This distribution is referred to as the Marshall-Olkin Gull Alpha Power Weibull (MOGAP-Wei) distribution. Three various structures, including symmetrical, positively skewed, and negatively skewed, are revealed by visually analyzing the probability density function (PDF) of the newly introduced distribution. Similarly, the hazard rate function of the MOGAP-Wei distribution can take several different forms, including increasing, decreasing, unimodal, and bathtub-shaped patterns, demonstrating its flexibility in modeling different types of failure behaviors. The unknown parameters of the proposed approach of the distributions are estimated through the popular maximum likelihood estimation method. The performance of the relevant estimation approach is assessed through a simulation study. The findings from the simulation showed that as sample size increases, the mean of the estimates approaches the true parameter value, the values of mean square errors consistently decrease, and the values of biases also decrease. Finally, the practical applicability and flexibility of the newly proposed distribution are demonstrated by three real-world datasets. The evaluation of the MOGAP-Wei model is made with the other existing probability distributions, including the alpha power Weibull distribution, the exponentiated Weibull distribution, the Weibull probability distribution, the modified Weibull distribution, and the Kumaraswamy Weibull distribution.

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Introduction

Many probability distributions are used in research on distribution theory to forecast and simulate real-life events in a variety of applied fields, such as engineering, survival, medical science, industrial, education, finance, and insurance sciences. However, due to the complexity of real-world phenomena, no single probability distribution has yet been proposed by researchers in the literature that can adequately model every phenomenon. For instance, among the classical distributions, the most well-known is the classical Weibull distribution. The classical Weibull model can model phenomena that have only a monotonic hazard rate function. Unfortunately, when the phenomena are complex, the Weibull distribution is not sufficiently flexible to provide a satisfactory representation of the data. Therefore, there is a clear and essential need for a modified and generalized version of the Weibull distribution to model a wide range of phenomena. In order to make the traditional models more versatile, the authors have consequently concentrated more on developing new methods for statistical probability distributions. To improve the flexibility level of the probability distribution to model and predict a wide range of real-world data, researchers often extend classical probability models by introducing one or more additional parameters. In this regard, some of the well-established approaches of probability distributions are the generalized class of distributions established through Medholkar and Srivastava [1], Marshall and Olkin [2] provided a novel class of distributions referred to as the Marshall-Olkin generated (MOG) class models, the Beta-Generated (B-G) family of probability distributions presented by Eugene et al. [3], the transmuted approach of probability distributions introduced by the authors Shaw and Buckley [4], Gamma-generated (G-G) family of probability distributions presented by the authors Zografos and Balakrishnan [5], the Kumaraswamy-Generalized (Ku-G) extension of distributions defined by the authors Cordeiro and de Castro [6], Generalized beta-generated method of distributions proposed by Alexander et al. [7], the odd gamma-generated type-3 distributions approach introduced by Trobi and Montazari [8], the T-X method to generate new versatile distributions by Alzaatreh et al. [9], the Weibull-G method of probability distributions presented by the authors Bourguignon et al. [10], the exponentiated-half-logistic approach of probability distributions presented by the authors Cordeiro et al. [11], the log-gamma generated (LGG) method of probability distributions presented by Amini et al. [12], the Kumaraswamy-odd log-logistic generator for generating novel distributions introduced by Alizadeh et al. [13], the Kumaraswamy-MO (Ku-MO) approach introduced by the authors Alizadeh et al. [14], the Odd Lomax generator (OLG) to introduce new distributions developed by Cordeiro et al. [15], and a new modified exponent power alpha (NMEPA) approach of probability distribution by the authors Shah et al. [16].

Marshall and Olkin [2] presented a general approach for generating novel probability distributions. Let $G(y)$ and $g(y)$ be the cumulative distribution function (CDF) and the PDF of any baseline distributions. Then, the CDF $K(y)$and PDF$k(y)$ of the Marshall-Olkin-generated family of probability distributions are given, respectively, by

$$K(y) = \frac{G(y)}{\omega + (1-\omega)G(y)}; \ \omega > 0, \tag{1}$$

and

$$k(y) = \frac{\omega g(y)}{[\omega + (1-\omega)G(y)]^2}. \tag{2}$$

Using (2), Marshall and Olkin [2] introduced two types of their special members, referred to as the MO-Exp (Marshall-Olkin exponential) and MO-Wei (Marshall-Olkin Weibull) probability distributions, along with the respective properties of these generated distributions. They further presented some meaningful physical interpretations. For $\omega = 1$, $k(y)$is equal to $g(y)$and for various values of ω, $k(y)$can be more versatile than $g(y)$.

Recently, the authors Ijaz et al. [17] developed another class of approach called the new Gull Alpha Power (GAP) approach of probability distributions. It is an effective tool to propose novel distributions by adding a new parameter to the classical models. The CDF$F(y)$ of the GAP family is

$$F(y) = \begin{cases} \dfrac{\alpha T(y;\phi)}{\alpha^{T(y;\phi)}} & if \alpha > 0, \alpha \neq 1 \\ T(y;\phi) & if \alpha = 1, \end{cases} \tag{3}$$

and the associated PDF$f(y)$takes the form

$$f(y) = \begin{cases} t(y;\phi)\alpha^{1-T(y;\phi)}(1 - \log(\alpha)T(y;\phi)) & if \alpha > 0, \alpha \neq 1 \\ t(y;\phi) & if \alpha = 1. \end{cases} \tag{4}$$

Ijaz et al. [17] considered the two-parameter Weibull model as a base model and introduced a new probability distribution termed the GAP-Weibull distribution.

The main aim of the present study is to develop a new approach to probability distributions that enhances the flexibility and adaptability of existing distributions. The newly proposed family is named the Marshall-Olkin Gull Alpha Power (MOGAP for short) family of distributions. The newly introduced family of probability distributions is constructed by incorporating the CDF of the GAP family, given in Eq. (3), as the baseline distribution in Eq. (1). By combining the Marshall-Olkin and Gull Alpha Power mechanisms, the proposed family provides greater flexibility in modelling diverse distributional shapes and hazard-rate behaviors than the

corresponding GAP and Marshall-Olkin families of distributions. In particular, this combined structure enables the newly proposed family to accommodate a broader range of hazard-rate patterns, making it suitable for complex real-life data. The study is organized into eight sections. Section 2 introduces the proposed family of probability distributions. In Section 3, a superior sub-model of the newly developed class, named the Marshall-Olkin Gull Alpha Power Weibull (MOGAP-Wei) distribution, is derived. The PDF and corresponding hazard function are also graphically illustrated in this section to demonstrate the flexibility of the proposed distribution. Section 4 investigates the distributional properties of the MOGAP approach. Maximum likelihood estimation is employed to estimate the model parameters of the MOGAP approach in Section 5. Section 6 presents a simulation study to evaluate the performance of the applied estimation method. Section 7 demonstrates the practical applicability of the proposed distribution using three real data sets. Finally, concluding remarks are presented in Section 8.

MOGAP family

Here, a novel method is proposed to incorporate two extra parameters into the probability distributions to bring more flexibility and generalize the existing distributions in terms of the hazard rate function (HRF) and PDF. The suggested method is known as the MOGAP family of distributions and is obtained by replacing the base distribution CDF$G(y)$in Eq. (1) by the CDF $F(y)$ of the GAP family given in Eq. (3). The CDF $K(y;\alpha,\omega,\phi)$ of the MOGAP approach is expressed as

$$K(y;\alpha,\omega,\phi)=\begin{cases}\dfrac{\alpha T(y;\phi)}{\omega\alpha^{T(y;\phi)}+\alpha(1-\omega)T(y;\phi)} & if\alpha>0,\alpha\neq 1\\ T(y;\phi) & if\alpha=1.\end{cases} \tag{5}$$

Its PDF$k(y;\omega,\alpha,\phi)=\frac{d}{dy}K(y;\omega,\alpha,\phi)$ can derived as

$$k(y;\omega,\alpha,\phi)=\begin{cases}\dfrac{\omega\alpha t(y;\phi)\alpha^{T(y;\phi)}\{1-\log(\alpha)T(y;\phi)\}}{(\omega\alpha^{T(y;\phi)}+\alpha(1-\omega)T(y;\phi))^2} & if\alpha>0,\alpha\neq 1\\ t(y;\phi) & if\alpha=1.\end{cases} \tag{6}$$

For $\omega>0$,$\alpha>0$,$\alpha\neq 1$, and $y\in\Re$, the SF (survival function)$S(y;\omega,\alpha,\phi)=1-K(y;\omega,\alpha,\phi)$, the hazard function (HF) $h(y;\omega,\alpha,\phi)=\frac{k(y;\omega,\alpha,\phi)}{S(y;\omega,\alpha,\phi)}$, and the cumulative HF (CHF) $h(y;\omega,\alpha,\phi)=-\log(1-K(y;\omega,\alpha,\phi))$of the MOGAP family corresponding to Eq. (5) and Eq. (6) are given, respectively, by

$$S(y;\omega,\alpha,\phi)=\frac{\omega(\alpha^{T(y;\phi)}-\alpha T(y;\phi))}{\omega\alpha^{T(y;\phi)}+\alpha(1-\omega)T(y;\phi)}, \tag{7}$$

$$h(y;\omega,\alpha,\phi)=\frac{\alpha t(y;\phi)\alpha^{T(y;\phi)}\{1-\log(\alpha)T(y;\phi)\}}{(\alpha^{T(y;\phi)}-\alpha T(y;\phi))(\omega\alpha^{T(y;\phi)}+\alpha(1-\omega)T(y;\phi))}, \tag{8}$$

$$\tau(y;\omega,\alpha,\phi)=-\log\left(\frac{\omega(\alpha^{T(y;\phi)}-\alpha T(y;\phi))}{\omega\alpha^{T(y;\phi)}+\alpha(1-\omega)T(y;\phi)}\right), \tag{9}$$

respectively, where $T(y;\phi)$and $t(y;\phi)$are the corresponding CDF and PDF of the base distribution, depending on the parameter vectorϕ.

Sub-model description

Here, based on the MOGAP approach of probability distributions, we create a new flexible distribution termed the Marshall-Olkin Gull Alpha Power Weibull, for short, the MOGAP-Wei distribution. The MOGAP-Wei distribution is defined by taking the traditional Weibull distribution as the base distribution. The CDF and PDF of the Weibull distribution with their corresponding parameters $\beta=1$and$\theta\geq 0$for $y\geq 0$are given by $T(y;\phi)=1-e^{-y^\theta}andt(y;\phi)=\theta y^{\theta-1}e^{-y^\theta}$. Substituting this CDF$T(y;\phi)$of the Weibull distribution in Eq. (5), we obtain the $K(y;\omega,\alpha,\phi)$ of the MOGAP-Wei distribution as follows

$$K(y;\omega,\alpha,\phi)=\frac{\alpha\left(1-e^{-y^\theta}\right)}{\omega\alpha^{\left(1-e^{-y^\theta}\right)}+\alpha(1-\omega)(1-e^{-y^\theta})};\ y\geq 0, \tag{10}$$

where,$\omega,\theta,\alpha>0$and$\alpha\neq 1$. The PDF $k(y;\omega,\alpha,\phi)$corresponding to Eq. (10) is given by

$$k(y;\omega,\alpha,\phi)=\frac{\alpha\theta\omega y^{\theta-1}e^{-y^\theta}\alpha^{\left(1-e^{-y^\theta}\right)}\left\{1-\log(\alpha)\left(1-e^{-y^\theta}\right)\right\}}{\left(\omega\alpha^{\left(1-e^{-y^\theta}\right)}+\alpha(1-\omega)(1-e^{-y^\theta})\right)^2};\ y>0, \tag{11}$$

where,ω,α, and$\theta$are called the tilt and shape parameters, respectively.

