International Journal of Modern Sciences and Engineering Technology (IJMSET)
ISSN 2349-3755; Available at https://www.ijmset.com
Volume 4, Issue 7, 2017, pp.11-20

Dynamics of a Non-Autonomous Difference Equation

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Abstract

In this paper we investigate the boundedness, the periodicity character and the global behavior of the positive solutions of the difference equation

$$x_{n+1} = a_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ...,$$

where $\{a_n\}$ is a sequence of nonnegative real numbers and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers.

Key words and phrases. Boundedness character, dynamics, periodic solution, global stability.

1.INTRODUCTION:

The subject of difference equations is a very important topic in the real life where they have been applied in several mathematical models in biology, economics, genetics, population dynamics, ets. See for example [2,5-11]. For this reason, there exists a great increasing interest in the study of the qualitative properties of the solutions of the difference equations. See [1,3,4,13-16].

Our aim in this paper is to study the boundedness character and the global asymptotic behavior of the positive solutions of the difference equation

$$x_{n+1} = a_n + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ..., \tag{1.1}$$

where $\{a_n\}$ is a sequence of nonnegative real numbers and the initial condition x_{-1}, x_0 are arbitrary positive real numbers.

Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for every set of initial condition $x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Theorem A [12]: A (linearized stability).

The linearized equation of Eq.(1.2) is

$$y_{n+1} = p_1 y_n + p_2 y_{n-1}. \tag{1.3}$$

(a) If both roots of the quadratic equation

$$\lambda^2 - p_1\lambda - p_2 = 0, \quad (1.4)$$

lie in the open unit disk, $|\lambda| < 1$, then the equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable.

(b) If at least of the roots of Eq.(1.3) has absolute value greater that one , then the equilibrium $\overline{x}$ of Eq.(1.2) is unstable.

(c) A necessary and sufficient condition for both roots of Eq.(1.3) to lie in the open unit disk $|\lambda| < 1$, is

$$|p_1| < 1 - p_2 < 2.$$

In this case the locally asymptotically stable equilibrium $\overline{x}$ is also called a sink.

(d) A necessary and sufficient condition for one root of Eq.(1.3) to have absolute value great than one and for the other to have absolute values less than one is

$$|p_1| > |1 - p_2| \text{ and } p_1^2 + 4p_2 > 1.$$

In this case $\overline{x}$ is called a saddle point.

Theorem B [12]: Let $[a, b]$ be an interval of real numbers and assume that

$$f : [a, b]^2 \longrightarrow [a, b]$$

is a continuous function satisfying the following properties.

(a) $f(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$.

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$m = f(m, M), \quad M = f(M, m),$$

then $m = M$. Then Eq.(1.2) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(1.2) converges to $\overline{x}$.

2.MATERIALS AND METHODS:

1- Some specialized scientific books.
2 - Some periodicals and specialized scientific journals.
3 - Some programs ready for mathematical calculations such as Maple, Mathematica, Matlab.
4- High-capacity computers connected to the web.
5. Printer.
6- Communicating with the different scientific centers

3.RESULTS AND DISCUSSION:

2. Permanence of Eq.(1.1)

In this section we investigate the boundedness of Eq.(1.1).

Theorem 1. *Suppose that* $\lim_{n\to\infty} a_n = a \geq 1$, *then every positive solution of Eq.(1.1) is bounded and persists.*

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1.1). Then

$$x_n \geq a > 1 \text{ for all } n \geq 1.$$

Let $\epsilon \in (0, a-1)$, we see from Eq.(1.1) that

$$x_n \geq a - \epsilon \text{ for all } n \geq -1.$$

ISSN 2349-3755; Available at https://www.ijmset.com
Volume 4, Issue 7, 2017, pp.11-20

Let $\epsilon \in (0, a-1)$, we see from Eq.(1.1) that

$$x_n \geq a - \epsilon \quad \text{for all } n \geq -1.$$

Then we can find $L \in (a+\epsilon, a+\epsilon+1)$ such that

$$L - \epsilon \leq x_{-1}, x_0 \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Since $a > 1$, then we get

$$a \leq \frac{L-\epsilon-1}{L-\epsilon-a}. \tag{2.1}$$

Set

$$f(u,v) = a + \frac{u}{v}.$$

Then

$$f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = a + \frac{L-\epsilon}{\frac{L-\epsilon}{L-a-\epsilon}} = L - \epsilon,$$

and

$$f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) = a + \frac{\frac{L-a-\epsilon}{L-\epsilon}}{L-\epsilon} \leq a + \frac{1}{L-a-\epsilon} \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Now it follows from Eq.(1.1) that

$$x_1 = f(x_0, x_{-1}) \leq f(\frac{L-\epsilon}{L-a-\epsilon}, L-\epsilon) \leq \frac{L-\epsilon}{L-a-\epsilon}.$$

