University of Tripoli

Faculty of science
Department of mathematics

Some Inequalities for the Riemann–Stieltjes Integral

Submitted by:

Fadia Muftah Banur Hejaj

Supervised by:

Prof. Taufik Abdussalam Bulati

Submitted in partial fulfillment of the requirement for the degree of master of science in mathematics.

Tripoli – 2017

نموذج (17) أ

جامعة طرابلس

إدارة الدراسات العليا والتدريب

قرار لجنة المناقشة والحكم

الكلية العلوم **القسم** الرياضيات

قامت اللجنة التي شُكلت بناءً على قرار الأخ رئيس الجامعة رقم (486) لسنة 2017 ف من الإخوة:

1. أ.د. توفيق عبد السلام البولاطي مشرفاً أولاً، ومقرراً
2. ـــ ـــ ـــ ـــ مشرفاً ثانياً، عضواً
3. أ.د. سالم ابراهيم القوي ممتحناً داخلياً، عضواً
4. أ.د. ابراهيم عبدالله تنتوش ممتحناً خارجياً، عضواً

بمناقشة الرسالة المقدمة من الطالبه/ غادة مفتاح منصور حباج لنيل درجة الإجازة العالية (الماجستير)

في الرياضيات وعنوانها:

((بعض المتباينات لتكامل ريمان - استيلتجز))

على تمام الساعة 12 من يوم الخميس الموافق 13 / 4 / 2017 ف بمبنى كلية العلوم

وقررت ما يلي:-

القرار

بعد إتمام الطالب لمتطلبات درجة الإجازة العالية وبمناقشة وتقييم الرسالة العلمية المقدمة وحسب ما تنص عليه اللوائح تقرر:

☐ إجازة الرسالة بدون ملاحظات.

☑ إجازة الرسالة بملاحظات ويُمنح الطالب فترة لا تزيد عن ثلاثة أشهر لاستكمال الملاحظات.

☐ عدم إجازة الرسالة ويُمنح الطالب فرصة أخرى لمناقشتها في مدة أقصاها ثلاثة أشهر.

☐ رفض الرسالة.

أعضاء اللجنة	التوقيع
1. أ.د. توفيق عبدالسلام البولاطي	
2. ـــ ـــ	
3. أ.د. سالم ابراهيم القوي	
4. أ.د. ابراهيم عبدالله تنتوش	

منسق الدراسات العليا بالقسم

الاسم والتوقيع أ.د. منصور محمد الأسطى

رئيس القسم

الاسم والتوقيع أ.د. علي محمد صقر

رئيس قسم الرياضيات

مدير مكتب الدراسات العليا والتدريب بالكلية

الاسم والتوقيع أ.د. كمال البوالقاسم البوديه

عميد الكلية

الاسم والتوقيع أ.د. علي حسين العجيلي

د.سعيد امنة

جامعة طرابلس **نموذج (18) أ**

إدارة الدراسات العليا والتدريب

نموذج الاعتماد النهائي لرسالة الإجازة العالية (الماجستير)

كلية العلوم **قسم** الرياضيات

أنا أ.د. توفيق عبدالسلام البولاطم . عضو هيئة تدريس بقسم الرياضيات والمشرف على رسالة الطالب فاطمه مفتاح البشير حجاج التي نوقشت بتاريخ 13/4/2017تحت عنوان: بعض المتباينات لتكامل ريمان - استيلتجز

بعد مراجعة كل التصويبات والتعديلات والتغييرات والنواقص التي اقترحتها لجنة المناقشة أقر بأن الطالب التزم بتنفيذ كافة ما طلب منه تنفيذاً كاملاً.

توقيع الأستاذ المشرف:

التـــــاريخ : 19 / 4 / 2017 ف

أعضاء لجنة المناقشة:

الاسم: أ.د. توفيق عبدالسلام البولاطم التوقيع:

الاسم: التوقيع:

الاسم: أ.د. سالم ابراهيم القوى التوقيع:

الاسم: أ.د. ابراهيم عبدالله تنتوش التوقيع:

منسق الدراسات العليا بالقسم

الإسم: أ.د. منصور محمد الأسطى

التوقيع:

رئيس القســـــــم

الإسم:-: أ.د. علي محمد صقر

التوقيع: :

رئيس قسم الرياضيات

مدير مكتب الدراسات العليا والتدريب بالكلية

الإسم: أ.د. [illegible] أبوديه

التوقيع:

عميـــد الكليـــة

الاسم : أ.د. علي حسين العبيلي

التوقيع :

مدير إدارة الدراسات العليا والتدريب

الاسم :

التوقيع :

مدير الإدارة

✍د.سعيد🖳 امنة

﴿ بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ ﴾

﴿ يَرۡفَعِ ٱللَّهُ ٱلَّذِينَ ءَامَنُواْ مِنكُمۡ وَٱلَّذِينَ أُوتُواْ ٱلۡعِلۡمَ
دَرَجَٰتٖۚ وَٱللَّهُ بِمَا تَعۡمَلُونَ خَبِيرٞ ﴾

صدق الله العظيم

سورة المجادلة : الآية (11)

الإهداء

إن كان وراء كل إمراءة رجل عظيم فأبي أعظم الرجال

إلى روح أبي الطاهرة رحمه اللّه أهدي كل ثمار جهدي.

شكر وتقدير

الحمد لله على نعمة العلم والأدب، والشكر لله أن فتح أمامي أبواب العلم في أحلك الظروف، والشكر موصول لأستاذي د. توفيق البولاطي الذي تفضل بالإشراف على هذا العمل و منحني من وقته الكثير.

وشكري يخلوه الوفاء بدون ذكر أمي وأبي فلولا دعواتهم ما وفقني الله، ولا أنسى أن أشكر زوجي على اهتمامه وسعة صدره معي، وكل التقدير لمن رفعوا من همتي في ساعات الإحباط أخواتي العزيزات وكل من ساندني من أهلي وأهل زوجي.
وللحبيبة التي كانت انيسة درب الدراسة صديقتي أ. مبروكة الفضيل.

وختاماً الشكر لكل من علمني ولو حرف من نعومة أظافري إلى يومي هذا.

الباحثة✍

Contents

ABSTRACT

During the last seventh decades ago, there has been ongoing interest concering different types of inequalities integration.

Our aim in this project is to study the important type of these inequalities, from Riemann-Stieltjes integral, which are well known in the literature as the Ostrowski and trapezoid inequalities.

In this study we focused our attention on the results to find the Riemann-Stieltjes Integral of product integrators and here applied on some inequalities.

ملخص

خلال السبعة العقود الماضية كان هناك اهتمام مستمر بشأن أنواع مختلفة من المتباينات التكاملية.

والهدف الأساسي في هذا البحث هو دراسة اهم أنواع تلك المتباينات من خلال تكامل ريمان-اشتيلتجز، وهي ما يعرف بمتباينة أوسترسكي ومتباينة شبه المنحرف.

وأهم ما تطرقت له هذه الدراسة هو مبرهنة لإيجاد تكامل ريمان-اشتيلتجز
وتطبيقها على بعض المتباينات. [$\int_a^b fd(gh)$]ذو المكاملات المضروبة

INTRODUCTION

T. J. Stieltjes (1856–1894) introduced a generalization of the Riemann integral, Stieltjes himself died before the appearance of his paper, and the idea at traced almost no attention for the next 15 years, the type of integration considered here is somewhat more general, and the added generality makes it very useful in certain applications, especially in statistics and numerical integration.

We shall consider bounded functions on closed intervals of real number system, define the integral of one such function with respect to another, and derive the main properties of this integral,

In this study we shall focus our attention on two integral inequalities which are well known in the literature as the trapezoid and Ostrowski inequalities and depended in her proofs on the Riemann-Stieltjes integral, the trapezoid inequality is deals with the estimation of the magnitude of the difference,

$$\int_a^{\mathrm{b}} f dt - \left[(x - a) f(a) + (b - x) f(b) \right],$$

and the Ostrowski inequality provides an error analysis for the quantity

$$\int_a^b f\, dt - (b - a)\, f(x).$$

Since the writing of the classical book by Hardy, Litlewood and Polya in (1934), the subject of differential and integral inequalities has grown by about 800%. Ten years on, we can confidently assert that this growth will increase even more significantly Inequalities have proved to be an applicable tool for the development of many branches of mathematics.