Remark 1. The parameters (ω, α, θ) of the MOGAP-Wei distribution are identifiable: distinct parameter combinations cannot

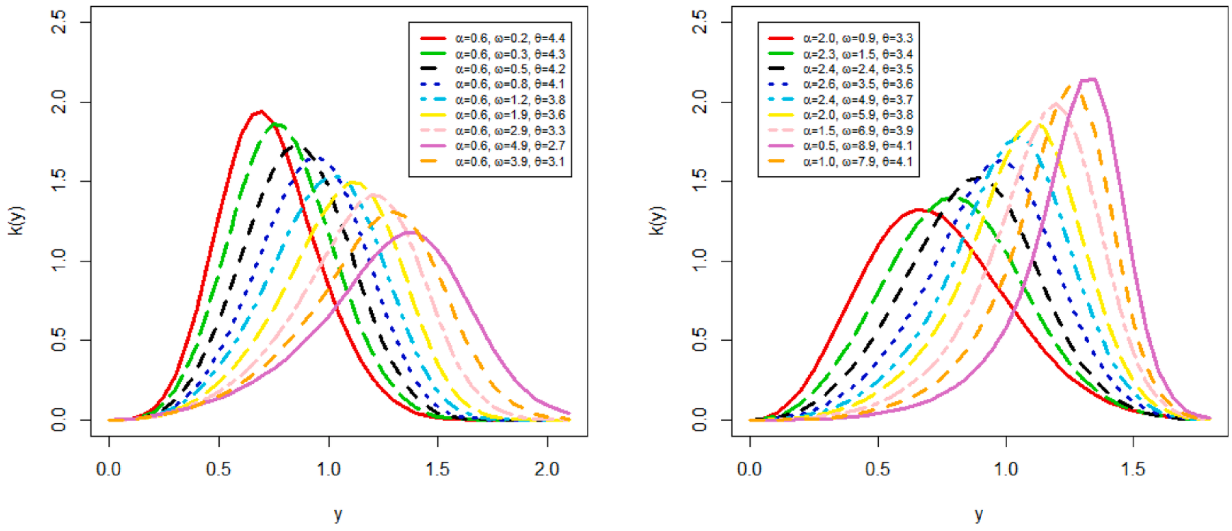

Fig. 1. Diverse shapes of the $k(y;\alpha,\omega,\phi)$ of the MOGAP-Wei distribution.

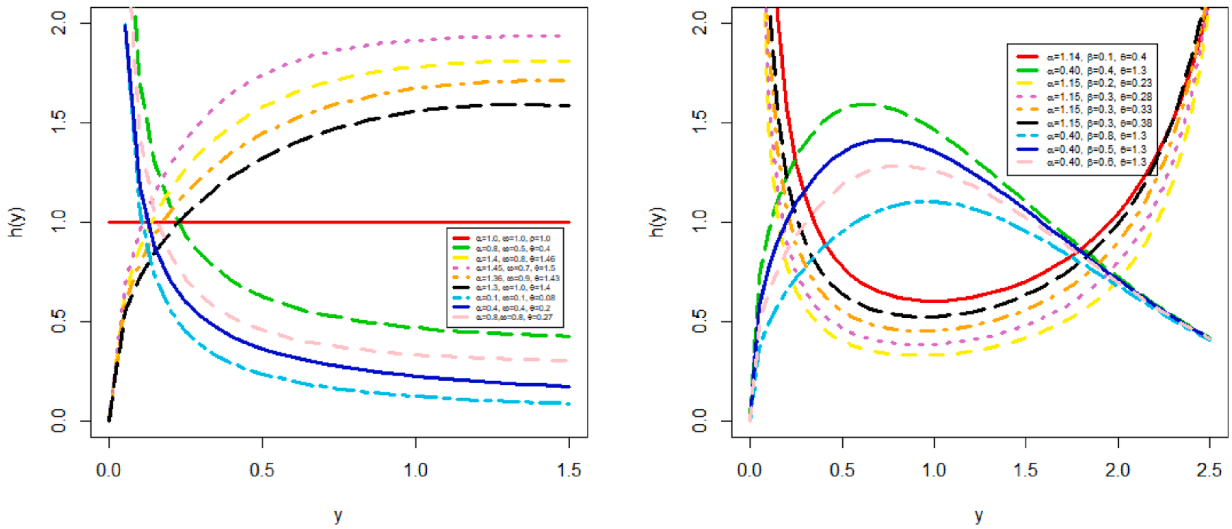

Fig. 2. Diverse shapes of the $h(y;\alpha,\omega,\phi)$ of the MOGAP-Wei distribution.

generate the same distribution. To see this, let $z = y^{\theta}$, so that z traces out $(0, \infty)$ bijectively as y ranges over $(0, \infty)$, for any fixed $\theta > 0$. Expanding $K(y; \omega, \alpha, \theta)$ from Eq. (10) as a series in z near $z = 0$ gives

$$K(y;\,\omega,\,\alpha,\theta) = z/\omega + z^2\left[log(\alpha)\,/\,\omega + (\omega-2)\,/\,\left(2\omega^2\right)\right] + O\left(z^3\right), z = y^{\theta}$$

Suppose two parameter vectors $(\omega_1, \alpha_1, \theta_1)$ and $(\omega_2, \alpha_2, \theta_2)$ yield the same CDF for every $y > 0$. As $y \to 0^{+}$, the CDF behaves like y^{θ}/ω to leading order. Since this power-law term dominates all others near the origin, equality of the two CDFs forces equality of the exponent $(\theta_1 = \theta_2 = \theta)$ and equality of its coefficient $(\omega_1 = \omega_2 = \omega)$. With θ and ω fixed, the coefficient of the next-order term, $\log(\alpha)/\omega + (\omega - 2)/(2\omega^2)$, is strictly increasing in $\log(\alpha)$ since its slope $1/\omega$ is positive; matching this coefficient across the two parameterizations therefore forces $\alpha_1 = \alpha_2$. Hence the map $(\omega, \alpha, \theta) \mapsto K(y; \omega, \alpha, \theta)$ is injective on $\{\omega, \theta > 0, \alpha > 0, \alpha \neq 1\}$, and the parameters of the MOGAP-Wei distribution are identifiable.

The SF$S(y;\omega,\alpha,\phi)$, HF$h(y;\omega,\alpha,\phi)$, and CHF$H(y;\omega,\alpha,\phi)$ of the MOGAP-Wei distribution are given, respectively, by

$$S(y;\omega,\alpha,\phi) = 1 - \frac{\alpha\left(1-e^{-y^{\theta}}\right)}{\omega\alpha^{\left(1-e^{-y^{\theta}}\right)} + \alpha(1-\omega)\left(1-e^{-y^{\theta}}\right)}; y > 0, \tag{12}$$

$$h(y;\omega,\alpha,\phi)=\frac{\alpha\theta y^{\theta-1}e^{-y^{\theta}}\alpha^{\left(1-e^{-y^{\theta}}\right)}\left\{1-\log(\alpha)\left(1-e^{-y^{\theta}}\right)\right\}}{\left(\alpha^{\left(1-e^{-y^{\theta}}\right)}-\alpha(1-e^{-y^{\theta}})\right)\left[\omega\alpha^{\left(1-e^{-y^{\theta}}\right)}-\alpha(\omega-1)(1-e^{-y^{\theta}})\right]}, \tag{13}$$

$$H(y;\omega,\alpha,\phi)=-\log\left(\frac{\omega\left(\alpha^{\left(1-e^{-y^{\theta}}\right)}-\alpha\left(1-e^{-y^{\theta}}\right)\right)}{\omega\alpha^{\left(1-e^{-y^{\theta}}\right)}+\alpha(1-\omega)(1-e^{-y^{\theta}})}\right). \tag{14}$$

Fig. 1 shows some graphical representations using various combinations of the corresponding parameter values of the PDF$k(y;\omega,\alpha,\phi)$ of the newly proposed MOGAP-Wei probability distribution. The PDF of the proposed distribution can adopt diverse forms, including symmetrical, negatively skewed, and positively skewed.

Similarly, the HF$h(y;\omega,\alpha,\phi)$of the newly proposed MOGAP-Weibull distribution with several combinations of different parameter values are graphically illustrated in Fig. 2. Fig. 2 demonstrates that the HF of the proposed distribution is very flexible and can exhibit a diversity of shapes, including constant, increasing, reverse-J, decreasing, unimodal, and bathtub.

One of the most important advantages of the proposed family is that it reduces to several distributions when the parameter$(\beta=1)$. Hence, the CDF in Eq. (10) generalizes many existing probability distributions that exist in the literature as the following special cases:

1. The two-parameter MO-Wei distributions proposed by Marshall and Olkin [2], when$(\alpha=1)$, and its CDF is defined as follows

$$K(y;\omega,\theta)=\frac{\left(1-e^{-y^{\theta}}\right)}{\omega+(1-\omega)(1-e^{-y^{\theta}})};\ y\geq 0,\ \omega,\theta>0.$$

2. The two-parameter GAP-Wei distributions proposed by Ijaz et al. [17], when$(\omega=1)$, and its CDF is defined as follows

$$K(y;\alpha,\theta)=\frac{\alpha\left(1-e^{-y^{\theta}}\right)}{\alpha^{\left(1-e^{-y^{\theta}}\right)}};\ y\geq 0,\alpha,\theta>0.$$

3. The one-parameter Weibull distribution proposed by Weibull [18], when $(\omega=1)$and$(\alpha=1)$, and its CDF is defined as follows

$$K(y;\theta)=1-e^{-y^{\theta}};\ y\geq 0,\ \theta>0.$$

Properties of the MOGAP family

The statistical characteristics of the MOGAP method of probability distributions, such as moments, quantile function (QF), Order Statistics, residual life (RL), reverse residual life (RRL), and moment generating function (MGF), are investigated in the next subsections.

Quantile function

The QF can generate a random number from any probability distribution. It is applied in several statistical measure computations, notably skewness and kurtosis. The QF is also referred to as the inverse distribution function (IDF), resulting in the following expression

$$u\omega\alpha^{T(y;\phi)}+\alpha(u(1-\omega)-1)T(y;\phi)=0. \tag{15}$$

By using u as a uniform random number, the equation can generate a random sample from any subcase (or base member) of the MOGAP family.

r^{th} *moment*

The moment of a probability distribution is a crucial tool used to describe its locations and shapes. Specifically, the first moment is used to describe the central tendency of the distribution, the second moment corresponds to the variance, and similarly, the higher-order moments describe the skewness and kurtosis of the proposed distribution. Let Y be a random variable that follows the MOGAP family; then the *rth* moment of Y can be defined as

$$\mu_r^{/}=\int_{-\infty}^{\infty}y^r k(y;\omega,\alpha,\phi)dy. \tag{16}$$

Using some algebraic expansion,

Table 1
Numerical explanation of the key metrics of the MOGAP-Wei model.

