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Sumudu Transform to Solve Fuzzy Impulsive Function

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Abstract:
This paper proposes a method for solving the fuzzy impulsive equation by using the Sumudu transform, an integral transform that can successfully handle distributions and special functions. This approach is shown by giving an example of a differential equation involving the fuzzy Dirac delta function, which has several applications in the real world, and solving this differential equation by applying the Sumudu transform. After that, the obtained solution from using the Sumudu transform is compared with the solution from using the Laplace transform in the same example.

Keywords: Fuzzy Impulsive, Fuzzy Dirac Delta Function, Sumudu Transform.

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تحويل سومودو لحل دالة الاندفاعية الضبابية

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الملخص
تقترح هذه الورقة طريقة لحل المعادلة الاندفاعية الضبابية باستخدام تحويل سومودو، وهو تحويل تكاملي يمكن أن ينجح في معالجة التوزيعات والدوال الخاصة. ويظهر هذا النهج من خلال إعطاء مثال لمعادلة تفاضلية تتضمن دالة دلتا ديراك الضبابية التي لها العديد من التطبيقات في العالم الحقيقي، وحل هذه المعادلة التفاضلية باستخدام تحويل سمودو، وبعد ذلك، يتم مقارنة الحل الذي تم الحصول عليه من استخدام تحويل سومودو مع الحل الناتج من استخدام تحويل لابلاس على نفس المثال .

الكلمات المفتاحية: اندفاعية ضبابية، دالة دلتا ديراك الضبابية، تحويل سمودو.

1. Introduction

The concept of the Kronecker delta function for the discrete case was generalized to the continuous case by Dirac [7,8]. This is known as the Dirac delta function.

The differential equations containing the fuzzy Dirac delta function widely used in economy, thermal physics, quantum mechanics, etc.

The fuzzy function can be defined in order to solve any fuzzy logical problem mathematically.

$$\tilde{\delta}_a(t-t_0) = \begin{cases} 0 & 0 \le t \le t_0 - a, \\ \frac{1}{2a} & t_0 - a \le t \le t_0 + a, \\ 0 & t_0 + a \le t, \end{cases} \quad (1)$$

The function $\tilde{\delta}_a(x-x_0)$ is named a unit fuzzy impulse since it has the integration property.

$$\int_0^{\infty} \tilde{\delta}(t-t_0)dt = \tilde{1} \quad (2)$$

where $\tilde{1}$ is about 1.

Working with a different form of unit fuzzy impulse that is determined by the limit is appropriate in practice

$$\tilde{\delta}(t-t_0) = \lim_{a \to 0} \tilde{\delta}(t-t_0) \quad (3)$$

The fuzzy Dirac delta function formulation $\tilde{\delta}(t-t_0)$, is useful for describing an instantaneous impulse at time $t = t_0$.

The formal assumption that used is the same approach in Kwun et al [9] but with the Sumudu transform or S-transform [3] of the fuzzy Dirac delta function which is:

$$S\{\tilde{\delta}(t-t_0)\} = \lim_{a \to 0} S\{\tilde{\delta}_a(t-t_0)\} \quad (4)$$

In this research, the S-transform of the fuzzy Dirac delta function and a fuzzy impulsive differential equation example are studied.

2. Preliminaries

Definition 2.1. [4] Let is a mapping $u: \mathbb{R} \to [0,1]$ called fuzzy number with the following criteria:

1. u is normal, that is there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.
2. u is convex, that is for all $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$

$$u(\alpha x + (1-\alpha)y) \ge \min\{u(x), u(y)\}$$

3. u is upper semi continuous, that is for any $x_0 \in \mathbb{R}$, $u(x_0) \ge \lim_{x \to x_0^{\pm}} u(x)$
4. Supp $u = \{x \in \mathbb{R}: u(x) > 0\}$ is the support of u, and its closure cl(supp u) is compact.

As the fuzzy number is resolved by the interval $[u]_\alpha$, researchers Friedman et al. and Ma et al. [5] defined another representation of fuzzy numbers parametrically as in the following definition.

Definition 2.2. A fuzzy number $u = [\underline{u}_\alpha, \overline{u}_\alpha]$ of functions $\underline{u}_\alpha$ and $\overline{u}_\alpha$ for any $\alpha \in [0,1]$, which satisfies the following requirements.

1. $\underline{u}_\alpha$ is a bounded non-decreasing left continuous function in $(0,1]$.
2. $\overline{u}_\alpha$ is a bounded non-increasing left continuous function in $(0,1]$.
3. $\underline{u}_\alpha \le \overline{u}_\alpha$

Definition 2.3. [1] Let $u \in E$. u is called triangular fuzzy number if its membership function has the following form:

$$u(x) = \begin{cases} 0, & if\ x < a, \\ \frac{x-a}{b-a}, & if\ a \le x < b \\ \frac{c-x}{c-b}, & if\ b \le x \le c \\ 0, & if\ x > c \end{cases} \quad (5)$$

And its α-level are simply : $[u]_\alpha = [a + (b-a)\alpha, c - (c-b)\alpha]$, for any $\alpha \in [0,1]$.