In 1938 Ostrowski proved the integral inequality which is known in the literature as Ostrowski's inequality which is provides an error analysis for the quantity $\int_a^b f\,dt - (b - a)\,f(x)$, by formula

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2}}{(b-a)^2} \right](b-a)] \,\|f'\|_{\infty},$$

In the year 1995 G, A, Anastassion [5] gave a different proof to Ostrowski's inequality and using concept of the optimal function to establish optimal upper bounds on the deviation of a function from its averages, these lead to sharp inequalities. In 1976, Milovanovic et al. proved a generalization of the trapezoid and Ostrowski inequalities for n-time differentiable mappings. In 1998, Dragomir [17] presented a new results to the classical Ostrowski's inequality and for the first time applied it to the estimation of error bounds for some special means and for some numerical quadrature rules, the monographs {[19], [23], [28], [29] and [30]} were written from 1999 -2004 to present some selected results on Ostrowski type inequalities and their applications. In 2000, Cerone et al. [32], in 2004, Ujevic [57], and in 2011, Alomari [3] were Presented very useful results by concept of the perturbation. In 2014, Dragomir [22] proved the results to find the Riemann-Stieltjes Integral of product integrators and applied on some inequalities.

The basic idea for proofs of main results in this thesis by using the integration by parts formula for Riemann-Stieltjes Integral with the help of the Peano kernels theorem, for example

$$\int_a^b (x-t)df(t) = (x-t)f(t)\Big|_a^b + \int_a^b f(t)dt.$$

The material of this thesis is organized as follows:

In the first chapter, there will be basic concepts which will be used throughout the thesis. Among them, the definitions of functions of bounded variation**,** Riemann- Stieltjes integral and their fundamental properties.

In chapter two, we will give a different generalization of the trapezoid and Ostrowski inequalities.

Chapter three, contains some types and results of the Riemann-Stieltjes Integral of product integrators and the trapezoid and Ostrowski inequalities for the Riemann-Stieltjes Integral.

CHAPTER1

Preliminaries

1.1 Some concepts

Let f and g denote real-valued functions defined on a closed interval $[a, b]$ of the real line. We shall suppose that both f and g are bounded on $[a, b]$, this standing hypothesis will not be repeated a lot.

Definition 1.1 [51]

A mapping f is said to be bounded function if there is real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Proposition 1.2 [10]

i. If a, b are real numbers, then

$$\text{Sup}\{a, b\} = \frac{1}{2}\{a + b + |a - b|\}, \text{ and } \inf\{a, b\} = \frac{1}{2}\{a + b - |a - b|\}.$$

ii. If f, g are continuous real-valued functions on $[a, b]$, then

$$\text{Sup}\{f, g\} = \frac{1}{2}(f + g + |f - g|), \text{ and } \inf\{f, g\} = \frac{1}{2}(f + g - |f - g|).$$

Definition 1.3

A function f is said to be monotonic increasing on $[a, b]$ if $f(x_2) \geq f(x_1)$ for $x_2 > x_1$, and monotonic decreasing if $f(x_2) \leq f(x_1)$ for $x_2 > x_1$.

Definition 1.4

A real-valued function f is continuous at $x_0 \in [a, b]$ if given ε > o, there exists $\delta > 0$, such that $|x - x_0| < \delta$ and $x_0 \in [a, b]$ implies that

$$|f(x) - f(x_0)| < \varepsilon.$$

Definition 1.5 [55]

A real-valued function f is absolutely continuous on $[a, b]$ if given ε > o, there exists $\delta > 0$, such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

Whenever $\{(a_{i,}\ b_i)\}$ is a finite collection of disjoint intervals with

$$\sum_{i=1}^{n} |b_i - a_i| < \delta.$$

Mean Value Theorem 1.6 [10]

Suppose that f is continuous on a closed interval $[a, b]$ and that f has a derivative interval (a, b). Then there exists at least one point c in (a, b), such that

$$f(b) - f(a) = f'(c)\,(b - a).$$

Theorem 1.7 [36]

If f is continuous on $[a, b]$ and f' exists and is bounded on (a, b), then f is absolutely continuous on $[a, b]$.

proof:

Suppose that $|f'(x)| \leq M$ for $x \in (a,\ b)$, M is real number

let $\varepsilon > 0$, consider

$\sum_{i=1}^{n} |\ f(d_i) - f(\ c_i)\ |$ when $\{(d_{i,}\ c_i):\ 1 \leq i \leq n\}$ is a finite collection of disjoint intervals in $[a, b]$, such that $\sum_{i=1}^{n} |d_i - c_i| < {}^{\varepsilon}/_{M}$,

Now, observe that

$$\sum_{i=1}^{n} |\ f(d_i) - f(\ c_i)\ | = \sum_{i=1}^{n} \frac{|\ f(d_i) - f(\ c_i)\ |}{|d_i - c_i|}\,|d_i - c_i|$$

The mean value theorem tells us that for $1 \leq i \leq n$ there exists $x_i \in [c_i,\ d_i]$ such that,

$$\frac{|\ f(d_i) - f(\ c_i)\ |}{|d_i - c_i|} = |f'(\ x_i)| \leq M.$$

Therefore

$$\sum_{i=1}^{n} \frac{|\, f(d_i) - f(\ c_i)\ |}{|\, d_i \ \ -c_i\ |}\ |\, d_i - c_i\,| \ \le\ M \sum_{i=1}^{n} |\, d_i - c_i\,| < M\ (\varepsilon/_{M}) = \varepsilon.$$

Hence f is absolutely continuous on$[a, b]$.

Definition 1.8 [11]

The mapping $f\ : [a,\ b] \ \rightarrow\ R$ is said to be L−Lipschitzian on $[a, b]$ if

$$|\, f(x) - f\ (y)| \ \le\ L\, |x\ -\ y| \quad \text{For } \ x, y\ \in\ [a,\ b].$$

Proposition 1.9 [36]

Let f: $[a,\ b]\ \rightarrow\ R$ be a function that is L − Lipschitzian for some constant $L > 0$. Then f is absolutely continuous on $[a, b]$.

Proof

let $\varepsilon > 0$, and choose $\delta = \frac{\varepsilon}{L}$.

now, if $\{\ (d_i, c_i):\ 1 \le\ i\ \le n\ \}$ is a finite collection of disjoint intervals in $[a, b]$, such that $\sum_{i=1}^{n} |\, d_i\ - c_i\ \ | < \delta$,

So by using the Lipschitz condition for $(d_{i,}\ c_i)$, we obtain

$$|\ f(d_i) - f(\ c_i)\,| \le\ \ L\ |\ d_{\ i} -\ c_i|, \text{ for all } 0 \le i \le n$$

Therefor

$$\sum_{i=1}^{n}\ |\ f(d_i) - f(\ c_i)\ | \le\ \ L\ \sum_{i=1}^{n}\ |\ d_{\ i} -\ c_i| < L\ \frac{\varepsilon}{L} = \varepsilon.$$

(Sequence of Taylor) 1.10 [38]

The Taylor formula for continuous function f on $I \subset R$ and $a \in I$,

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}\, f''(a) + \cdots + \ \ \frac{(x-a)^k}{k!}\, f^k(a).$$

our point the Taylor formula with an integral remainder term,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^k}{k!} f^{(k)}(a)$$

$$+ \frac{1}{k!}\int_a^x (x-t)^k f^{(k-1)}(t)d. \qquad 1.1$$

(It can be verified by integration by parts).

Suppose that we are given an approximant (e. g. of a function, a derivative and an integral). Whose error vanishes for $f \in \mathbb{P}_K[x]$, where

$$\mathbb{P}_K[x] = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k.$$

Notation 1.11 [38]

The Taylor formula produces an expression for the error that depends on $f^{(k+1)}$.