ω	α	θ	$\mu_1{}'$	$\mu_2{}'$	σ^2	**skewness**	**kurtosis**
0.350	0.520	0.420	3.177939	86.18804	76.08875	9.682885	205.5498
0.350	0.520	0.550	1.856081	15.19643	11.75136	5.267879	54.56898
0.350	0.520	0.770	1.267663	4.026033	2.419064	2.894917	16.86026
0.350	0.520	1.150	1.024706	1.738026	0.688004	1.553889	6.578377
0.350	0.520	1.450	0.968809	1.333936	0.395344	1.070897	4.540803
0.350	0.520	1.950	0.937549	1.093852	0.214853	0.605881	3.337802
0.350	0.520	2.120	0.933676	1.054918	0.183169	0.495686	3.158529
0.350	0.520	2.620	0.929873	0.988739	0.124075	0.246863	2.901529
0.350	0.520	2.740	0.929898	0.979194	0.114483	0.199036	2.875671
0.350	0.520	2.860	0.930131	0.971187	0.106043	0.154649	2.858507
0.350	0.520	3.220	0.931648	0.953916	0.085948	0.038459	2.844924
0.420	0.350	0.520	1.403356	12.96671	10.99731	7.151432	100.5061
0.550	0.450	0.520	1.484262	14.10657	11.90354	6.905725	93.66477
0.770	0.550	0.520	1.422972	13.56619	11.54134	7.025373	96.62037
1.150	0.660	0.520	1.214394	11.20511	9.730356	7.625782	113.3077
1.450	0.750	0.520	1.066312	9.437688	8.300668	8.201003	131.0228
1.950	0.840	0.520	0.768389	5.902393	5.311968	9.944903	194.9988
2.120	0.950	0.520	0.720469	5.169466	4.650389	10.41498	216.7061
2.620	0.970	0.520	0.393367	1.422792	1.268054	13.76011	477.6708
2.700	1.130	0.520	0.386197	1.007416	0.858267	9.698234	280.1409
0.860	1.440	0.520	2.676616	30.36444	23.20017	5.042589	51.25672
0.220	1.770	0.520	5.193107	73.07883	46.11048	3.630373	28.3917
0.750	2.240	3.550	1.069169	1.216924	0.073801	-0.3580443	3.029064
0.880	0.550	4.440	0.849164	0.775829	0.054749	-0.0023627	2.74643
0.960	0.220	5.540	0.760252	0.6162143	0.038231	0.1731388	2.87282

$$(1-y)^{-\upsilon} = \sum_{i=0}^{\infty} \frac{\Gamma(\upsilon+i)}{\Gamma(\upsilon)i!} y^i;\ |y|<1,\ \ \upsilon>0.$$

The denominator in Eq. (6) of the MOGAP family can be expressed as

$$\omega^{-2}\alpha^{-2T(y;\phi)}\left(1+\left(\frac{1}{\omega}-1\right)T(y;\phi)\alpha^{(1-T(y;\phi))}\right)^{-2}$$

$$= \omega^{-2}\alpha^{-2T(y;\phi)}\sum_{i=0}^{\infty}(i+1)\left(1-\frac{1}{\omega}\right)^i T(y;\phi)^i\alpha^{i(1-T(y;\phi))}, \tag{17}$$

after substituting the above expression, Eq. (6) can be rewritten as follows

$$k(y;\omega,\alpha,\phi) = \sum_{i=0}^{\infty}\alpha^{1+i}\omega^{-1}t(y;\phi)(i+1)\left(1-\frac{1}{\omega}\right)^i \times(1-\log(\alpha)T(y;\phi))T(y;\phi)^i\alpha^{T(y;\phi)(-1-i)}. \tag{18}$$

Further, using the series representation,

$$\alpha^y = \sum_{j=0}^{\infty}\frac{(\log(\alpha))^j}{j!}y^j. \tag{19}$$

Using $y = T(y;\phi)(-1-i)$ in Eq. (19), we get

$$\alpha^y = \sum_{j=0}^{\infty}\frac{(\log(\alpha))^j}{j!}T(y;\phi)^j(-1)^j(i+1)^j. \tag{20}$$

Using Eq. (20) in Eq. (18), the PDF of the MOGAP approach can be expressed as

$$k(y;\omega,\alpha,\phi) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j!}(\log(\alpha))^j\alpha^{1+i}\omega^{-1}t(y;\phi)(-1)^j(i+1)^{j+1}\left(1-\frac{1}{\omega}\right)^i \times(1-\log(\alpha)T(y;\phi))T(y;\phi)^{i+j}. \tag{21}$$

Now, if Y follows the MOGAP family of distributions, and inserting Eq. (21) in (16), then the rth moments can be obtained as

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j!}(\log(\alpha))^j\alpha^{1+i}\omega^{-1}(-1)^j(i+1)^{j+1}\left(1-\frac{1}{\omega}\right)^i \mathrm{Y}_{r,i+j} - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j!}(\log(\alpha))^{j+1}\alpha^{1+i}\omega^{-1}(-1)^j(i+1)^{j+1}\left(1-\frac{1}{\omega}\right)^i \mathrm{Y}_{r,i+j+1}. \tag{22}$$

Where, $\mathrm{Y}_{r,i+j} = \int_{-\infty}^{\infty} y^r t(y;\phi)T(y;\phi)^{i+j}dy$ and $\mathrm{Y}_{r,i+j+1} = \int_{-\infty}^{\infty} y^r t(y;\phi)T(y;\phi)^{i+j+1}dy$

Table 1 uses Eq. (22) to provide a numerical representation of the recommended statistical properties of combining different values for each parameter.

Next, the MGF for the MOGAP approach of distributions, say $M_y(t)$, is derived as

$$M_y(t) = \int_{-\infty}^{\infty} e^{ty}k(y;\omega,\alpha,\phi)dy, \tag{23}$$

$$M_y(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} y^r k(y;\alpha,\omega,\phi)dy, \tag{24}$$

$$M_y(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r. \tag{25}$$

Using Eq. (22) in Eq. (25), we can easily acquire the MGF of the MOGAP approach of probability distributions.

Order statistics

Order statistics play a fundamental role in distribution theory, as they provide valuable information about the behavior of sample observations and help analyze the variability and characteristics of data based on the sample size.

Let $y_i(i = 1,2,3....m)$be the ordered random sample of size m that follows the MOGAP family approach, then the probability density function (PDF) of the rth order statistics corresponding to Eq. (5) and Eq. (6) is given by

$$f_{r:m}(y) = \frac{m!}{(r-1)!(m-r)!}k(y;\omega,\alpha,\phi)[K(y;\omega,\alpha,\phi)]^{r-1}[1-K(y;\omega,\alpha,\phi)]^{m-r}. \tag{26}$$

The 1st and m^{th} order statistics of the MOGAP family approach can be stated as $y_{1:m} = \min(y_1, \mathrm{y}_2, ...,\mathrm{y}_m)$ and $y_{m:m} = \max(y_1, \mathrm{y}_2, ..,\mathrm{y}_m)$. Since, $0 < K(y;\omega,\alpha,\phi) < 1$for $y \in \Re^+$, by utilizing the well-known binomial expansion of $[1-K(y;\omega,\alpha,\phi)]^{m-r}$as follow

$$[1-K(y;\omega,\alpha,\phi)]^{m-r} = \sum_{i=0}^{m-r}\binom{m-r}{i}(-1)^i[K(y;\omega,\alpha,\phi)]^i, \tag{27}$$

using Eq. (27) into Eq. (26), we get

$$f_{r:m}(y) = \frac{k(y;\omega,\alpha,\phi)}{B(r,m-r+1)}\sum_{i=0}^{m-r}\binom{m-r}{i}(-1)^i[K(y;\omega,\alpha,\phi)]^{r+i-1}. \tag{28}$$

Using Eq. (5) and Eq. (6) in Eq. (28), we can obtain the PDF of the$f_{r:m}$.

Residual and reverse residual lifetime

The RL of the MOGAP family of distribution, say$R_{(Y)}(z)$, where z represents time, can be expressed as

$$R_{(y)}(z) = \frac{S(y+z)}{S(z)}, \tag{29}$$

$$R_{(y)}(z) = \frac{\omega(\alpha^{T(y+z;\phi)} - \alpha T(y+z;\phi))/(\omega\alpha^{T(y+z;\phi)} + (1-\omega)\alpha T(y+z;\phi))}{\omega(\alpha^{T(z;\phi)} - \alpha T(z;\phi))/(\omega\alpha^{T(z;\phi)} + (1-\omega)\alpha T(z;\phi))}; y \in \Re. \tag{30}$$

Similarly, the RRL of the MOGAP family approach, say$\overline{R}_{(y)}(z)$, can be expressed as

$$\overline{R}_{(y)}(z) = \frac{K(z-y)}{K(z)}, \tag{31}$$

$$R_{(y)}(z) = \frac{[\alpha T(z-y)/(\omega\alpha^T(z-y) + \alpha(1-\omega)T(z-y))]}{[\alpha T(z)/(\omega\alpha^T(z) + \alpha(1-\omega)T(z))]}; y \in \Re. \tag{32}$$

Table 2
The MCSA numerical outcomes of the MOGAP-Wei model for Case I and Case II.

		Case I: $\omega = 0.4$, $\alpha = 0.1$, $\theta = 1.9$			Case II: $\omega = 0.1$, $\alpha = 0.3$, $\theta = 4.3$		
n	Est.	MLE	MSE	Bias	MLE	MSE	Bias
25	$\widehat{\omega}$	1.0292227	1.27658472	0.62922275	0.2345374	0.41103202	0.13453741
	$\widehat{\alpha}$	0.4600982	0.33616861	0.36009820	0.5350543	0.47158989	0.23505430
	$\widehat{\theta}$	2.0746750	0.22676886	0.17467501	4.1628500	0.62017985	-0.13715026
50	$\widehat{\omega}$	0.7851621	0.67006489	0.38516210	0.1827335	0.14115649	0.08273353
	$\widehat{\alpha}$	0.3487141	0.29093545	0.24871412	0.5231112	0.28496607	0.22311116
	$\widehat{\theta}$	1.9868240	0.10077300	0.08682357	4.2095380	0.43527110	-0.09046249
75	$\widehat{\omega}$	0.6744675	0.46326609	0.27446748	0.1525923	0.04161411	0.05259226
	$\widehat{\alpha}$	0.2865127	0.23359952	0.18651269	0.4984740	0.22256670	0.19847398
	$\widehat{\theta}$	1.9398670	0.07753981	0.03986685	4.2468160	0.36609416	-0.05318401
100	$\widehat{\omega}$	0.6487399	0.39209156	0.24873986	0.1504841	0.06041232	0.05048406
	$\widehat{\alpha}$	0.2530969	0.12345413	0.15309689	0.4841772	0.21251822	0.18417720
	$\widehat{\theta}$	1.9493760	0.05450321	0.04937550	4.2470440	0.31071089	-0.05295623
200	$\widehat{\omega}$	0.5286234	0.16582800	0.12862336	0.1474118	0.01219457	0.04741180
	$\widehat{\alpha}$	0.1788549	0.05172992	0.07885491	0.4987245	0.18847508	0.19872455
	$\widehat{\theta}$	1.9235210	0.02611851	0.02352127	4.2717170	0.21508451	-0.02828254
300	$\widehat{\omega}$	0.4796436	0.09087166	0.07964359	0.1437787	0.03326079	0.04377867
	$\widehat{\alpha}$	0.1506905	0.02678560	0.05069046	0.4732988	0.16807978	0.17329883
	$\widehat{\theta}$	1.9168230	0.01801652	0.01682262	4.2747710	0.17178361	-0.02522892
400	$\widehat{\omega}$	0.4568615	0.04685780	0.05686148	0.1385533	0.00869257	0.03855329
	$\widehat{\alpha}$	0.1352171	0.01394601	0.03521714	0.4688050	0.14976917	0.16880499
	$\widehat{\theta}$	1.9168620	0.01393419	0.01686214	4.3023140	0.13112183	0.00231357
500	$\widehat{\omega}$	0.4455846	0.03356859	0.04558460	0.1341864	0.00763238	0.03418639
	$\widehat{\alpha}$	0.1281588	0.00992807	0.02815884	0.4468557	0.13120640	0.14685565
	$\widehat{\theta}$	1.9151900	0.01155650	0.01519028	4.2928110	0.10757481	-0.00718902
600	$\widehat{\omega}$	0.4314294	0.02312397	0.03142941	0.1278558	0.00645944	0.02785579
	$\widehat{\alpha}$	0.1203862	0.00619829	0.02038619	0.4216832	0.11208111	0.12168318
	$\widehat{\theta}$	1.9065950	0.00938070	0.00659517	4.2873470	0.09424482	-0.01265293
700	$\widehat{\omega}$	0.4323931	0.01960062	0.03239305	0.1272370	0.00576599	0.02723703
	$\widehat{\alpha}$	0.1200690	0.00524835	0.02006901	0.4214468	0.10495607	0.12144684
	$\widehat{\theta}$	1.9080670	0.00808324	0.00806738	4.3054560	0.08295766	0.00545624
800	$\widehat{\omega}$	0.4256374	0.01389089	0.02563743	0.1223676	0.00518469	0.02236755
	$\widehat{\alpha}$	0.1159299	0.00341290	0.01592992	0.4025116	0.09316769	0.10251156
	$\widehat{\theta}$	1.9092460	0.00675072	0.00924550	4.3028960	0.07512072	0.00289608
900	$\widehat{\omega}$	0.4196618	0.01302919	0.01966183	0.1233283	0.00483712	0.02332825
	$\widehat{\alpha}$	0.1134499	0.00345268	0.01344994	0.4016782	0.08579048	0.10167817
	$\widehat{\theta}$	1.9040430	0.00588746	0.00404322	4.3081850	0.06218262	0.00818512
1000	$\widehat{\omega}$	0.4175913	0.00978013	0.01759133	0.1218379	0.00426388	0.02183785
	$\widehat{\alpha}$	0.1111768	0.00242352	0.01117681	0.3970113	0.07789322	0.09701128
	$\widehat{\theta}$	1.9039620	0.00535116	0.00396186	4.3070410	0.06205324	0.00704092