Again we see from Eq.(1.1) that

$$x_1 = f(x_0, x_{-1}) \geq f(L-\epsilon, \frac{L-\epsilon}{L-a-\epsilon}) = L - \epsilon.$$

By induction we obtain that

$$L - \epsilon \leq x_n \leq \frac{L-\epsilon}{L-a-\epsilon} \quad \text{for all} \quad n = -1, 0, 1, \ldots.$$

Second assume that $a = 1$ and let $\epsilon \in (0, \delta)$ and $\delta \in (0,1)$, it follows from Eq.(1.1) that

$$x_n \geq 1 - \epsilon + \delta \quad \text{for } n \geq 1.$$

Then one can find $L \in (1+\epsilon+\delta, 2+\epsilon+\delta)$ such that

$$L - \epsilon + \delta \leq x_{-1}, x_0 \leq \frac{L-\epsilon+\delta}{L-\epsilon-1+\delta}.$$

Therefore the rest of the proof is similar to the above and it is omitted. □

3. Global Attractity of Eq.(1.1)

In this section we show that if $a > 1$, then every positive solution of Eq.(1.1) converges to $(a+1)$.

Theorem 2. *Assume that $a > 1$. Then every positive solution of Eq.(1.1) converges to the unique positive equilibrium point $\overline{x} = (a+1)$ of Eq.(1.1).*

Proof. Note that when $a > 1$, it was shown in Theorem 1 that every positive solution of Eq.(1.1) is bounded. Then we have the following

$$s = \lim_{n\to\infty} \inf x_n \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_n.$$

It is clear that $s \le S$. We want to proof that $s \ge S$. Now it is easy to see from Eq.(1.1) that

$$s \ge a + \frac{s}{S} \quad \text{and} \quad S \le a + \frac{S}{s}.$$

Thus we have

$$sS \ge aS + s \quad \text{and} \quad sS \le as + S.$$

This implies that

$$aS + s \ \le as + S.$$

Then we get

$$a(S - s) \le (S - s),$$

or

$$(a-1)(S-s) \le 0 \Leftrightarrow s \ge S.$$

Thus the proof is complete. □

In order to confirm the obtained results of this section we consider the following numerical examples:

Example 1. *Figure 1 shows the global attractivity of the equilibrium point $\overline{x} = 2$ of Eq.(1.1) whenever $x_{-1} = 1.21, x_0 = 1.32$, and $a = 1$.*

Example 2. *Figure 2 shows the global attractivity of the equilibrium point $\overline{x} = 6$ of E.q(1.1) whenever $x_{-1} = 4, x_0 = 9$, and $a = 5$.*

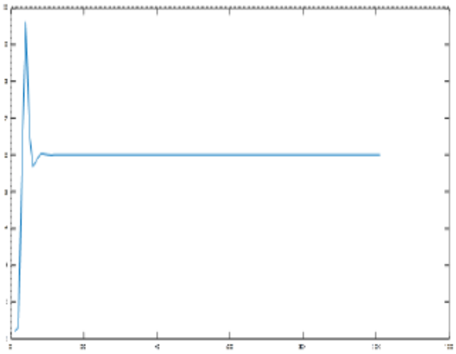

Figure (1)

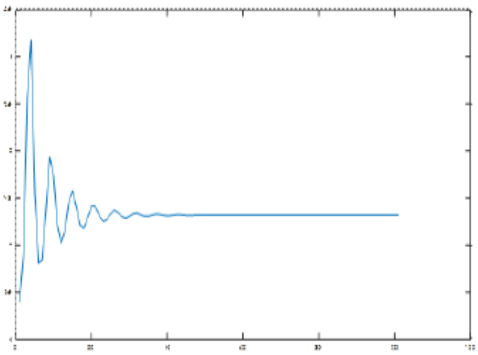

Figure (2)

4. When a_n is periodic

In this section we investigate the periodicity character of the positive solutions of Eq.(1.1) whenever $\{a_n\}$ is a periodic sequence of period two of the form $\{\alpha, \beta, \alpha, \beta, ...\}, \alpha \ne \beta$. Assume that $a_{2n} = \alpha$ and $a_{2n+1} = \beta$. Then we have