This is the basis for the Peano kernel theorem,

Formally, let $L(f)$ be an error of an approximant, thus L maps from $C[a,b]$ to R,

And L is linear, so $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ for α, β ∈ R, and that $L(f) = 0$ for $f \in \mathbb{P}_K[x]$,

Thus, from (1.1) we have

$$L(f) = \frac{1}{k!} L\{\int_a^x (x-t)^k f^{(k-1)}(t)dt\}, \quad a \le x \le b.$$

To make the range of integration independent of x, we introduce the notation

$$(x-t)_+^k = \begin{cases} (x-t)^k & \text{if} \quad x > t \\ 0 & \text{if} \quad x \le t, \end{cases}$$

Whence $L(f) = \frac{1}{k!} L\{\int_a^b (x-t)_+^k f^{(k+1)}(t)dt\}$.

Now, let $K(t) = L\{ (x-t)_+^k \}$, for $x \in [a,b]$, then K is independent of f.

Suppose that it is allowed to exchange the order of action of $\int$ and L,

So $$L(f) = \frac{1}{k!}\int_a^x K(t)\, f^{(k+1)}(t)dt. \qquad 2.1$$

Theorem 1.12 [8] (the Peano kernel)

Let L be a linear functional from a space of functions to R such that $L(f) = 0$ for$f \in \mathbb{P}_K[x]$, provided that $f \in C^{k+1}[a,b]$ and the above exchange of L with integration sign is valid, the formula (2.1) is true.

Example 1.13

We approximate a derivative by a linear combination of function values,

$$f'(0) = -\tfrac{3}{2}f(0) + 2f(1) - \tfrac{1}{2}f(2).$$

Therefore, $L(f) = f'(0) - [-\tfrac{3}{2}f(0) + 2f(1) - \tfrac{1}{2}f(2)]$.

And it is easy to check that $L(f) = 0$ for $f \in \mathbb{P}_2[x]$, [Verify by trying

$f(x) = 1,\ x, x^2$ and using linearity of L]. Thus, for $f \in C^3[0,2]$ we have,

$$L(f) = \tfrac{1}{2}\int_0^2 K(t)\, f^{(3)}(t)dt.$$

To evaluate the Peano kernel K, we fix t. Letting $g(x) = (x-t)_+^2$.

We have, $K(t) = L(g) = g'(0) - [-\tfrac{3}{2}g(0) + 2g(1) - \tfrac{1}{2}g(2)]$

$$= 2\,(0-t)_+ - [-\tfrac{3}{2}(0-t)_+^2 + 2(0-t)_+^2 - \tfrac{1}{2}(0-t)_+^2].$$

So

$$K\,(t) = \begin{cases} 0, & t \le 0 \\ 2t - \tfrac{3}{2}t^2\,, & 0 \le t \le 1 \\ \tfrac{1}{2}(2-t)^2\,, & 1 \le t \le 2. \end{cases}$$

It is obvious that $K(t) = 0$ for t$\notin [0,\ 2]$, since then L acts on a quadratic polynomial

1.2 - Functions of Bounded Variation

Definition 1.14 [35]

A function $f : [a, b] \rightarrow R$ is said to be of bounded variation on $[a, b]$ if and only if there is a constant $M \geq 0$, such that

$$\sum_{i=1}^{n} \left| f(x_{\mathrm{i}}) - f(x_{\mathrm{i-1}}) \right| \leq M,$$

for all partitions $\mathrm{p} = \{x_0, x_1, x_2, \ldots, x_{\mathrm{n}}\}$ of $[a, b]$.

If f is of bounded variation on $[a, b]$, then the total variation of

f is defined to be

$$\bigvee_a^b f = \sup\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : P = \{x_\circ, x_1, \ldots, x_n\}$$

is a partition of $[a, b]\}$.

Lemma 1.15 [54]

Let f: $[a , b] \rightarrow R$ be a function, Let $\{x_{\mathrm{i}}: 0 \leq i \leq n\}$ and $\{y_{\mathrm{i}} : 0 \leq i \leq m\}$ any partitions of $[a, b]$ such that

$$\{x_{\mathrm{i}} : 0 \leq i \leq n\} \subseteq \{y_{\mathrm{i}} : 0 \leq i \leq m\},$$

Then,

$$\sum_{i=1}^{n} \left| f(x_{\mathrm{i}}) - f(x_{i-1}) \right| \leq \sum_{i=1}^{m} \left| f(y_{\mathrm{i}}) - f(y_{i-1}) \right|.$$

Theorem 1.16 [36]

Let f and g be functions of bounded variation on $[a, b]$,

and let k be a constant. Then

(1) f is bounded on $[a, b]$.

(2) f is of bounded variation on every closed subinterval of $[a, b]$.

(3) kf is of bounded variation on $[a, b]$.

(4) $f + g$ and $f - g$ are of bounded variation on $[a, b]$.

(5) $f\ g$ is of bounded variation on $[a,\ b]$.

(6) If $1/g$ is bounded on $[a, b]$, then $f\ /g$ is of bounded variation on $[a,\ b]$.

Proof:

(1) Suppose f is not bounded on $[a,\ b]$,
so there exist $x\ \in\ [a\ ,b]$, such that $|f(x)|\ >\ r$ for r ∈ R.

Now, let $x = x_m$ for $0 \leq m\ \leq n$, such that $\{x_{\text{i}} : 1\leq \text{i} \leq \text{n}\}$ be a partition of $[a, b]$.

Then $\sum_{\text{i}=1}^{\text{n}}\ |f(\ x_i)-\ f(\ x_{i-1})| > r,$

Therefor $\bigvee_a^b f > r,$ for some partition $\{x_{\text{i}} : 1\leq \text{i} \leq \text{n}\ \}$ of $[a, b]$.

Hence, if f be functions of bounded variation on $[a,\ b]$, then f is bounded.

(2) We begin by assuming that f is of bounded variation on $[a,\ b]$ Thus

$$\bigvee_a^b f = \sup \{\sum_{\text{i}=1}^{\text{n}}\ |f(x_i\ ) - f(\ x_{i-1})|\} = \text{r},$$

Let $[c, d] \subseteq [a, b]$ and $\{x_{\text{i}} : 1\leq i \leq n\}$ be a partition of $[c, d]$,

Then extend this partition to $[a, b]$ by adding the points a and b, and relabeling

So $\{x_{\text{i}}\ :\ 0\ \leq i \leq n + 2\ \}$ is a partition of $[a,\ b]$ such that $x_1 = c$ and $x_{\text{n+1}} = d$.

Then

$$\sum_{\text{i}=2}^{\text{n}+1}\ |f\ (x_i) - f(\ x_{i-1})| \leq |\ f(x_1) - f(a)\ |$$

$$+ \sum_{\text{i}=2}^{\text{n}+1}\ |f(x_{\text{i}}) - f(x_{\text{i-1}})\ | + |\ f\ (b) - f\ (x_{\text{n}})\ | \leq\ r.$$

Because original partition of $[c,\ d]$ was arbitrary we can conclude that,

$$\bigvee_c^d f\ \leq\ r.$$

(3) Let $\{x_{\text{i}}: 1 \leq \text{i} \leq \text{n}\}$ be a partition of $[a,\ b]$ consider

$$\sum_{\text{i}=1}^{\text{n}}\ |k\ f(x_{\text{i}}) - kf(x_{\text{i-1}})\ | = k \sum_{\text{i}=1}^{\text{n}}\ |\ f(\ x_i) - \ f(x_{\text{i-1}})|$$

$$\leq |k|\ \bigvee_a^b\ f\ \leq |k|r, \qquad \text{for r} \in \text{R}$$

Then kf is bounded variation and $\bigvee_a^b (kf) = |k| \bigvee_a^b f$.

(4) Let $\{x_i: 1 \leq i \leq n\}$ be a partition of $[a, b]$.