Maximum likelihood estimation

The maximum likelihood estimation (MLE) method is one of the most widely used methods in the literature for estimating the unknown parameters of probability distributions. The associated confidence intervals and additional evaluations to confirm statistical significance may be defined using the MLE strategy, which provides estimates with an important number of features. For a detailed study, please see Dong et al. [19] and Doornik [20]. The MLE scheme for estimating the associated unknown parameters of the MOGAP family is discussed in this section. Let $y_1, y_2, ..., y_n$ be the vector of a sample randomly selected from the MOGAP family of distributions with parameters ω, α, and ϕ. Then, the LLF (Log-likelihood function) equivalent to Eq. (6) is given by

$$\begin{aligned}\ell(y;\alpha,\omega,\phi) &= n\log(\alpha)+n\log(\omega)+\sum_{i=1}^{n}\log t(y_i;\phi)+\log(\alpha)\sum_{i=1}^{n}T(y_i;\phi)\\ &+\sum_{i=1}^{n}\log(1-\log(\alpha)T(y_i;\phi))-2\sum_{i=1}^{n}\log\left\{\omega\alpha^{T(y_i;\phi)}+(1-\omega)\alpha T(y_i;\phi)\right\}.\end{aligned} \tag{33}$$

Differentiating Eq. (33) with respect to the corresponding parameters ω, α, and ϕ, we obtain

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{T(y_i;\phi)}{\alpha}-\sum_{i=1}^{n}\frac{T(y_i;\phi)}{\{\alpha(1-\log(\alpha)T(y_i;\phi))\}}-2\sum_{i=1}^{n}\frac{T(y_i;\phi)\{\omega\alpha^{T(y_i;\phi)-1}+(1-\omega)\}}{\{\omega\alpha^{T(y_i;\phi)}+(1-\omega)\alpha T(y_i;\phi)\}}, \tag{34}$$

Table 3
The MCSA numerical outcomes of the MOGAP-Wei model for Case III and IV.

		Case III: $\omega = 0.9, \alpha = 0.2, \theta = 1.3$			Case IV: $\omega = 3.4, \alpha = 2.7, \theta = 3.9$		
n	Est.	MLE	MSE	Bias	MLE	MSE	Bias
25	$\widehat{\omega}$	1.7579186	2.82678054	0.85791863	4.066605	2.9411259	0.6666048
	$\widehat{\alpha}$	0.5226617	0.39151311	0.32266165	2.913120	0.6653344	0.2131198
	$\widehat{\theta}$	1.3378260	0.05756461	0.03782597	3.482610	1.2618453	-0.4173901
50	$\widehat{\omega}$	1.4778664	1.80990637	0.57786638	3.954692	2.6842908	0.5546918
	$\widehat{\alpha}$	0.4254820	0.23822254	0.22548196	2.814680	0.4087799	0.1146797
	$\widehat{\theta}$	1.3240560	0.02684345	0.02405584	3.528116	1.1088105	-0.3718844
75	$\widehat{\omega}$	1.4013898	1.47610106	0.50138978	3.930678	2.5079621	0.5306782
	$\widehat{\alpha}$	0.3977817	0.19000273	0.19778169	2.785425	0.2944021	0.0854249
	$\widehat{\theta}$	1.3079800	0.01818554	0.00797958	3.560665	0.9658091	-0.3393349
100	$\widehat{\omega}$	1.4145408	1.35912545	0.51454084	3.933870	2.3345405	0.5338697
	$\widehat{\alpha}$	0.4058291	0.18625974	0.20582905	2.778674	0.2354464	0.0786739
	$\widehat{\theta}$	1.3101710	0.01479374	0.01017069	3.585095	0.8849405	-0.3149045
200	$\widehat{\omega}$	1.1770499	0.69929932	0.27704994	3.905489	2.0067279	0.5098395
	$\widehat{\alpha}$	0.3140292	0.10079313	0.11402921	2.753340	0.1419276	0.0639147
	$\widehat{\theta}$	1.3032630	0.00649051	0.00326279	3.610068	0.6960584	-0.3105665
300	$\widehat{\omega}$	1.1602443	0.58746176	0.26024426	3.845058	1.7276333	0.5054887
	$\widehat{\alpha}$	0.3066205	0.08539034	0.10662052	2.742819	0.0983569	0.0533400
	$\widehat{\theta}$	1.3059320	0.00527131	0.00593189	3.648640	0.5931059	-0.2899315
400	$\widehat{\omega}$	1.0831037	0.38187388	0.18310365	3.795202	1.5367758	0.4450583
	$\widehat{\alpha}$	0.2776035	0.05905321	0.07760353	2.727949	0.0840670	0.0428188
	$\widehat{\theta}$	1.3035200	0.00385198	0.00352010	3.692485	0.5141556	-0.2513602
500	$\widehat{\omega}$	1.0289639	0.26257405	0.12896392	3.768231	1.3213718	0.3952021
	$\widehat{\alpha}$	0.2557495	0.04049113	0.05574946	2.730412	0.0588917	0.0279493
	$\widehat{\theta}$	1.3023500	0.00298704	0.05574946	3.714629	0.4315485	-0.2075149
600	$\widehat{\omega}$	1.0027589	0.20569561	0.10275886	3.731619	1.1055454	0.3682308
	$\widehat{\alpha}$	0.2464563	0.03190481	0.04645627	2.725571	0.0412726	0.0304119
	$\widehat{\theta}$	1.3029090	0.00264336	0.00290853	3.729928	0.3310903	-0.1853707
700	$\widehat{\omega}$	0.9915607	0.15538122	0.09156066	3.727886	1.0851028	0.3316189
	$\widehat{\alpha}$	0.2401052	0.02460631	0.04010525	2.722716	0.0416709	0.0255705
	$\widehat{\theta}$	1.3027900	0.00212474	0.00279023	3.732158	0.3327875	-0.1700719
800	$\widehat{\omega}$	1.0030628	0.17350253	0.10306280	3.689708	0.8765720	0.3278860
	$\widehat{\alpha}$	0.2447084	0.02776339	0.04470842	2.722393	0.0291005	0.0227158
	$\widehat{\theta}$	1.3035090	0.00194953	0.00350879	3.760584	0.2604865	-0.1678423
900	$\widehat{\omega}$	0.9733815	0.11928729	0.07338148	3.712694	0.8238148	0.2897078
	$\widehat{\alpha}$	0.2308719	0.01861520	0.03087187	2.726837	0.0239529	0.0223931
	$\widehat{\theta}$	1.3015710	0.00180948	0.00157118	3.748538	0.2328219	-0.1394163
1000	$\widehat{\omega}$	0.9697824	0.10290886	0.06978242	3.666723	0.7728227	0.2667234
	$\widehat{\alpha}$	0.2304923	0.01616472	0.03049231	2.715699	0.0262113	0.0156992
	$\widehat{\theta}$	1.3028160	0.00160824	0.00281609	3.777700	0.2206382	-0.1222995

$$\frac{\partial \ell}{\partial \omega} = \frac{n}{\omega} - 2\sum_{i=1}^{n} \frac{\{\alpha^{T(y_i;\phi)} - \alpha T(y_i;\phi)\}}{\{\omega\alpha^{T(y_i;\phi)} + (1-\omega)\alpha T(y_i;\phi)\}}, \tag{35}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \phi} = \sum_{i=1}^{n} \frac{\partial t(y_i;\phi)/\partial\phi}{t(y_i;\phi)} + \log(\alpha)\sum_{i=1}^{n}\partial T(y_i;\phi) \Big/ \partial\phi - \log(\alpha)\sum_{i=1}^{n}\frac{\partial T(y_i;\phi)/\partial\phi}{\{1-\log(\alpha)T(y_i;\phi)\}} \\ - 2\sum_{i=1}^{n}\frac{\{\omega\log(\alpha)\alpha^{T(y_i;\phi)}\partial T(y_i;\phi)/\partial\phi + (1-\omega)\alpha\partial T(y_i;\phi)/\partial\phi\}}{\{\omega\alpha^{T(y_i;\phi)} + (1-\omega)\alpha T(y_i;\phi)\}},\end{aligned} \tag{36}$$

equating $\frac{\partial \ell}{\partial \alpha}$, $\frac{\partial \ell}{\partial \omega}$, and $\frac{\partial \ell}{\partial \phi}$to zero, and by solving numerically, we can obtain the MLEs$(\widehat{\alpha}, \widehat{\omega}, \widehat{\phi})$for ω, α, and ϕ.