(4.1) $$x_{2n+1} = \alpha + \frac{x_{2n}}{x_{2n-1}}, \quad n = 0, 1, ...,$$

and

(4.2) $$x_{2n+2} = \beta + \frac{x_{2n+1}}{x_{2n}}, \quad n = 0, 1, ...$$

International Journal of Modern Sciences and Engineering Technology (IJMSET)
ISSN 2349-3755; Available at https://www.ijmset.com
Volume 4, Issue 7, 2017, pp.11-20

Theorem 3. *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then Eq.(1.1) has periodic solutions of prime period two.*

Proof. To prove that Eq.(1.1) possess a periodic solution $\{x_n\}$ of prime period two, we must find positive numbers x_{-1}, x_0 such that

(4.3) $$x_{-1} = x_1 = \alpha + \frac{x_0}{x_{-1}}, \quad \text{and} \quad x_0 = x_2 = \beta + \frac{x_{-1}}{x_0}.$$

Let $x_{-1} = x$, $x_0 = y$, then we obtain from (4.3)

(4.4) $$x = \alpha + \frac{y}{x}, \quad \text{and} \quad y = \beta + \frac{x}{y}.$$

Now we want to prove that (4.4) has a solution (x^*, y^*), $x^* > 0$, $y^* > 0$. From the first relation of (4.4) we have

(4.5) $$y = (x - \alpha)x.$$

From (4.5) and the second relation of (4.4) we get

$$x(x-\alpha) = \beta + \frac{x}{x(x-\alpha)},$$

or

$$x(x-\alpha)^2 - \beta(x-\alpha) - 1 = 0.$$

Now define the function

(4.6) $$f(x) = x(x-\alpha)^2 - \beta(x-\alpha) - 1, \quad x > \alpha.$$

Then

$$\lim_{x \to \alpha^+} f(x) = -1, \quad \lim_{x \to \infty} f(x) = \infty.$$

Hence Eq.(4.6) has at least one solution $x^* > \alpha$. Then if $y^* = (x^* - \alpha)x^*$, we have that the solution $\{x_n\}_{n=-1}^{\infty}$ is periodic of prime period two. □

Theorem 4. *Assume that* $\{x_n\}_{n=-1}^{\infty}$ *be a periodic solution of period two of Eq.(1.1) and consider Eq.(1.1) when the case* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$ *with* $\alpha \neq \beta$. *Suppose that*

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x^*},$$

where x^* *is defined in Theorem 3. Then* $\{x_n\}_{n=-1}^{\infty}$ *is locally asymptotically stable.*

Proof. It was shown in Theorem 3 that there exist x^*, y^* such that

(4.7) $$x^* = \alpha + \frac{y^*}{x^*}, \quad y^* = \beta + \frac{x^*}{y^*}.$$

Now Eq.(1.1) can be rewritten in the following form by splitting the even-indexed and odd-indexed terms:

(4.8) $$u_{n+1} = \alpha + \frac{v_n}{u_n}, \quad \text{and} \quad v_{n+1} = \beta + \frac{\alpha u_n + v_n}{u_n v_n}.$$

Now, we consider the map T on $[0, \infty) \times [0, \infty)$ such that

$$T(u,v) = \begin{bmatrix} T_1(u,v) \\ T_2(u,v) \end{bmatrix} = \begin{bmatrix} \alpha + \frac{v}{u} \\ \beta + \frac{\alpha u + v}{uv} \end{bmatrix}.$$

Then we have

$$\frac{\partial T_1}{\partial u} = \frac{-v}{u^2}, \quad \text{and} \quad \frac{\partial T_1}{\partial v} = \frac{1}{u},$$

$$\frac{\partial T_2}{\partial u} = \frac{-v^2}{v^2 u^2}, \quad \text{and} \quad \frac{\partial T_2}{\partial v} = \frac{-\alpha u^2}{u^2 v^2},$$

Therefore the Jacobian matrix of T at (x^*, y^*) is

$$J_T(x^*, y^*) = \begin{bmatrix} \frac{-y^*}{x^{*2}} & \frac{1}{x^*} \\ \frac{-1}{x^{*2}} & \frac{-\alpha}{y^{*2}} \end{bmatrix},$$

and its characteristic equation associated with (x^*, y^*) is

$$\lambda^2 + \lambda\left(\frac{\alpha}{y^{*2}} + \frac{y^*}{x^{*2}}\right) + \frac{\alpha}{x^{*2}y^*} + \frac{1}{x^{*3}} = 0. \tag{4.9}$$