By repeated use of the triangle inequality

We have

$$\sum_{i=1}^{n} | f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1}) | \leq \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) | + \sum_{i=1}^{n} | g(x_i) - g(x_{i-1}) |$$

$$\leq \bigvee_a^b f + \bigvee_a^b g$$

And notice that $\bigvee_a^b f + \bigvee_a^b g$ is finite, the partition we choose was arbitrary hence $f + g$ is bounded variation to prove $f - g$ is of bounded variation simply note that $f - g = f + (-g)$, by (3), $(-g)$ is bounded variation.

(5) To prove fg is bounded variation

Let $\{x_i : 1 \leq i \leq n\}$ be arbitrary partition of $[a, b]$ then,

By repeated use of the triangle inequality, we get

$$\sum_{i=1}^{n} | f(x_i) g(x_i) - f(x_{i-1}) g(x_{i-1}) | = \sum_{i=1}^{n} | f(x_i) g(x_i) - f(x_{i-1}) g(x_{i-1}) + (f(x_i) g(x_{i-1}) - f(x_i) g(x_{i-1}) |$$

$$= \sum_{i=1}^{n} | |f(x_i)| | g(x_i) - g(x_{i-1}) |$$

$$+ \sum_{i=1}^{n} | g(x_{i-1}) | | f(x_i) - f(x_{i-1}) |$$

$$\leq (nM) \bigvee_a^b g + (nN) \bigvee_a^b f,$$

Where $|f(x)| < M$ and $|g(x)| < N$, for $x \in [a, b]$.

Since $(nM) \bigvee_a^b g + (nN) \bigvee_a^b f$ is finite,
Then fg is bounded variation.

(6) Since $1/g$ is bounded so there exists $M \in R$ such that

$$1/g(x) \leq M, \quad \text{for } x \in [a, b].$$

Now let $\{x_i: 0 \le i \le n\}$ be a partition, then

$$\sum_{i=1}^{n} \left| \frac{1}{g(x_i)} - \frac{1}{g(x_{i-1})} \right| = \sum_{i=1}^{n} \left| \frac{g(x_{i-1}) - g(x_i)}{g(x_i) g(x_{i-1})} \right|$$

$$\le M^2 \sum_{i=1}^{n} \left| g(x_i) - g(x_{i-1}) \right|$$

$$\le M^2 \bigvee_a^b g < \infty$$

Thus $\frac{1}{g}$ is bounded variation so by (5) $(f)\left(\frac{1}{g}\right) = \frac{f}{g}$ it is also.

Lemma 1.17 [47]

If $f : [a, b] \to$ R is a function and f is of bounded variation on $[a, c]$ and $[c, b]$, then f is of bounded variation on $[a, b]$ and

$$\bigvee_a^b f = \bigvee_a^c f + \bigvee_c^b f.$$

Theorem 1.18 [54]

If f is monotone increasing on $[a, b]$, then f is of bounded variation on $[a, b]$, and

$$\bigvee_a^b f = f(b) - f(a).$$

Proof:

Let $\{x_i, 1 \le i \le n\}$ be a partition of $[a, b]$, we know $f(x_i) \ge f(x_{i-1})$ for i

and so $f(x_i) - f(x_{i-1}) \ge 0$, and $\left| f(x_i) - f(x_{i-1}) \right| = \left(f(x_i) - f(x_{i-1}) \right)$.

Hence

$$\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1}) \right)$$

$$= \left(f(x_n) - f(x_{n-1})\right) + \left(f(x_{n-1}) - f(x_{n-1})\right) + \dots$$

$$+ \left(f(x_3) - f(x_2)\right) + \left(f(x_2) - f(x_1)\right)$$

$$= f(x_n) - f(x_1) = f(b) - f(a)$$

Easily noting that

$$x_n = b \text{ and } x_1 = a.$$

It is the same for every partition of $[a,\ b]$. So

$$\bigvee_a^b f\ = f(b) - f(a) < \infty$$

Thus f is of bounded variation

Lemma 1.19 [55]

If $f : [a, b] \to \mathrm{R}$ is a function, then $\bigvee_a^b f = 0$ if and only if f is constant.

Proof :

Suppose that f is constant then f is monotone function, so by

(1.18) $\bigvee_a^b f\ = f\ (b) - f\ (a)$

However $f(b) = f\ (a)\ =\ c \in R.$

So $\bigvee_a^b f\ \ = 0$

Now suppose that f is not constant on $[a,\ b]$, so there exists $x_1,\ x_2 \in [a,\ b]$ such that $x_1 = x_2$ and $f\ (x_1) \neq f\ (x_2)$.

If we take these points as a partition of $[a,\ b]$, we have

$$\bigvee_a^b f\ \ \geq \left| f(x_1) - f\ (a) \right| + \left| f\ (x_2) - f(x_1) \right| + \left| f\ (b) - f(x_2) \right| \geq 0.$$

But $\left| f\ (x_2) - f\ (x_1) \right| > 0$

Thus $\bigvee_a^b f\ \ > 0$ and $\bigvee_a^b f\ \neq 0.$

Lemma 1.20 [36]

If f is a function of bounded variation on $[a,\ b]$ and $x\ \in [a,\ b]$, then

$g(x) = \bigvee_a^x f$ is an increasing function on $[a,\ b]$.

Proof:

Let $x_1, x_2 \in [a\ , b]$ and $x_1\ \leq\ x_2$, because f is of bounded variation so by

(1.17) we have

$$\bigvee_a^{x_2} f = \bigvee_a^{x_1} f + \bigvee_{x_1}^{x_2} f$$

$$\bigvee_a^{x_2} f - \bigvee_a^{x_1} f = \bigvee_{x_1}^{x_2} f$$

$$g(x_2) - g(x_1) = \bigvee_{x_1}^{x_2} f \geq 0$$

So $$g(x_2) \geq g(x_1)$$

Hence $g(x)$ is an increasing.

Theorem 1.21 [47]

If $f : [a, b] \to R$ is a function of bounded variation, then there exist two increasing functions, f_1 and f_2 such that $f = f_1 - f_2$.

Proof

Let $f_1 = \bigvee_a^x f$ for $x \in [a, b]$,

And $f_1(a) = 0$, so by (1.20) f_1 is increasing.

Now,

define $f_2 = f_1 - f$,

We need show that f_2 is increasing.

Let $x, y \in [a, b]$ such that $x < y$, then

$$f_1(y) - f_1(x) = \bigvee_x^y f$$

$$\geq |f(y) - f(x)| \geq f(y) - f(x).$$

[Because $f_1(y) - f_1(x) = \bigvee_a^y f - \bigvee_a^x f = \bigvee_x^y f$].

Then

$$f_1(y) - f_1(x) \geq f(y) - f(x)$$
$$f_1(y) - f(y) \geq f_1(x) - f(x)$$

So $$f_2(y) \geq f_2(x)$$

Thus f_2 is increasing on $[a, b]$, and $f = f_1 - f_2$.

Lemma 1.22 [56]

If $f : [a,\ b] \rightarrow R$ is absolutely continuous, then it is of bounded variation.

Proof

Let $\delta > 0$ such that $\sum_{i=1}^{n} \mid f(d_{\mathrm{i}}) - f(c_{\mathrm{i}}) \mid < 1$ when $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mid d_{\mathrm{i}} - c_{\mathrm{i}} \mid <$ δ, and $\{ (d_{\mathrm{i}}, c_{\mathrm{i}}): 1 \leq i \leq n \}$ is a finite collection of disjoint intervals in $[a,\ b]$,

Round up ($\frac{b-a}{\delta}$) to the nearest integer value and call it k.

Now, construct a partition of $[a,\ b]$ as follows, $\{x_{\mathrm{i}} = a + i\left(\frac{b-a}{k}\right) : 0 \leq i \leq k \}$.