Simulation

This part utilizes a Monte Carlo simulation analysis (MCSA) to examine the practical performance of the maximum likelihood estimator (MLE) for the newly suggested distribution. The equations in Section 5 can mimic any particular model of the MOGAP approach for probability distributions. The present simulation analysis is conducted using the MOGAP-Wei distribution. Four different sets of parameter values are analyzed to investigate the entire simulation results. The four identified combinations of true parameter values are as follows: Case I: ($\omega = 0.4, \alpha = 0.1, \theta = 1.9$), Case II: ($\omega = 0.1, \alpha = 0.3, \theta = 4.3$), Case III: ($\omega = 0.9, \alpha = 0.2, \theta = 1.3$), and Case IV: ($\omega = 3.4, \alpha = 2.7, \theta = 3.9$). In general, there is no restriction on the selection of the initial values of the parametersω, α, and θ

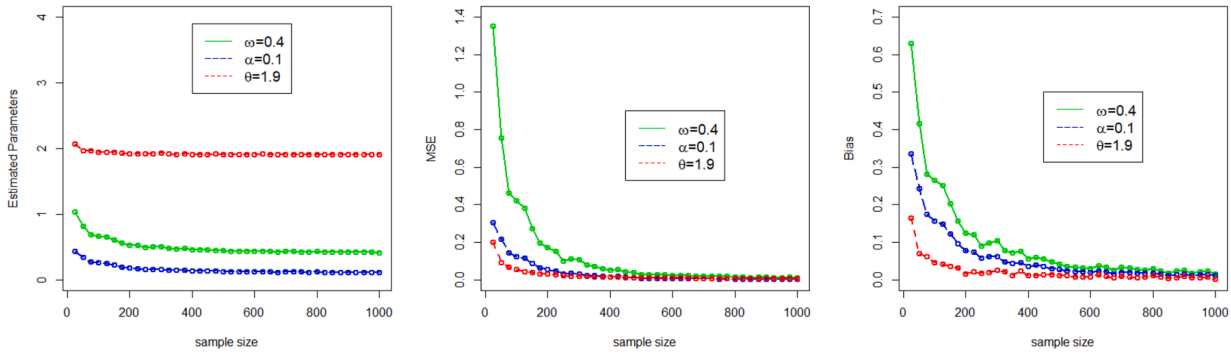

Fig. 3. The average of the estimates, the MSE, and the Bias Plots for Case I.

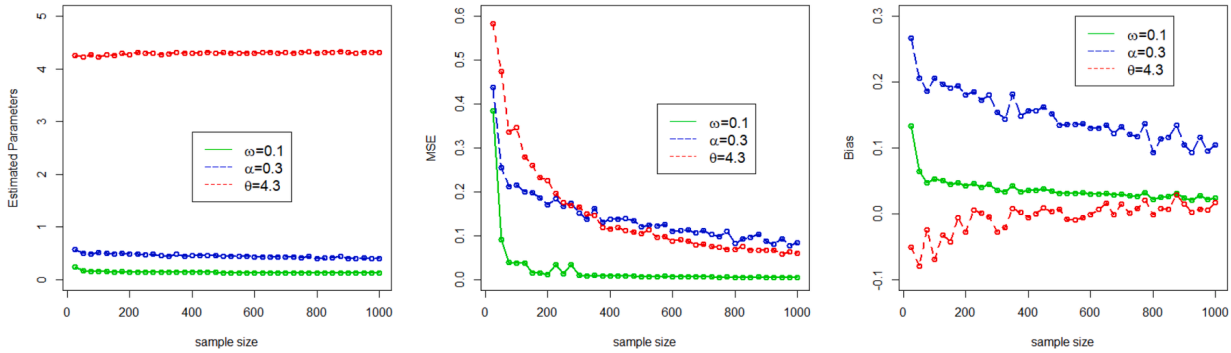

Fig. 4. The average of the estimates, the MSE, and the Bias Plots for Case II.

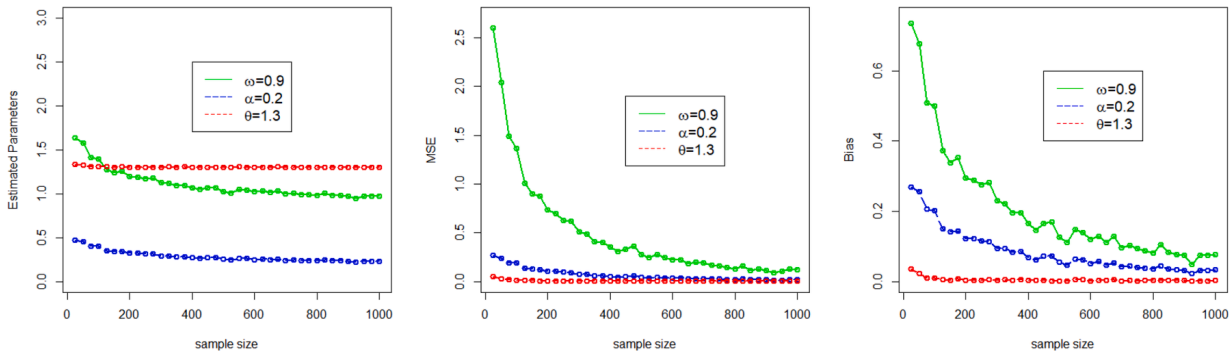

Fig. 5. The average of the estimates, the MSE, and the Bias plots for Case III.

for the simulation analysis. As established previously, the prescribed conditions permit the selection of arbitrary initial parameter values satisfying the range of $\omega \geq 0$, $\alpha \geq 0$, and $\theta \geq 0$. Accordingly, the initial values used in the simulation analysis may be chosen freely which do not violate the imposed parameter limits. The sample sizes of 25, 50, 75, 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 are involved in the simulation analysis. Each procedure of the MCSA is repeated a thousand (1000) times, and the average of the estimates, Biases, and the average of mean squared errors (MSEs) are calculated. The model parameters were estimated by the maximum likelihood estimation method. The entire simulation study is conducted in R, with parameter estimation performed using the optim() function and the L-BFGS-B optimization method with 1,000 iterations. The mathematical expressions for Biases and MSEs are, respectively, calculated as

$$Bias(\widehat{\omega}) = \frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\omega}_i - \omega);$$

and

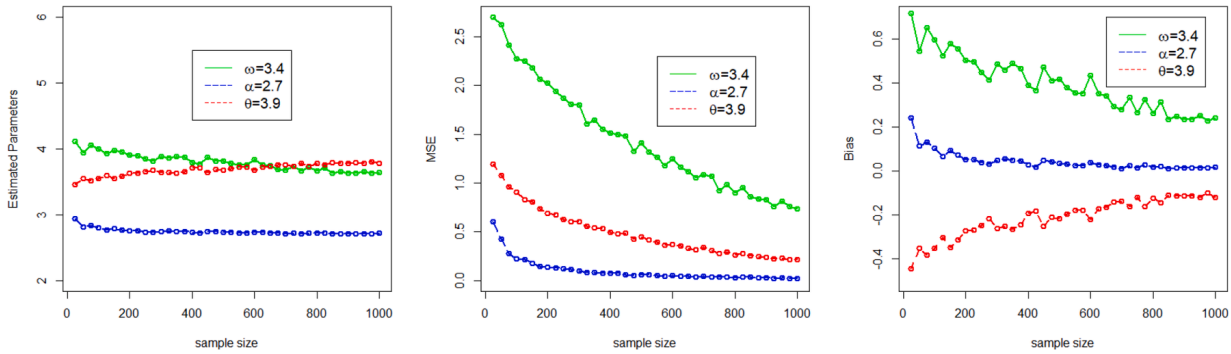

Fig. 6. The average of the estimates, the MSE, and the Bias Plots for Case IV.

$$MSE(\widehat{\omega}) = \frac{1}{1000}\sum_{i=1}^{1000}(\widehat{\omega}_i - \omega)^2;$$

the process is also repeated forαandθ. The numerical results of the MCSA for the average of the estimates, Biases, and MSEs are recorded in Tables 2 and 3. Linked to Table 2, the simulation descriptions (or results) for Case I are graphically shown in Fig. 3, and for Case II in Fig. 4. Similarly, linked to Table 3, the simulation results for Case III are graphically visualized in Fig. 5 and for Case IV in Fig. 6. From the entire MCSA, we have numerically as well as graphically observed the following output

- When n ↑, the average of the estimates approaches the true values.
- When n ↑, the average of the MSEs decreases and tends to be zero.
- When n ↑, the average of the biases decreases.

Applications to real data sets

This section demonstrates the superiority and adaptability of the newly proposed distribution by analyzing three real datasets from the field of engineering and biomedical sectors. The datasets are selected because they represent lifetime, strength, and failure-time observations and have been previously analyzed in the literature, providing a suitable basis for evaluating the empirical performance, flexibility, and practical utility of the proposed distribution.

Description of the real data sets

The first dataset is an engineering dataset consisting of 63 observations representing the strength of 1.5 cm glass fibers. The dataset was originally reported by Merovci et al. [21] and was subsequently utilized by Dey et al. [22]. The observations represent the measured strength of the glass fibers and are analyzed as positive continuous lifetime measurements. This dataset is selected to assess the performance of the proposed model in describing engineering strength data. The observations of the first dataset are 0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61,1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73,1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, and 2.24.

The second dataset is another engineering dataset comprising 76 observations and representing the fatigue fracture life of Kevlar 373/epoxy under continuous pressure at the 90-stress level. The observations correspond to the fracture times recorded until all specimens failed. The dataset was originally reported by Paul and Barlow [23] and was subsequently also used by Dey et al. [22]. The observations of the second dataset are 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, and 9.0960.

Similarly, the third dataset is obtained from the biomedical field and consists of 128 observations representing the diminution time of bladder cancer patients. The dataset was originally reported by Lee and Wang [24] and was subsequently analyzed by Liu et al. [25]. The observations of the third data set are 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93,

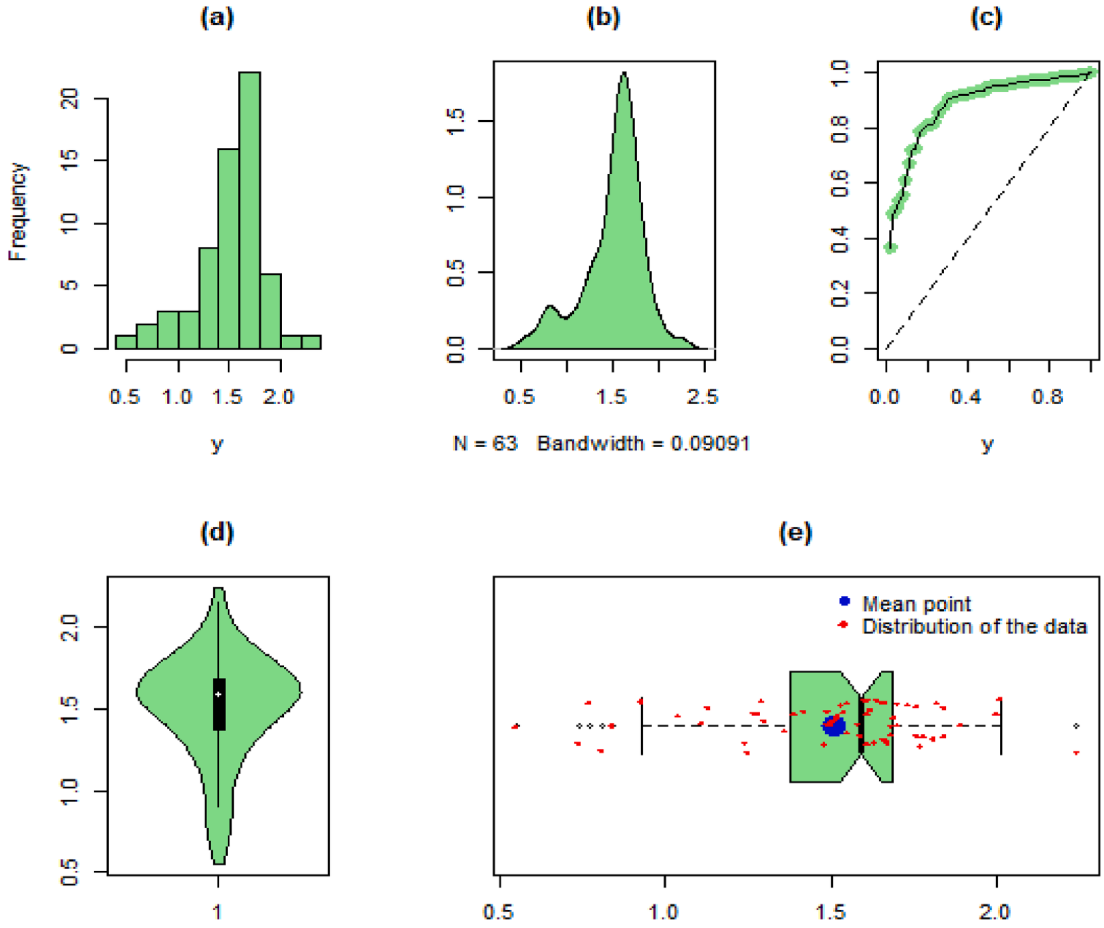

Fig. 7. The plots (a) His-P, (b) KDP, (c) TTP, (d) Vio-P, and (e) BP for Data 1.