It follows from (4.7) that $\frac{y^*}{x^{*2}} = 1 - \frac{\alpha}{x^*}$ and since $x^* > \alpha, \quad y^* > \beta$ we have

$$\frac{\alpha}{y^{*2}} + \frac{y^*}{x^{*2}} + \frac{\alpha}{x^{*2}y^*} + \frac{1}{x^{*3}} < \frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} + 1 - \frac{\alpha}{x^*} < 1.$$

Thus

$$\frac{\alpha}{\beta^2} + \frac{1}{\alpha\beta} + \frac{1}{\alpha^3} < \frac{\alpha}{x^*} < 1.$$

Then all roots of Eq.(4.9) have modulus less than 1. Therefore by Theorem A, System (4.8) is asymptotically stable. The proof is complete. □

Theorem 5. *Assume that* $\{a_n\} = \{\alpha, \beta, \alpha, \beta, ...\}$, *with* $\alpha \neq \beta$. *Then every solution of Eq.(1.1) converges to a period two solution of Eq.(1.1).*

Proof. We know by Theorem 1 that every positive solution of Eq.(1.1) is bounded, it follows that there are some positive constants l, L, s and S such that

$$l = \lim_{n\to\infty} \inf x_{2n+1} \quad \text{and} \quad L = \lim_{n\to\infty} \sup x_{2n+1},$$

$$s = \lim_{n\to\infty} \inf x_{2n} \quad \text{and} \quad S = \lim_{n\to\infty} \sup x_{2n}.$$

Then it is easy to see from Eq.(4.1) and Eq.(4.2) that

$$l \geq \alpha + \frac{s}{L} \quad \text{and} \quad L \leq \alpha + \frac{S}{l},$$

and

$$s \geq \beta + \frac{l}{S} \quad \text{and} \quad S \leq \beta + \frac{L}{s}.$$

Then we obtain

$$Ll \geq \alpha L + s \quad \text{and} \quad Ll \leq \alpha l + S,$$

and

$$Ss \geq \beta S + l \quad \text{and} \quad Ss \leq \beta s + L.$$

Thus we get

$$\alpha L + s \leq Ll \leq \alpha l + S \quad \text{and} \quad \beta S + l \leq Ss \leq \beta s + L.$$

ISSN 2349-3755; Available at https://www.ijmset.com
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Thus it is clear from (4.10) that $s = S$ and $l = L$. Now assume that $\lim_{n\to\infty} x_{2n+1} = S$ and $\lim_{n\to\infty} x_{2n} = L$. We want to proof that $S \neq L$. From Eq.(4.1) and Eq.(4.2) we get

$$S = \alpha + \frac{L}{S} \quad \text{and} \quad L = \beta + \frac{S}{L}.$$

As thae sake of contraduction assume that $L = S$, then

$$L = \alpha + 1 \quad \text{and} \quad S = \beta + 1$$

thus $\alpha = \beta$ which is a contraduction.
So $\lim_{n\to\infty} x_{2n+1} \neq \lim_{n\to\infty} x_{2n}$. The proof is so complete. □

Example 3. *Figure 3 shows that the solution of Eq.(1.1) is periodic solution of period two when* $x_{-1} = 2.3, x_0 = 1.3$ *and* $\alpha = .73827543, \beta = .6763772$.

Example 4. *Figure 4 shows that the solution of Eq.(1.1) is periodic solution of period two when* $x_{-1} = 15.30,\ x_0 = 10.30$ *and* $\alpha = 6, \beta = 1$.

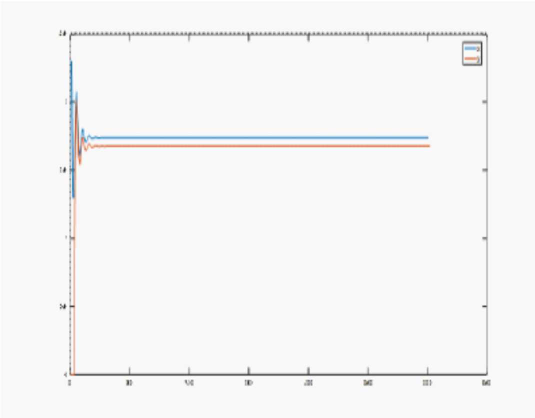

Figure (3)