Then

$$x_{\mathrm{i}} - x_{i-1} = \left(a + i\left(\frac{b-a}{k}\right)\right) - \left(a + (i-1)\left(\frac{b-a}{k}\right)\right) = \frac{b-a}{k} \leq \delta,$$

So, by the absolute continuity condition, we have

$$\mathsf{V}_{x_i}^{x_{i-1}} f \leq 1,$$

Now, by Summing over *i* from 0 to k and using the (1.17), we have

$$\mathsf{V}_{a}^{b}(f) \leq \sum_{i=1}^{k} \mathsf{V}_{x_i}^{x_{i-1}} f \leq 1 + 1 + \ldots + 1 = k$$

Therefore f is of bounded variation.

Example 1.23

Define the function $f : [0\,,1] \rightarrow R$, by $f\,(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\cos\frac{\pi}{x} & \text{if } x \neq 0 \end{cases}$

We know that $\cos\frac{\pi}{x}$ is bounded, and too

$$\left|\cos\frac{\pi}{x}\right| \leq 1, \quad \text{where } x \neq 0,$$

then by use of definition of continuity in(1.4) we have,

$$|f(x) - f(0)| = \left|x\cos\frac{\pi}{x} - 0\right| = |x| \left|\cos\frac{\pi}{x}\right| \leq |x|$$

Choose $\delta = \varepsilon$.

If $|x - 0\,| < \delta$ implies that $\mid f(x) - f(0) \mid \leq |x| < \varepsilon$, then f is continuous on [0, 1]

but is not of bounded variation, to see this, for each $m \in$ N, let the partition

$$P_m = \{\, 0,\ \frac{1}{2m},\ \frac{1}{2m-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$$

The values of f at the points of this partition one

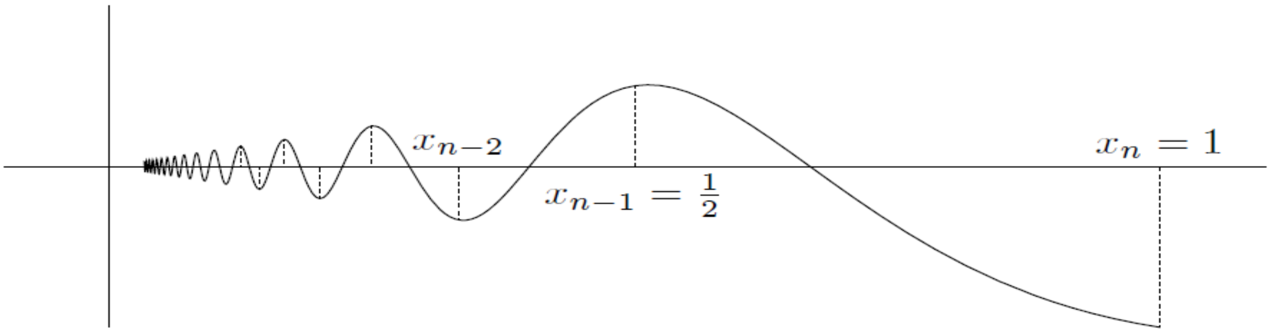
$$f(P_m) = \{0, \frac{1}{2m}, -\ \frac{1}{2m-1},\ \frac{1}{2m-2}, \dots, \frac{1}{3},\ \frac{1}{2}, -1\}, \text{ then}$$

$$\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| = \left| \frac{1}{2m} - 0 \right| + \left| -\frac{1}{2m-1} - \frac{1}{2m} \right| + \left| \frac{1}{(2m-2)} + \frac{1}{2m-1} \right|$$

$$+ \dots + \left| -\frac{1}{3} - \frac{1}{4} \right| + \left| \frac{1}{2} + \frac{1}{3} \right| + \left| -1 - \frac{1}{2} \right|$$

$$= \frac{1}{2m} + \frac{1}{2m-1} + \frac{1}{2m} + \frac{1}{(2m-2)} + \frac{1}{2m-1} + \dots + \frac{1}{3} + \frac{1}{4} + \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{2}$$

$$= 2\left(\frac{1}{2m} + \frac{1}{2m-1} + \dots + \frac{1}{2} \right) + 1,$$



We have the series $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges, then given any M, there is a partition P_m for which

$$\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| > M.$$

So by lemma1.22 f is not absolutely continuous.

Corollary 1.24 [47]

If f is continuous on $[a, b]$ and f' exists and is bounded on (a, b).

Then f is of bounded variation on $[a, b]$.

Corollary 1.25 [35]

If $f : [a,b] \rightarrow R$ is a function that is L-Lipschitzian for some finite constant L > 0, then f is of bounded variation on $[a, b]$.

Remark 1.26 [36]

If f is a continuous function from $[a,\ b]$ to R, and if f is differentiable on $(a,\ b)$ with $|f^{\prime}(x)| \leq M$ for $x \epsilon (a,b)$, then

$$| f\,(x) - f\,(y)| \leq M\,|x - y| \quad for \ \ x, y \in [a,\ b],$$

in this case, f is a Lipschitz continuous function on $[a,\ b]$.

1.3 - Riemann- Stieltjes integral

Definition 1.27 [47]

Let $f, g : [a,\ b] \rightarrow R$ be bounded functions, suppose that there exists a real number A such that for every $\varepsilon > 0$ there is $\delta > 0$ for which,

$$\left| \sum_{i=1}^{n} f(\xi_i)\,[g\,(x_\mathrm{i}) - g\,(x_{i-1})] - A \right| < \varepsilon,$$

For every subdivision P of mesh size less then δ and for $\{\xi_i\}$ with $(x_{i-1} \leq \xi_i \leq x_\mathrm{i})$, $i = 1,2,\dots,n$, then we say that f is Riemann – Stieltjes integrable with respect to g on $[a,\ b]$ or $f \in R(g)$, and we write $\int_a^b f dg = A$.

[mesh P = $\|P\|$ = $\underset{0\leq i \leq 1}{\mathrm{Max}}$ $|\ x_\mathrm{i} - x_\mathrm{i\text{-}1}|$, for $i = 1,2,\dots,n$].

Example 1.28

Let $f, g : [0,\ 1] \rightarrow R$ given by $f(x) = 1$, and

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \frac{1}{3} \\ 1 & \text{for} \quad \frac{1}{3} \leq x \leq 1, \end{cases}$$

Then the sum $\sum_{i=1}^{n} f(\xi_i)\left(g\,(x_\mathrm{i}) - g\,(x_{i-1})\right) = \sum_{i=1}^{n} \left(g\,(x_\mathrm{i}) - g\,(x_{i-1})\right)$,

for any partition $\{x_0,\ x_1,\ x_2, \dots,\ x_n\}$ of $[a, b]$ and any $\xi_i \in (x_\mathrm{i\text{-}1},\ x_\mathrm{i})$, there is m such that $0 \leq m \leq n$, and $\frac{1}{3} \in (x_{\mathrm{m}-1}, x_\mathrm{m})$, so

$$\sum_{i=1}^{n} \left(g\,(x_\mathrm{i}) - g\,(x_{i-1})\right) = 0 + \dots + (g(x_\mathrm{m}) - g(x_{\mathrm{m}-1})) + \dots + 0 = 1 - 0 = 1,$$

Then however $\sum_{i=1}^{n} f(\xi_i)\left(g\,(x_\mathrm{i}) - g\,(x_{i-1})\right) = 1$, Then $f \in R(g)$, and $\int_0^1 f dg = 1$.

Remark 1.29 [9]

If f is Riemann – Stieltjes integrable with respect to g then,

$$\int_a^b f dg = \lim_{\| P\| \to o} S(P, f, g),$$

Where

$S(P, f, g) = \sum_{i=1}^{n} f(\xi_i) \left(g(x_i) - g(x_{i-1})\right)$, is called Riemann – Stieltjes sum, for $\xi_i \in (x_{i-1}, x_i)$, where $P = \{x_1, x_2, \ldots, x_n\}$ any partition of $[a, b]$, and

$$\| P\| = \underset{0 \le i \le n}{\text{Max}} | x_i - x_{i-1} |, \quad \text{for} \quad i = 1,2,3, \ldots, n.$$

Definition 1.30 [51]

A partition P^* is said to be a refinement of P, if $P^* \supseteq P$.