Definition 2.4. [3] Consider the function f in A is defined as

$$A = \{f(t) : \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{if } \ t \in (-1)^j \times [0, \infty)\} \quad (6)$$

The Sumudu transform defined by

$$S\{f(t)\} = G(u) = \int_0^\infty f(ut)e^{-t}dt, \qquad u \in (\tau_1, \tau_2) \qquad (7)$$

2.5. Linear property of Sumudu transform [2]
If S-transform of functions $f_1(t)$ and $f_2(t)$ are $G_1(u)$ and $G_2(u)$, $a, b \in \mathbb{R}$, then

$$S\{af_1(t) + bf_2(t)\} = aG_1(u) + bG_2(u) \qquad (8)$$

2.6. Sumudu transform of derivatives of the function $f(t)$ [2]

$$S\{f'(t)\} = \frac{S\{f(t)\}}{u} - \frac{f(0)}{u} \qquad (9)$$

$$S\{f''(t)\} = \frac{S\{f(t)\}}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u} \qquad (10)$$

3. Sumudu transform of the fuzzy Dirac delta function

Theorem 3.1. For $t_0 > 0$ and $a > 0$

$$\int_0^\infty \tilde{\delta}_a(t - t_0)dt = \tilde{1} \qquad (11)$$

$$S[\tilde{\delta}(t - t_0)] = \frac{1}{u}e^{\frac{-t}{u}}.\tilde{1} \qquad (12)$$

Proof:

Firstly, by the fuzzy impulse unit's definition we obtain

$$\left[\int_0^\infty \tilde{\delta}_a(t - t_0)dt\right]_\alpha = \int_{t_0-a}^{t_0+a} \left[\frac{1}{\tilde{2}a}\right]_\alpha dt$$

$$= \int_{t_0-a}^{t_0+a} \left[\frac{1}{a(3-\alpha)}, \frac{1}{a(\alpha+1)}\right] dt$$

$$= \left[\frac{2}{3-\alpha}, \frac{2}{\alpha+1}\right] = [\tilde{1}]_\alpha$$

Through utilizing resolution identity,

$$\int_0^\infty \tilde{\delta}_a(t - t_0)dt = \tilde{1}$$

Secondly, we can express $\tilde{\delta}_a(t - t_0)$ using the unit function by

$$[\tilde{\delta}_a(t - t_0)]_\alpha = \left[\frac{1}{\tilde{2}a}\left(v(t - (t_0 - a)) - v(t - (t_0 + a))\right)\right]_\alpha$$

$$= \begin{bmatrix} \frac{1}{a(3-\alpha)}\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right) \\ , \frac{1}{a(\alpha+1)}\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right) \end{bmatrix}$$

where:

$$v(t-b) = \begin{cases} 0 & t < b, \\ 1 & t \geq b. \end{cases}$$

From the S-transform 's properties we get:

$$\left[S(\tilde{\delta}(t-t_0))\right]_\alpha = \begin{bmatrix} S\left(\frac{1}{a(3-\alpha)}\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right)\right), \\ S\left(\frac{1}{a(\alpha+1)}\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a(3-\alpha)} S\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right), \\ \frac{1}{a(\alpha+1)} S\left(v\left(t-(t_0-a)\right)-v\left(t-(t_0+a)\right)\right) \end{bmatrix}$$

$$= \left[\frac{1}{a(3-\alpha)} e^{\frac{-t_0}{u}}\left(ue^{\frac{a}{u}}-ue^{\frac{-a}{u}}\right), \frac{1}{a(\alpha+1)} e^{\frac{-t_0}{u}}\left(ue^{\frac{a}{u}}-ue^{\frac{-a}{u}}\right)\right]$$

$$= \left[\frac{1}{a(3-\alpha)} e^{\frac{-t_0}{u}}\left(\frac{e^{\frac{a}{u}}-e^{\frac{-a}{u}}}{\frac{1}{u}}\right), \frac{1}{a(\alpha+1)} e^{\frac{-t_0}{u}}\left(\frac{e^{\frac{a}{u}}-e^{\frac{-a}{u}}}{\frac{1}{u}}\right)\right]$$

$$= e^{\frac{-t_0}{u}}\left(\frac{e^{\frac{a}{u}}-e^{\frac{-a}{u}}}{\frac{a}{u}}\right)\left[\frac{1}{(3-\alpha)}, \frac{1}{(\alpha+1)}\right].$$