8.65, 12.63, and 22.69.

In the present analysis, the observations of all the considered datasets are assumed to be independent realizations for the statistical analysis, consistent with the modeling assumptions adopted in this study. No additional censoring or modification of the reported observations is considered in the present study; the datasets are analyzed in their original form. Some descriptive measures of the first dataset (onward denoted by Data 1), second dataset (onward denoted by Data 2) and third dataset (onward denoted by Data 3) are min=[0.550, 0.0251, 0.080], max=[2.240, 9.0960, 79.050], 1st Qu=[1.375, 0.9048, 3.348], 3rd Qu=[1.685, 2.2959, 11.838], median=[1.590, 1.7362, 6.395], mean=[1.507, 1.9592, 9.366], range=[1.69, 9.0709, 78.97], variance=[0.1050575, 2.477415, 110.425], skewness=[-0.8999263, 1.979558, 3.286569], and kurtosis=[3.923761, 8.160792, 18.48308]. In accordance with the appropriate information sets, a few graphical representations include His-P (Histogram plot), KDP (Kernel density plot), TTP (TTT-transformation plot), Vio-P (Violin plot), and BP (Box plot), which are plotted in Figs. 7, 8, and 9.

Competing distributions and analytical measures

In this section, we discuss several existing distributions to contrast their flexibility and suitability with the newly proposed probability distribution using three real data sets. For comparison, we compare the newly introduced probability distribution with five competing distributions, where the two-parameter Weibull distribution is used as the baseline distribution. The prominent existing competing distributions are the following: (i) the Alpha Power Wei (AP-Wei) distribution introduced by Dey et al. [22], (ii) the Exponentiated Wei (Exp-Wei) distribution introduced by Medholkar and Sarivastava [1], (iii) the classical Wei distribution introduced by Weibull [18], (iv) the Modified Wei (Modi-Wei) distribution proposed by Sarhan and Zaindin [26] and the Kumaraswamy Wei (Kumar-Wei) probability distribution introduced by Cordeiro et al. [27]. The SFs of these competing distributions are

- The three parameters $(\omega, \alpha, \theta > 0)$ AP-Wei model:

$$S(y;\omega,\vartheta) = 1 - \frac{\omega^{\left(1-e^{-\alpha y^{\theta}}\right)} - 1}{\omega - 1};\ \vartheta = (\alpha,\theta), y \geq 0,$$

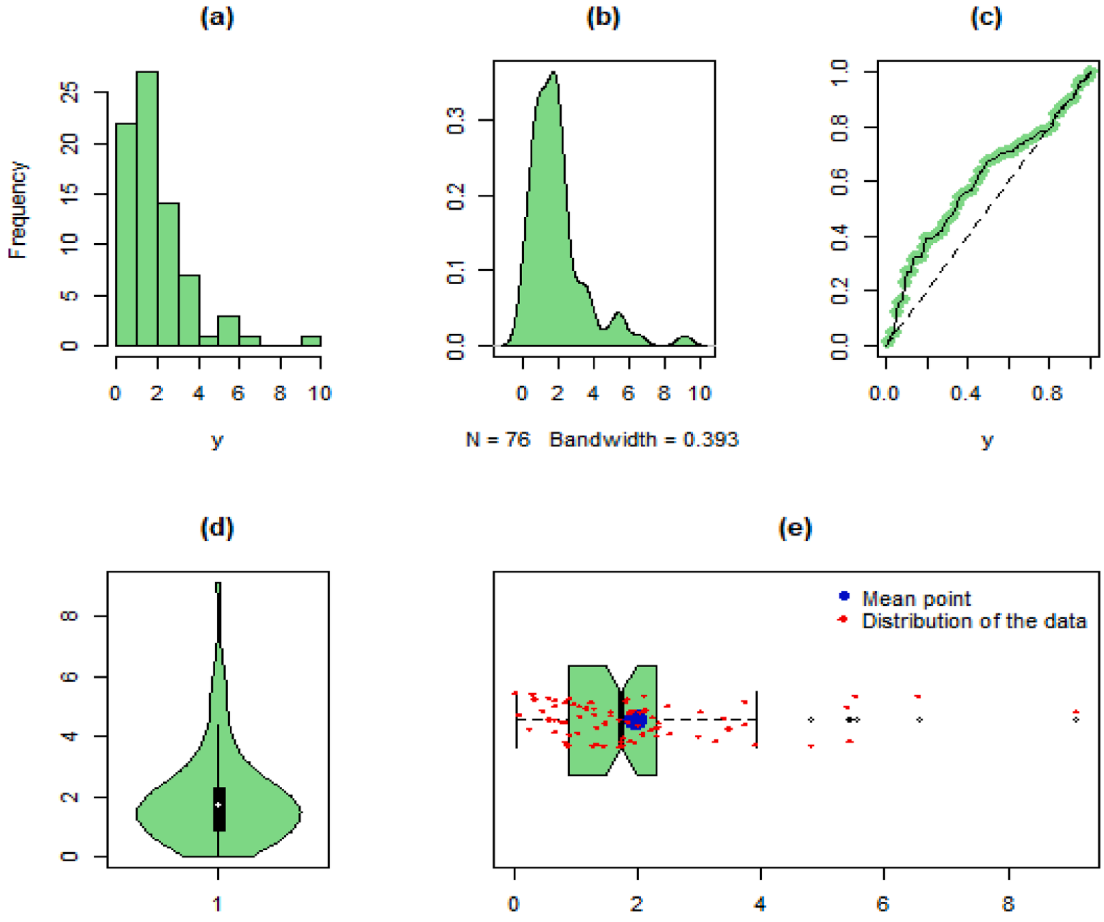

Fig. 8. The plots (a) His-P, (b) KDP, (c) TTP, (d) Vio-P, and (e) BP for Data 2.

- The three parameters $(\delta, \alpha, \theta > 0)$ Exp-Wei model:

$$S(y;\delta,\vartheta) = 1 - \left(1 - e^{-\alpha y^{\theta}}\right)^{\delta};\ \vartheta = (\alpha,\theta), y \geq 0,$$

- The two parameters $(\alpha, \theta > 0)$ classical Wei model:

$$S(y;\vartheta) = e^{-\alpha y^{\theta}};\ \vartheta = (\alpha,\theta), y \geq 0,$$

- The three parameters $(\alpha, \theta, \rho > 0)$ Modi-Wei model:

$$S(y;\alpha,\vartheta) = e^{-\alpha y^{\theta} - \rho y};\ \vartheta = (\alpha,\theta), y \geq 0,$$

- The four parameters $(\alpha, \theta, a, b > 0)$ Kumar-Wei model:

$$S(y;b,a,\vartheta) = \left(1 - \left(1 - e^{-\alpha y^{\theta}}\right)^{a}\right)^{b};\ \vartheta = (\alpha,\theta), y \geq 0.$$

Next, after considering the competing distribution, the practical performance of the newly introduced MOGAP-Wei distribution is compared with the competing probability distribution by applying the particular analytical measures (AMs). These AM measures are generally classified into two categories: (i) GFMs (goodness-of-fit measures) and (ii) DMs (discrimination measures). In addition to the GFMs and DMs, we further considered the P-values of the fitted probability distributions. The GFMs consist of three test statistics and are calculated as follows

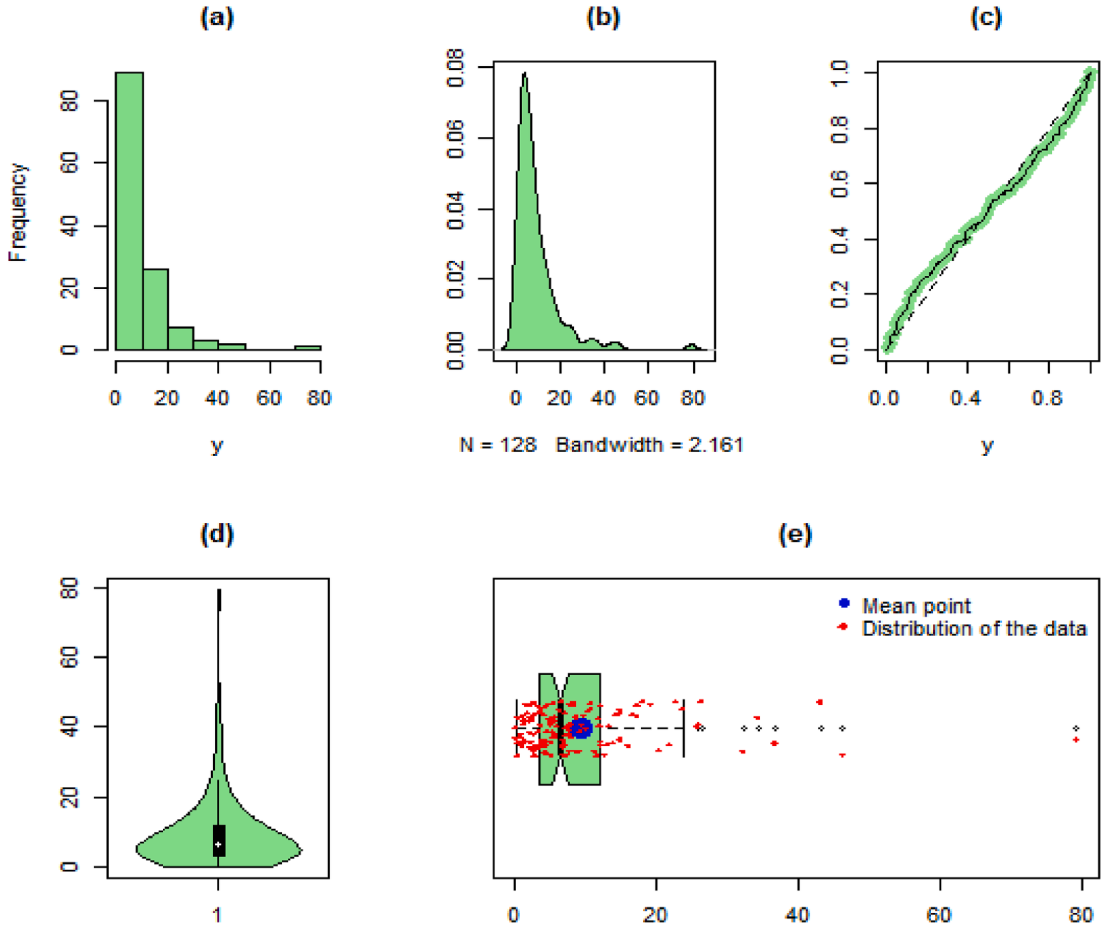

Fig. 9. The plots (a) His-P, (b) KDP, (c) TTP, (d) Vio-P, and (e) BP for Data 3.