Notation 1.31

If P^* is refinement of P then, mesh $P^* \le$ mesh P

So, if mesh $P^* < \delta^*$ and mesh $P < \delta$ for $\delta, \delta^* > 0$

Then, $\delta \ge \delta^*$.

Remark 1.32 [51]

Given two partition P_1 and P_2 of $[a, b]$, then their common refinement is

$$P^* = P_1 \cup P_2.$$

Remark 1.33 [10]

$f \in R(g)$ if each number $\varepsilon > 0$, there is a number A and a partition P_ε of $[a, b]$, such that if P is refinement of P_ε and if $S(P, f, g)$ is any corresponding Riemann – Stieltjes sum, then $\left|S(P, f, g) - A\right| < \varepsilon$.

Theorem1.34 (Cauchy criterion for integrality) [10]

$f \in$ R (g) if and only if each number ε > 0, there is a partition P_ε of $[a, b]$ such that if P_1, P_2 are refinements of P_ε and if $S\,(P_{1,} f, g)$ and S (P_2, f, g) are any corresponding Riemann – Stieltjes sums, then| S (P_1, f, g) – S (P_2, f, g) |< ε.

Proof

If $f \in$ R (g) and $\int_a^b f dg = \mathrm{A}$ there is P_ε such that if $P_1$and P_2 are refinements of P_ε.

Then $\left| S(P_1,\ f,\ g) - A \right| < \varepsilon/2$, and $\left| S(P_2,\ f,\ g) - A \right| < \varepsilon/2$,

So

$$\left| S(P_1, f,\ g)\ \text{-}\ S\,(P_2,\ f, g) \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely,

Let P_1 be a partition of $[a, b]$ such that if $P$and Q are refinements P_1, then

$$\left| S\,(P, f,\ g) - S\,(\,Q, f, g) \right| < 1.$$

Inductively, we choose P_n to be a refinement of P_{n-1} such that if P, Q are refinements of P_n.

Then

$$\left| S\,(P,\ f,\ g) - S\,(\,Q,\ f,\ g) \right| < 1/\mathrm{n}.$$

Let $(S\,(P_n,\ f,\ g\,)\,)$ be a sequence of real numbers obtained in this way,

since P_n is refinement of P_m for $n \geq m$, so this sequence of sums is Cauchy sequence.

The names that $(\,S\,(P_n, f,\ g)\,) \rightarrow L$ where L is real number,

So if ε > o, there is N, such that 2/N < ε and

$$\left| S\,(\,P_N, f, g) - L \right| < \varepsilon/2.$$

If P is a refinement of P_N, then

$$\left| S(\,P,\ f,\ g) - S(\,P_N,\ f,\ g) \right| < 1/N < \varepsilon/2.$$

Hence $|S\ (P, f, g) - L\ | < \varepsilon$.

Then, by remark 1.33 $f \in R(g)$ on $[a,\ b]$, and $\int_a^b f dg = \mathrm{L}$.

Theorem 1.35 [47]

If $f_1 \in R\ (g)$ and $f_2 \in R\ (g)$ on $[a, b]$, *then* $\alpha f_1 + \beta f_2 \in R(g)$ *on* $[a, b]$, and

$$\int_a^b (\alpha f_1 + \beta f_2)\, dg = \alpha \int_a^b f_1 dg + \beta \int_a^b f_2 dg.$$

Proof

let $\varepsilon > 0$ and let P_1 and P_2 be partitions of $[a,\ b]$ such that if P is refinement of both P_1 and P_2, then for any corresponding Riemann – Stieltjes sums, $S(P, f_1,\ g)$ and $S(P,\ f_2,\ g)$ there exist A_1 and A_2, such that

$$|\ S(P, f_1,\ g)\ -\ A_1\ | < \frac{\varepsilon}{2|\alpha|} \qquad 3.1$$

and

$$|\ S(P, f_2, g)\ -\ A_2\ | < \frac{\varepsilon}{2|\ \beta|}, \qquad 4.1$$

Let $P_\varepsilon = P_1 \cup P_2$, then $P_\varepsilon \subseteq P$ and both of relations above still hold.

When the same intermediate points are used, we have

$$\begin{aligned} S(P, \alpha f_1 + \beta f_2,\ g) &= \sum_{i=1}^{n} (\alpha f_1 + \beta f_2)(\xi_i)\,(\Delta_{\mathrm{i}} g) \\ &= \sum_{i=1}^{n} (\alpha f_1)(\xi_i)\,(\Delta_{\mathrm{i}} g) \\ &+ \sum_{i=1}^{n} (\beta f_2)(\xi_i)\,(\Delta_{\mathrm{i}} g) \\ &= \alpha \sum_{i=1}^{n} f_1(\xi_i)\,(\Delta_{\mathrm{i}} g) + \beta \sum_{i=1}^{n} f_2(\xi_i)\,(\Delta_{\mathrm{i}} g) \\ &= \alpha\, S(P, f_1,\ g) + \beta\, S(P, f_2,\ g). \qquad 5.1 \end{aligned}$$

So by 3.1, 4.1 and 5.1, we have

$$|\alpha A_1 + \beta A_2 \text{ - } S(P, \alpha f_1 + \beta f_2, g)| = |\alpha(A_1 \text{ - } S(P, f_1, g)) + \beta(A_2 - S(P, f_2, g))|$$

$$\leq |\alpha| \frac{\varepsilon}{2|\alpha|} + |\beta| \frac{\varepsilon}{2|\beta|} = \varepsilon$$

Then $\int_a^b (\alpha f_1 + \beta f_2) dg = \alpha A_1 + \beta A_2$.

Theorem 1.36 [10]

If $f \in R(g_1)$ and $f \in R(g_2)$ on $[a, b]$, then $f \in R(\gamma g_1 + \mu g_2)$, and

$$\int_a^b f d(\gamma g_1 + \mu g_2) = \gamma \int_a^b f d g_1 + \mu \int_a^b f d g_2 ,$$

When γ, μ are real numbers.

Proof:

Let $g = \gamma g_1 + \mu g_2$, then for any partition $\{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$, then

$$\Delta_i g = \Delta_i (\gamma g_1 + \mu g_2) = (\gamma g_1 + \mu g_2) (x_i) - (\gamma g_1 + \mu g_2)(x_{i-1})$$

$$= (\gamma g_1) (x_i) - (\gamma g_1)(x_{i-1}) + (\mu g_2)(x_i) - (\mu g_2) (x_{i-1})$$

$$= \gamma \Delta_i g_1 + \mu \Delta_i g_2.$$

Now,

let $\varepsilon > 0$, and let P_1 and P_2 be partitions of $[a, b]$, such that if P is refinement of both P_1 and P_2, then

$$\left| \sum_{i=1}^{n} f(\xi_i) (\Delta_i g_1) \text{ - } B_1 \right| < \frac{\varepsilon}{2|\gamma|},$$

And

$$\left| \sum_{i=1}^{n} f(\xi_i) (\Delta_i g_2) \text{ - } B_2 \right| < \frac{\varepsilon}{2| \mu|}.$$

If $P_\varepsilon = P_1 \cup P_2$, then P is refinement of P_ε and

Clearly, if $\{x_{i-1} \leq \xi_i \leq x_i\}$ is the same intermediate points are used, then

$$S(P,f,\ \gamma g_1+\ \mu g_2)\ =\ \sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(\gamma g_1+\ \mu g_2)$$

$$=\ \sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(\gamma g_1)+\sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(\mu g_2)$$

$$=\gamma\sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(g_1)+\mu\sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(g_2)$$

$$=\ \gamma\, S(P,f,g_1\ )\ +\mu\, S(P,f,\ g_2).$$

But we have

$$\int_a^b f dg_1 = B_1 \ \text{ and } \int_a^b f dg_2 =\ B_2,\ \text{ then}$$

$$|\sum_{i=1}^{n} f(\xi_i)\,(\Delta_{\text{i}}\,)(\gamma g_1+\ \mu g_2) - (\gamma B_1+\ \mu B_2)| = |[\ \gamma\, S(P,f,g_1) + \mu\, S(P,f,\ g_2)]$$

$$-\ (\gamma B_1+\ \mu B_2)\ |$$

$$\leq\ \ |\gamma[\ \ S(P,f,g_1) \text{ - } B_1]|+$$

$$|\mu[\ \ S(P,f,g_2) \text{ - } B_2]| <\ \varepsilon.$$

Hence, $f\in R(\gamma g_1+\ \mu g_2),$ and $\int_a^b f d(\ \gamma g_1{+}\mu\ g_2) = \gamma B_1+\ \mu B_2.$