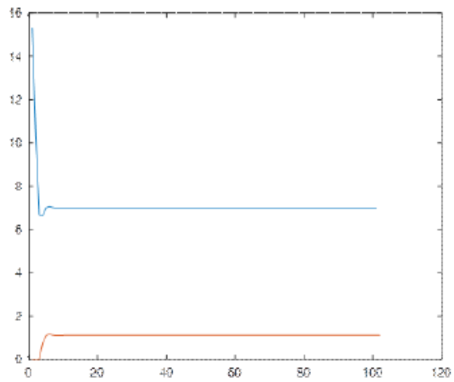

Figure (4)

5. The autonomous case of Eq.(1.1)

Consider Eq.(1.1) with $a_n = a$ where $a \in (0, \infty)$ then Eq.(1.1) has the form

$$x_{n+1} = a + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, ..., \tag{5.1}$$

where the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Clearly, the only equilibrium point of Eq.(5.1) is $\overline{x} = a + 1$.

The linearized equation of Eq.(5.1) about the equilibrium point $\overline{x} = a + 1$ is

$$y_{n+1} - \frac{1}{a+1} y_n + \frac{1}{a+1} y_{n-1} = 0, \qquad n = 0, 1, ... \tag{5.2}$$

Lemma 1. *The following statements are true.*

1. *The equilibrium point* $\overline{x} = a + 1$ *of Eq.(5.1) is locally asymptotically stable if* $a > 1$.
2. *The equilibrium point* $\overline{x} = a + 1$ *of Eq.(5.1) is unstable if* $0 \le a \le 1$.

ISSN 2349-3755; Available at https://www.ijmset.com
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Proof. The proof is followed directly by Theorem A and so will be omitted. □

Theorem 6. *Suppose that $a > 1$, then every positive solution of Eq.(5.1) is bounded.*

Proof. It follows from Eq.(5.1) that

$$x_{2n+1} = a + \frac{x_{2n}}{x_{2n-1}} \quad \text{and} \quad x_{2n} = a + \frac{x_{2n-1}}{x_{2n-2}}.$$

Therefore

$$x_{2n-1} > a \quad \text{and} \quad x_{2n-2} > a, \quad \text{for every } n \geq 1.$$

Then

$$x_{2n+1} = a + \frac{x_{2n}}{x_{2n-1}} < a + \frac{x_{2n}}{a} \quad \text{and} \quad x_{2n} = a + \frac{x_{2n-1}}{x_{2n-2}} < a + \frac{x_{2n-1}}{a}$$

Then it follows by induction that

$$x_{2n+1} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_{-1}}{a^n} = a + \frac{a}{a-1} + \frac{x_{-1}}{a^n},$$

and

$$x_{2n} < a + (1 + \frac{1}{a} + \frac{1}{a^2} + ...) + \frac{x_0}{a^n} = a + \frac{a}{a-1} + \frac{x_0}{a^n}.$$

The result now follows. □

Theorem 7. *Assume that $a > 1$ then every solution of Eq.(5.1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(5.1), then

(5.3) $$x_{n+1} = a + \frac{x_n}{x_{n-1}} > a, \quad \text{for all } n \geq 1.$$

Again it follows from Eq.(5.1) that

$$x_{n+1} = a + \frac{x_n}{x_{n-1}} \leq a + \frac{x_n}{a}.$$

Then

(5.4) $$\limsup x_n \leq \frac{a}{1 - \frac{1}{a}} = \frac{a^2}{a-1}.$$

Then the result follows from (5.3) and (5.4). □

Theorem 8. *Assume that $a > 1$. Then the equilibrium point $\overline{x} = a + 1$ is a global attractor of Eq.(5.1).*

Proof. Let $f : [\alpha, \beta]^2 \longrightarrow [\alpha, \beta]$ be a function defined by

$$f(u, v) = a + \frac{u}{v}$$

Suppose that (m, M) is a solution of the system

$$m = f(m, M), \qquad M = f(M, m).$$

Then we get

$$(a-1)(M-m) = 0,$$

Since $a > 1$, then we obtain

$$M = m.$$

ISSN 2349-3755; Available at https://www.ijmset.com
Volume 4, Issue 7, 2017, pp.11-20

It follows by Theorem B that $\bar{x}$ is a global attractor of Eq.(5.1) and then the proof is complete. □

4. CONCLUSION:

In this paper, we have discussed some properties of the non-autonomous difference equation (1.1), namely the periodicity, the boundedness, and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions. Also, we have investigated the local stability, the boundedness, and the global attractor when the equation (1.1) is autonomous difference equation.

5. ACKNOWLEDGEMENTS:

The authors would like to thank the referee for careful reading and for his/her comments and suggestions, which greatly improved an earlier version of this paper.

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