- The Cramer-von Mises (onward, this test is denoted byΔ_1)

$$\Delta_1 = \sum_{i=1}^{n} \left[K(y_i) - \frac{2i-1}{2n}\right]^2 + \frac{1}{12n},$$

- The Anderson Darling (onward, this test is denoted byΔ_2)

$$\Delta_2 = -n - \frac{1}{n}\sum_{i=1}^{n} (2i-1) \times [\log K(y_i) + \log(1 - K(y_{n-i+1}))],$$

- The Kolmogorov–Smirnov (onward, this test is denoted byΔ_3)

$$\Delta_3 = sup[K_n(y) - K(y)].$$

Similarly, the DMs consist of four information criteria (IC) and are calculated as follows

- The Akaike IC (onward, this statistical measure is denoted by $\aleph_1$)

$$\aleph_1 = 2p - 2L(.),$$

- The Bayesian IC (onward, this statistical measure is denoted by $\aleph_2$)

$$\aleph_2 = p\log(n) - 2L(.),$$

Table 4
Using Data 1, MLEs along with SEs of the fitted distributions.

Distributions	Parameter (SE)	Parameter (SE)	Parameter (SE)	Parameter (SE)
MOGAP-Wei(ω,ϑ)	0.9997212 (1.0090285)	29.8568438 (31.7219552)	2.7858004 (0.1463425)	-
APT-Wei(ω,ϑ)	10.7914822 (12.6496574)	0.1943259 (0.1080013)	4.4863965 (0.7638190)	-
Wei(ϑ)	0.0597881 (0.0204959)	5.7796203 (0.5747253)	-	-
Ex-Wei(δ,ϑ)	0.6813285 (0.2308605)	0.0204663 (0.0227667)	7.2123305 (1.5219509)	-
Ku-Wei(a,b,ϑ)	0.5097832 (0.15432686)	0.2237844 (0.02839326)	0.1110254 (0.0023467)	7.1056630 (0.0035313)
Mod-Wei(ρ,ϑ)	0.0403457 (0.02469058)	6.37685845 (0.96387271)	0.0305466 (0.0434649)	-

Table 5
Using Data 2, MLEs along with SEs of the fitted distributions.

Distributions	Parameter (SE)	Parameter (SE)	Parameter (SE)	Parameter (SE)
MOGAP-Wei(ω,ϑ)	0.0321840 (0.0523533)	0.5426725 (0.3929253)	0.8515420 (0.0965029)	-
APT-Wei(ω,ϑ)	0.0214569 (0.0563683)	0.0940096 (0.0630552)	1.5904094 (0.1468908)	-
Wei(ϑ)	0.3663843 (0.0619753)	1.3256096 (0.1138158)	–	-
Ex-Wei(δ,ϑ)	1.4426694 (0.6436290)	0.5796388 (0.3002974)	1.1012912 (0.2628943)	-
Ku-Wei(a,b,ϑ)	1.7817038 (2.2229502)	3.7545724 (23.6198590)	0.3047426 (0.4161774)	0.8893005 (0.8440869)
Mod-Wei(ρ,ϑ)	0.4314375 (0.4676771)	1.2827576 (0.2912969)	0.0631934 (0.4461247)	-

Table 6
Using Data 3, MLEs along with SEs of the fitted distributions.

Distributions	Parameter (SE)	Parameter (SE)	Parameter (SE)	Parameter (SE)
MOGAP-Wei(ω,ϑ)	0.0030619 (0.0028033)	1.3112246 (0.4438979)	0.4947303 (0.0262517)	-
APT-Wei(ω,ϑ)	87.7207132 (80.8324366)	0.5822861 (0.1425726)	0.6224972 (0.0702306)	-
Wei(ϑ)	0.0939429 (0.0190786)	1.0476338 (0.0675428)	-	-
Ex-Wei(δ,ϑ)	2.7963064 (1.2632260)	0.4537522 (0.2398209)	0.6543754 (0.1346215)	-
Ku-Wei(a,b,ϑ)	1.8555199 (4.5034039)	0.2442317 (5.9077165)	0.6104610 (0.4212818)	0.9270508 (0.4210172)
Mod-Wei(ρ,ϑ)	1.205474 (4.2501953)	1.005854 (0.0313432)	1.11746556 (4.2551516)	-

- The corrected Akaike IC (onward, this statistical measure is denoted by $\aleph_3$)

$$\aleph_3 = \frac{2np}{n-p-1} - 2L(.),$$

- The Hannan Quinn IC (onward, this statistical measure is denoted by $\aleph_4$)

$$\aleph_4 = 2p\log(\log(n)) - 2L(.).$$

Although the MOGAP-Wei distribution contains more parameters than the classical Weibull probability distribution, its performance should be evaluated in the context of model complexity. The GFMs, DMs, and the P-values account for this complexity by penalizing additional parameters. Therefore, the favorable values obtained by the MOGAP-Wei distribution indicate that its improved fit is not solely due to its greater flexibility, but reflects an appropriate balance between goodness of fit and model complexity.

In the above equations, n denotes the sample size (i.e., the size of the data), p signifies the number of corresponding model parameters, and $L(.)$signifies the log-likelihood function, and it is assessed by the applied estimation method. All numerical computations and real-data analyses were conducted using the R statistical software using the "BFGS" system and the R package (AdequacyModel). Generally, a model with smaller values of the GFMs and DMs, along with their larger P-values, indicates a better empirical fit to the considered data. Based on these AMs, the MOGAP-Wei distribution demonstrates the lowest values of the considered evaluation criteria and the highest P-value compared with the competing probability distributions.

Analysis of the considered data sets

In relation to Data 1, 2, and 3, the estimated parameter values with their corresponding standard errors (SEs) in parentheses of the MOGAP-Wei probability and other competing distributions are recorded in Tables 4, 5, and 6. The popular MLE technique is used to get these estimated values. The profile log-likelihood plots of the MLEs (i.e., $\widehat{\omega}$, $\widehat{\alpha}$, and $\widehat{\theta}$) of the MOGAP-Wei probability model are illustrated in Figs. 10, 11, and 12. From these Figures, we can see that the associated estimated parameters of the MOGAP-Wei

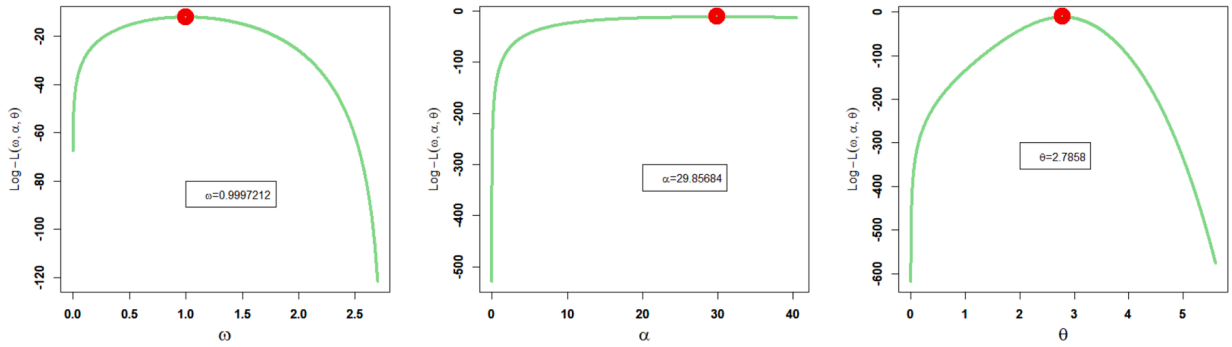

Fig. 10. The Profile plots for $\widehat{\omega}$, $\widehat{\alpha}$ and $\widehat{\theta}$of the MOGAP-Wei model for Data 1.

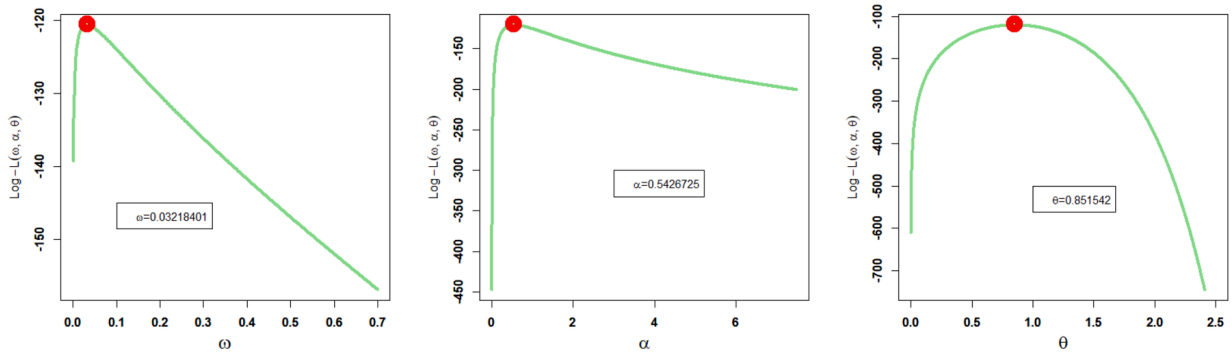

Fig. 11. The Profile plots for $\widehat{\omega}$, $\widehat{\alpha}$ and $\widehat{\theta}$of the MOGAP-Wei model for Data 2.

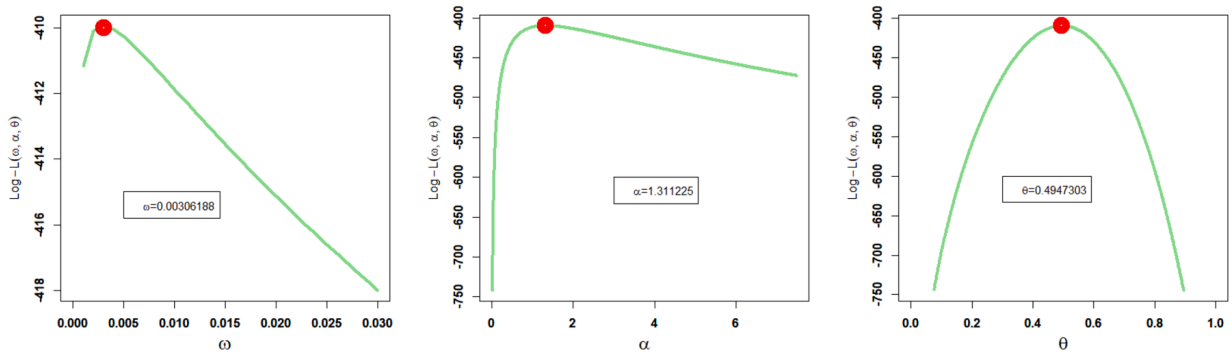

Fig. 12. The Profile plots for $\widehat{\omega}$, $\widehat{\alpha}$ and $\widehat{\theta}$of the MOGAP-Wei model for Data 3.

Table 7
The numerical illustrations of the fitted distributions for Data 1.