By using L'Hopital's rule, we get

$$\left[S(\tilde{\delta}(t-t_0))\right]_\alpha = \lim_{a\to 0}\left[S(\tilde{\delta}_a(t-t_0))\right]_\alpha$$

$$= \lim_{a\to 0} e^{\frac{-t_0}{u}}\left(\frac{e^{\frac{a}{u}}-e^{\frac{-a}{u}}}{a}\right)\left[\frac{1}{(3-\alpha)}, \frac{1}{(\alpha+1)}\right]$$

$$= e^{\frac{-t_0}{u}} \frac{2}{u}\left[\frac{1}{(3-\alpha)}, \frac{1}{(\alpha+1)}\right]$$

$$= \frac{1}{u} e^{\frac{-t_0}{u}}\left[\frac{2}{(3-\alpha)}, \frac{2}{(\alpha+1)}\right]$$

$$= \frac{1}{u} e^{\frac{-t_0}{u}}\left[\tilde{1}\right]_\alpha$$

Through utilizing resolution identity,

$$= \frac{1}{u} e^{\frac{-t_0}{u}} . \tilde{1}.$$

Examples 3.1. Let the initial value problem

$$y'' + y = k\tilde{\delta}(t-2\pi) \ , \quad \alpha \in [0,1] \qquad (13)$$

$$y(0) = 1, \ \ y'(0) = 0$$

Where $k \in \mathbb{R}$ is positive constant.

The movement of a fuzzy mass on a spring through a medium with discarded damping could be described using these models.

One unit below the equilibrium point, the fuzzy mass is set free from rest, and at time $t = 2$ seconds, it receives a sharp hit.

Solution. From the S-transform of the differential equation (13) is

$$\frac{y(u)}{u^2} - \frac{1}{u^2} + y(u) = k.\frac{e^{\frac{-2\pi}{u}}}{u}.\tilde{1}.$$

$$y(u) = \frac{1}{1+u^2} + \frac{kue^{\frac{-2\pi}{u}}}{1+u^2}\tilde{1}. \qquad (14)$$

The solution is obtained by using the translation's inverse form.

$$y(t) = \cos t + k\sin(t - 2\pi)v(t - 2\pi)\,\tilde{1}. \qquad (15)$$

Let $$[y(t)]_\alpha = \begin{bmatrix} \frac{2}{3-\alpha}(\cos t + k\sin(t-2\pi)v(t-2\pi)), \\ \frac{2}{\alpha+1}(\cos t + k\sin(t-2\pi)v(t-2\pi)) \end{bmatrix} \qquad (16)$$

Let $T > 0$. Examine this set of solutions.

$$X_\alpha = \left\{ \begin{matrix} [y(t)]_\alpha : [y(t)]_\alpha \text{ satisfies eq. } (16) \\ \text{for } t \in [0, T] \text{ and } \alpha \in [0,1] \end{matrix} \right\}$$

Non empty is obvious so we could select $\alpha \in [0,1]$.

Let $[y(t)]_\alpha \in X_\alpha$, then $\exists\ \alpha \in [0,1]$ for which

$$|[y(t)]_\alpha| \leq \sqrt{k^2\left|\frac{2}{\alpha+1} - \frac{2}{3-\alpha}\right|} \leq \frac{4}{3}k. \qquad (17)$$

So X_α is bounded. Let $[y]_{\alpha_c} \in X_\alpha$, then there is $\alpha_c \in [0,1]$ such that $\alpha_c \to \alpha \in [0,1]$ and

$$\lim_{c\to\infty}[y]_{\alpha_c} = \lim_{c\to\infty}\begin{bmatrix} \frac{2}{3-\alpha_c}(\cos t + k\sin(t-2\pi)v(t-2\pi)), \\ \frac{2}{\alpha_c+1}(\cos t + k\sin(t-2\pi)v(t-2\pi)) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3-\alpha}(\cos t + k\sin(t-2\pi)v(t-2\pi)) \\ , \frac{2}{\alpha+1}(\cos t + k\sin(t-2\pi)v(t-2\pi)) \end{bmatrix}$$

$$= [y(t)]_\alpha$$

Therefore X_α is compact because of $[\tilde{1}]_\alpha = \left[\frac{2}{3-\alpha}, \frac{2}{\alpha+1}\right]$ is closed.

From the (15) the solution can be written as

$$y(t) = \begin{cases} \cos t & 0 \leq t < 2\pi \\ \cos t + k.\tilde{1}.\sin t & t \geq 2\pi \end{cases}$$

It can be observed from the solution $y(t)$ that the mass is hit at $t = 2\pi$ when it is moving simply. The fuzzy unit impulse has an impact by increasing the vibration's amplitude to $\sqrt{(k\tilde{1})^2 + 1}$ for $t > 2\pi$. These results are the same as obtained by Kwun et al [9].

Conclusion

This study was investigated the Sumudu transform for solving the fuzzy Dirac delta function. It was addressed an example of a differential equation containing a fuzzy Dirac delta function. The results show the feasibility of the proposed method, where the obtained solution the same as the one were using Laplace transform by Kwun et al [9].

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