Theorem 1.37 [47]

Suppose that $a\leq c\leq b$, then $f\in R(g)$ on $[a,\ c]$ and $[c,\ b]$ if and only if $f\in R(g)$ on $[a,\ b]$, and

$$\int_a^c f dg\ +\ \int_c^b f dg =\ \int_a^b f dg.$$

Proof

If $\varepsilon > 0$, let $P_\varepsilon{}'$ be partitions of $[a,\ c]$ such that if P' is refinement of $P_\varepsilon{}'$, then

$$\left|\ S(P',f,g)\ -\ \ A\ '\ \ \right| < \frac{\varepsilon}{2}$$

Similarly for $[a,\ c]$ we can say

$$\left|\ S(P'',f,\ g)\ -\ \ A''\ \ \right| < \frac{\varepsilon}{2},\ \text{ for } P'' \text{ is refinement of } P_\varepsilon{}''$$

Then $\int_a^c fdg = A'$ and $\int_c^b fdg = A''$

Let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$ such that if P is refinement of P_ε, then

$$S(P, f, g) = S(P', f, g) + S(P'', f, g)$$

Where P' and P'' denote the portions of $[a, c]$ and $[c, b]$ induced by P, and the corresponding intermediate points are used, then

$$|(A' + A'') - S(P, f, g)| \leq |S(P', f, g) - A'| + |S(P'', f, g) - A''|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $f \in R(g)$ on $[a, b]$, and

$$\int_a^c fdg + \int_c^b fdg = \int_a^b fdg.$$

Conversely:

since $f \in R(g)$ on $[a, b]$, given $\varepsilon > 0$ there is a partition Q_ε of $[a, b]$ such that if P, Q are refinements of Q_ε, then (by Cauchy Criterion)

$$|\mathrm{S}(P, f, g) - \mathrm{S}(Q, f, g)| < \varepsilon$$

For any corresponding Riemann – Stieltjes sums, $\mathrm{S}(P, f, g)$ and $\mathrm{S}(Q, f, g)$

Now assume that $c \in Q_\varepsilon$,

let Q_ε' be the partition of $[a, c]$ such that $Q_\varepsilon' \subset Q_\varepsilon$,

Suppose that P' and Q' are partitions of $[a, c]$ such that $P' \supseteq Q_\varepsilon'$ and $Q' \supseteq Q_\varepsilon'$,

and

$$P' = P / \{[c, b] \cap Q_\varepsilon\} \quad \text{and} \quad Q' = Q / \{[c, b] \cap Q_\varepsilon\}, \text{ then}$$

P and Q are identical on $[c, b]$ that, if we use the same intermediate points, So

$$|\mathrm{S}\,(P', f, g) - \mathrm{S}\,(Q', f, g)| = |\,\mathrm{S}\,(P, f, g) - \mathrm{S}\,(Q, f, g)\,| < \varepsilon,$$

Therefore, $f \in R(g)$ on $[a,\ c]$, and a similar argument also applies to the interval $[c,\ b]$.

Theorem 1.38 (Integration by parts) [47]

A function f is integrable with respect to g over $[a, b]$ if and only if g is integrable with respect to f over $[a, b]$,

and

$$\int_a^b f dg + \int_a^b g df = f\,(b) g(b) - f\,(a) g(a).$$

Proof

Let $\varepsilon > 0,$ be given.

By definition (1-27), there is a $\delta' > 0$,

$$\left|\sum_{i=1}^{m} f(\xi'_i)\,[g(x'_i) - g(x'_{i-1})\,] - \int_a^b f\,dg \quad\right| < \varepsilon,$$

for partition P': $a = x'_0,\ x'_1,\ \dots,\ x'_m = b$of mesh ≤ δ', and $x'_{i-1} \leq \xi'_i \leq x'_i$.

Now,

let $\delta = \frac{1}{2}\delta'$, and choose $P\!: a = x_0,\ x_1,\ x_{2,}\ \dots, x_n = b$ of mesh ≤ $\delta,$ and

$x_{i-1} \leq \xi_i \leq x_i,\ i = 1,2,\dots,n,$ and we further select $\xi_0 = a,\ \xi_{n+1} = b$.

Then we obtain the partition P_ξ: $a = \xi_0, \xi_1, \dots, \xi_{n+1} = b.$

So, P_ξ is refinement of P' therefore mesh $P_\xi \leq \delta'$, and $\xi_{i-1} \leq x_{i-1} \leq \xi_i$, for $i = 1, 2, \dots, n{+}1$.

Then, we have

$$\sum_{i=1}^{n} g(\xi_i)\,[f(x_i) - f(x_{i-1})\,] = \sum_{i=1}^{n} g(\,\xi_i)\ f(x_i) - \sum_{i=1}^{n} g(\,\xi_i)\ f(x_{i-1})$$

$$= \sum_{i=2}^{n+1} g(\,\xi_{i-1})\ f(x_{i-1}) + g(a)f(a) - g(a)f(a)$$

$$- \sum_{i=1}^{n} g(\xi_i)\ f(x_{i-1}) - \ g(b)f(b) - g(b)f(b)$$

$$= \sum_{i=1}^{n+1} g(\xi_{i-1})\ f(x_{i-1}) - g(a)f(a)$$

$$- \sum_{i=1}^{n+1} g(\xi_i)\ f(x_{i-1}) + g(b)f(b),$$

$$= \sum_{i=1}^{n+1} f(x_{i-1})\,(\,g(\xi_{i-1}) - g(\xi_i)\,)$$

$$- g(a)f(a) \ + g(b)f(b),$$

Therefore

$$\sum_{i=1}^{n} g(\xi_i)\,[f(x_i) - f(x_{i-1})\,] = g(b)f(b) - g(a)f(a)$$

$$- \sum_{i=1}^{n+1} f(x_{i-1})\,[\,g(\xi_i) - g(\xi_{i-1})]$$

Then, by exists of $\int_a^b fdg$ and since $P$and P_ξ are refinements of P', we have $|\sum_{i=1}^{n+1} f(x_{i-1})\,[\,g(\xi_i) - g(\xi_{i-1})] - \int_a^b fdg\,| < \varepsilon,$

But,

$$|\sum_{i=1}^{n+1} f(x_{i-1})\,[\,g(\xi_i) - g(\xi_{i-1})] - \int_a^b fdg\,| = |\sum_{i=1}^{n} g(\xi_i)\,[f(x_i) - f(x_{i-1})\,]$$

$$- \{[g(b)f(b) - g(a)f(a)] - \int_a^b fdg\,| < \varepsilon.$$

Hence, $\int_a^b gdf$ exist and

$$\int_a^b fdg + \int_a^b gdf = f\,(b)g(b) - f\,(a)g(a).$$

Theorem 1.39 (Modification of the integral) [51]

Suppose that f, g and g' are continuous on $[a,\ b]$, then $\int_a^b fdg$ exists. And $\int_a^b f\,dg = \int_a^b fg'\,dx$.

Proof

Let $\boldsymbol{\varepsilon} > 0,$ be given.