Distributions	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	Δ_1	Δ_2	Δ_3	P-values
MOGAP-Wei	30.2776	36.7070	30.6844	32.8063	0.108154	0.603074	0.09935	0.5628
APT-Wei	32.9482	39.3776	33.3550	35.477	0.168637	0.927440	0.12256	0.3003
Wei	34.4136	38.6999	34.6136	36.0995	0.237268	1.303867	0.15232	0.1075
Ex-Wei	35.3528	41.7822	35.7596	37.8816	0.201030	1.116805	0.14701	0.1313
Ku-Wei	35.5131	44.0856	36.2027	38.8847	0.149198	0.847826	0.13379	0.2094
Mod-Wei	35.7896	42.2184	36.1967	38.3181	0.13819	0.79702	0.13311	0.2142

probability distribution have unique roots and maximize the likelihood function. Furthermore, the DMs (i.e.,$\aleph_1$, $\aleph_2$, $\aleph_3$, and $\aleph_4$), GFMs (i.e., Δ_1, Δ_2, and Δ_3), and P-values of the MOGAP-Wei and other competing models (or distributions) for numerical illustration purposes are recorded in Tables 7, 8, and 9. In addition, the visual representations of the fitted PDFs and fitted CDFs of the MOGAP-Wei and other competing distributions are visualized in Figs. 13, 14, and 15. From Tables 7, 8, and 9, it is observed that the MOGAP-Wei

Table 8
The numerical illustrations of the fitted distributions for Data 2.

Distributions	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	Δ_1	Δ_2	Δ_3	P-values
MOGAP-Wei	247.2712	254.2634	247.6045	250.0656	0.07381	0.42182	0.08271	0.6453
APT-Wei	248.7289	255.7211	249.0622	251.5233	0.09010	0.53750	0.0862	0.6329
Wei	249.0494	256.7108	249.2138	250.9123	0.13059	0.76724	0.10993	0.2953
Ex-Wei	250.3272	257.3194	250.6606	253.1216	0.11674	0.69124	0.09875	0.4217
Ku-Wei	252.1412	261.4642	252.7046	255.8671	0.11372	0.67373	0.09729	0.4403
Mod-Wei	251.0494	258.0416	251.3827	253.8438	0.13060	0.109930.76724	0.2953	

Table 9
The numerical illustrations of the fitted distributions for Data 3.

Distributions	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	Δ_1	Δ_2	Δ_3	P-values
MOGAP-Wei	825.8836	834.4397	826.0772	829.360	0.02671	0.17237	0.03890	0.9902
APT-Wei	827.5233	836.0794	827.7168	830.999	0.05275	0.31456	0.04969	0.9110
Wei	832.1738	837.8778	832.2698	834.491	0.13132	0.78619	0.06996	0.5580
Ex-Wei	827.3602	835.9163	827.5538	830.836	0.04367	0.28847	0.04502	0.9576
Ku-Wei	831.4379	842.846	831.7631	836.073	0.07166	0.45133	0.04956	0.9116
Mod-Wei	834.1738	842.7298	834.3673	837.6501	0.13137	0.78648	0.07002	0.5570

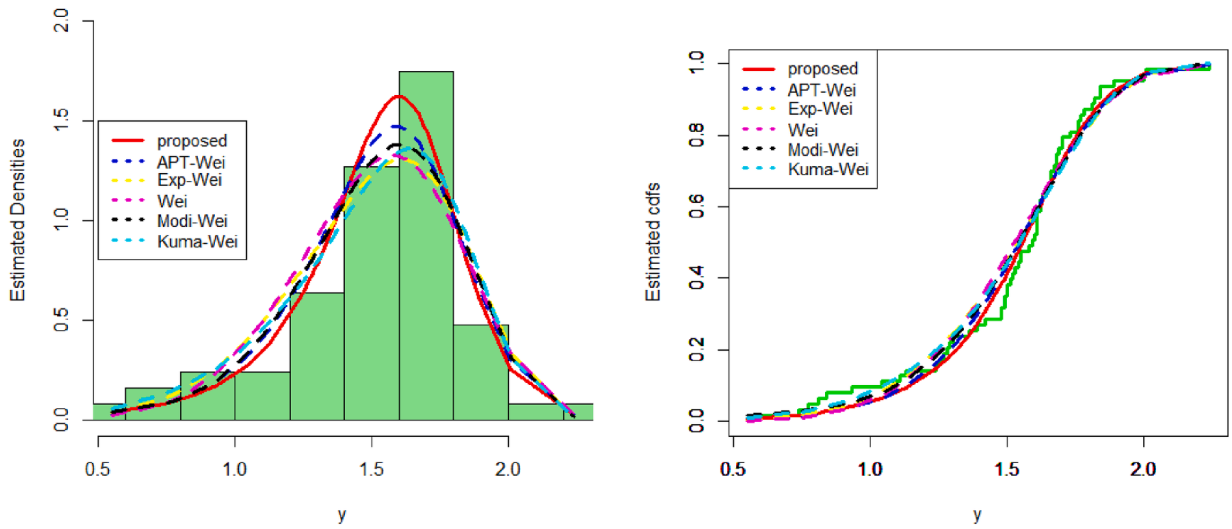

Fig. 13. The fitted PDFs and fitted CDFs plots of the fitted distributions for Data 1.

probability distribution has the lowest DMs and GFMs values, and the highest P-value as compared to the APT-Wei, Exp-Wei, Wei, Modi-Wei, and Kuma-Wei distributions. Similarly, from Figs. 13, 14, and 15, we can also see that the MOGAP-Wei distribution provides an extremely close fit to each considered dataset as compared to the other considered competing probability distributions. Based on the above comprehensive analysis and discussion, the newly proposed MOGAP-Wei probability distribution demonstrates superior performance in modeling all the considered datasets and can therefore be regarded as the most suitable candidate among the considered competing distributions.

Conclusions

In this study, we introduced the Marshall–Olkin Gull Alpha Power (MOGAP) family of probability distributions and developed its corresponding Weibull-based sub-model, referred to as the Marshall–Olkin Gull Alpha Power Weibull (MOGAP-Wei) distribution, by considering the classical Weibull distribution as the baseline model. The PDF of the MOGAP-Wei distribution is investigated with different combinations of the parameter values, showing that it can accommodate positively skewed, negatively skewed, and symmetric shapes. Moreover, the HF is also found to exhibit a wide range of shapes, including increasing, decreasing, constant, non-monotonic, unimodal, and bathtub-shaped patterns. From a theoretical perspective, several important distributional properties of the proposed family, such as the quantile function, moments, moment generating function, order statistics, residual life, and reverse residual life distributions, are investigated. The well-known estimation method, such as MLE, is applied to estimate the associated parameters of the proposed MOGAP-Wei distribution. The performance of the MLE method for the proposed distribution is further

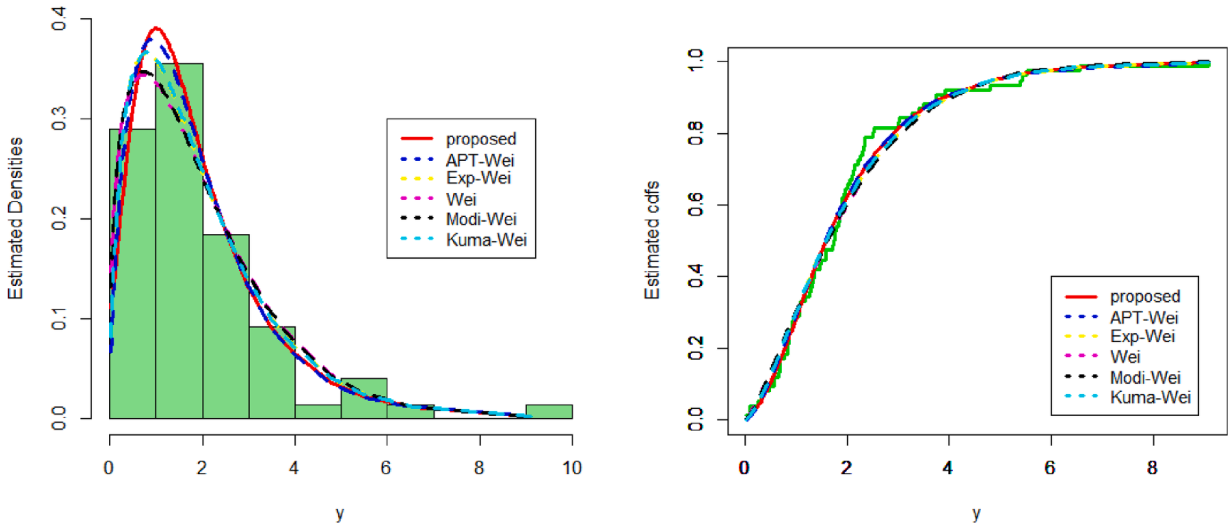

Fig. 14. The fitted PDFs and fitted CDFs plots of the fitted distributions for Data 2.

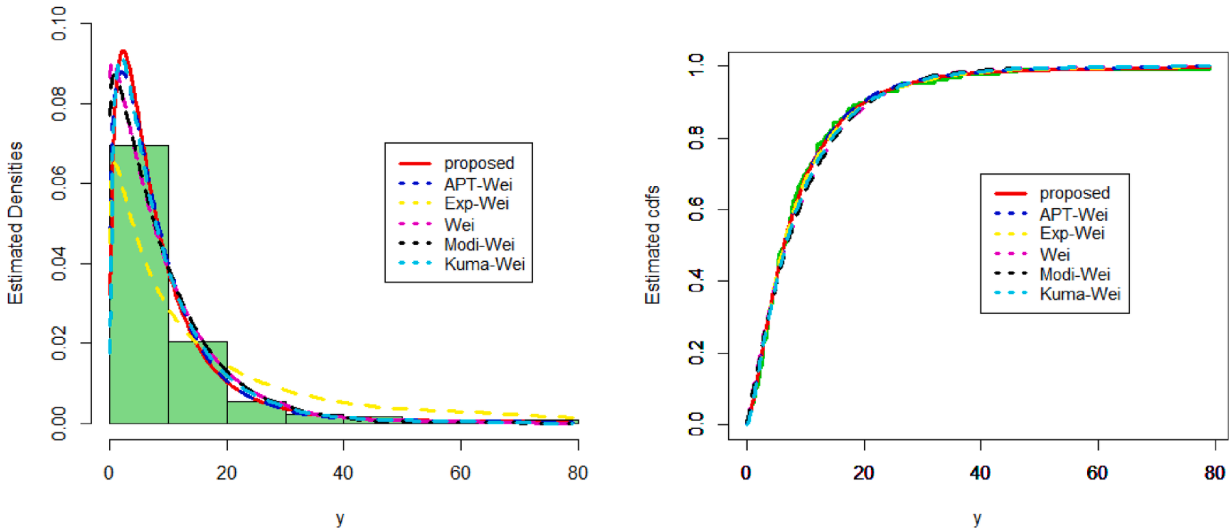

Fig. 15. The PDFs and CDFs plots of the fitted distributions for Data 3.

investigated through a simulation study under different parameter values and sample sizes. The simulation results indicated that the parameter estimates generally approached their corresponding true values as the sample size increased, while the associated biases and mean squared errors (MSEs) tended to decrease and approach zero. These findings provide numerical and graphical evidence regarding the behavior of the MLEs for the proposed distribution. Finally, the practical applicability and superiority of the MOGAP-Wei distribution are examined by using three real-world datasets from the field of engineering and the biomedical sectors. The empirical illustration demonstrated that the proposed model provided a good fit to the considered datasets in comparison to the competing probability distribution. Despite its flexibility and promising empirical performance, the newly proposed model has some limitations that should be acknowledged. The inclusion of additional parameters may increase the complexity of parameter estimation and make the interpretation of individual parameters more challenging. Moreover, numerical optimization for the MLEs may be more computationally demanding than that required for simpler probability models. These considerations provide useful directions for future research while not diminishing the flexibility and practical contribution of the proposed model.

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Data availability statement

The data supporting this study's findings are available within the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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