By definition (1-26) we have shown that,

$$\left|\sum_{i=1}^{n} f(\xi_i)\,[g(x_i)-g(x_{i-1})\,] - \int_a^b f(x)\ g'(x)\,dx\right| < \varepsilon, \qquad 6.1$$

for any partition $P = \{\ x_1,\ x_2,\ x_3, \dots , x_n\}$ of $[a, b]$, such that the mesh of P is Sufficiently small and $\xi_i \in [x_{i-1}, x_i\,]$.

by the mean – value theorem for any $i = 1,2,3, \dots , n$ there is

$\eta_i \in [x_{i-1}, x_i]$, such that

$$g'(\eta_i) = \frac{g(x_i)-g(x_{i-1})}{x_i - x_{i-1}}, \text{ so}$$

$$\sum_{i=1}^{n} f(\xi_i)\,[g(x_i)-g(x_{i-1})\,] = \sum_{i=1}^{n} f(\ \xi_i)\ g'(\eta_i)\,(x_i - x_{i-1}). \qquad 7.1$$

If $\eta_i = \xi_i$, then

$$\sum_{i=1}^{n} f(\ \xi_i)\ g'(\eta_i)[(x_i - x_{i-1})] = \sum_{i=1}^{n}(fg')(\xi_i)\,[(x_i)-(x_{i-1})\,]. \qquad 8.1$$

Now, since g' is continuous on $[a, b]$ (it is compact), then g' is uniformly continuous on $[a, b]$.

Therefore, there is a $\delta > 0$ such that for $|\xi_i - \eta_i| < \delta$ it follows that

$$|g'(\xi_i) - g'(\eta_i)| < \frac{\varepsilon}{2M\ (b-a)} \qquad 9.1$$

(Where $|f(x)| \le M\ \ for\ \ a \le x \le b$).

By definition of Riemann – Stieltjes integral (where $g(x) = x$ and for any partition P with mesh less than δ), and from [8.1] we have,

$$\left|\sum_{i=1}^{n}(fg')(\xi_i)\,[(x_i)-(x_{i-1})\,] - \int_a^b (fg')(x)dx\ \right| < \frac{\varepsilon}{2}. \qquad 10.1$$

If $\eta_i \neq \xi_i$ then from [9.1], we can say that

$$\left|\sum_{i=1}^{n}\ f(\xi_i)\,[g'(\eta_i) - g'(\xi_i)]\big((x_i)-(x_{i-1})\big)\right|$$

$$< \sum_{i=1}^{n} M\left|\frac{\varepsilon}{2M\ (b-a)}(x_i - x_{i-1})\right|$$

$$= \frac{\varepsilon}{2(b-a)}\sum_{i=1}^{n}\ (x_i - x_{i-1}) = \frac{\varepsilon}{2}. \qquad 11.1$$

Lastly, by [6.1], [10.1] and [11.1], for any $\xi_i, \eta_i \in [x_{i-1},\ x_i]$ we get

$$\left|\sum_{i=1}^{n} f(\xi_i)\,[g(x_i) - g(x_{i-1})\,] - \int_a^b fg'dx\ \right|$$

$$= \left|\sum_{i=1}^{n} f(\ \xi_i)\ \ g'(\eta_i)(\ x_i - x_{i-1}) - \int_a^b fg'\,dx\right|$$

$$\leq \left|\sum_{i=1}^{n}\ f(\xi_i)\,[g'(\eta_i) - g'(\xi_i)\,]\big((x_i) - (x_{i-1})\big)\right|$$

$$+ \left|\sum_{i=1}^{n}(fg')(\xi_i)\,[(x_i) - (x_{i-1})\,] \text{-} \int_a^b (fg')(x)dx\ \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\int_a^b f dg$ exists, and

$$\int_{\mathrm{a}}^{\mathrm{b}} f\ dg = \int_{\mathrm{a}}^{\mathrm{b}} fg'\,\mathrm{d}x$$

Example 1.40

Let : $[0,\ \frac{\pi}{2}\,] \rightarrow [0,\ 1]$, be a function define by $f(x) = \sin x$, then $f'(x) = \cos x$

And f, f' are continuous on $[0,\ \frac{\pi}{2}\,]$.

So, we can use [1.34] and [1. 33] to show that $\int_0^{\frac{\pi}{2}} fdf$ is exist and

$$\int_0^{\frac{\pi}{2}} fdf = \int_0^{\frac{\pi}{2}} \sin x\, d(\sin x) = \int_0^{\frac{\pi}{2}} ff'\,dx = \int_0^{\frac{\pi}{2}} \sin x \cos x\, dx$$

$$= \frac{1}{2}\ (\sin x)^2|_0^{\frac{\pi}{2}} = \frac{1}{2}.$$

And, from Integration by parts;

$$\int_0^{\frac{\pi}{2}} \sin x\, d(\sin x) = \sin x \sin x\ |_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\, d(\sin x)$$

So,

$$2\int_0^{\frac{\pi}{2}} \sin x\, d(\sin x) = (\sin x)^2|_0^{\frac{\pi}{2}} = 1$$

Or $\int_0^{\frac{\pi}{2}} \sin x \, d(\sin x) = \frac{1}{2}$.

Definition 1.41 [47]

Let f, g be functions defined on $[a, b]$, and g be a monotonically increasing function on $[a, b]$.

Corresponding to any partition P of $[a, b]$, $P = \{a = x_0, x_1, x_2 \dots, x_n = b\}$, and

$$\Delta_i g = g(x_i) - g(x_{i-1}), \quad \text{for } i = 1,2,\dots,n, \quad \text{then} \quad (\Delta_i g \geq 0).$$

Define the upper and lower Darboux – Stieltjes sums,

$$S^+(P, f, g) = \sum_{i=1}^n M_i \, \Delta_i g,$$

$$S_-(P, f, g) = \sum_{i=1}^n m_i \, \Delta_i g,$$

where

$$m_i = \inf \{ (x) : x_{i-1} \leq x \leq x_i \}$$

$$M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \},$$

Then the upper Darboux – Stieltjes integral of as

$$\overline{\int_a^b} f \, dg = \inf S^+(P, f, g),$$

and lower Darboux - Stieltjes integral of as

$$\underline{\int_a^b} f dg = \sup S^-(P, f, g).$$

If $\overline{\int_a^b} f dg = \underline{\int_a^b} f dg$, then f is Darboux – Stieltjes integrable with respect to g, and

$$\overline{\int_a^b} f dg = \underline{\int_a^b} f dg = (\text{S} - \text{D})\int_a^b f dg.$$

Examples 1.42

(1) If g is constant on $[a, b]$, then any bounded function f is Riemann – Stieltjes integrable with respect to g.

Clearly;

$\Delta_i g = g(x_i) - g(x_{i-1}) = 0$, for any partition $p = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$,

and

$$S^+(P, f, g) = \sum_{i=1}^{n} M_i \Delta g_i = 0 = \sum_{i=1}^{n} m_i \Delta g_i = S^-(P, f, g).$$

So $$\overline{\int_a^b} f dg = \underline{\int_a^b} f dg = 0.$$

(2) Suppose g increases on $[a, b]$ $x_0 \in [a, b]$, and continuous at x_0,

$f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$, $x_0 \in [a, b]$, then $f \in R(g)$.

Since if $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|g(x) - g(x_0)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x - x_0| < \delta,$$

Let P any partition of $[a, b]$, such that $x_{i-1} \le x_0 \le x_i$ and $|x_i - x_{i-1}| < \delta$

Then $$\Delta_i g = g(x_i) - g(x_{i-1}) = g(x_i) - g(x_0) + g(x_0) - g(x_{i-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$0 \le S^+(P, f, g) = \sum_{i=1}^{n} M_i \Delta_i g = \Delta_i g_i < \varepsilon$$

thus $$0 \le \overline{\int_a^b} f dg = \inf S^+(P, f, g) < \varepsilon$$

Since ε is arbitrary, so $\overline{\int_a^b} f dg = 0$.

also for any P partition of $[a, b]$, $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\} = \{0\}$

Therefore $$\underline{\int_a^b} f dg = 0$$

Then $$\overline{\int_a^b} f dg = \underline{\int_a^b} f dg = (\text{S} - \text{D}) \int_a^b f dg = 0.$$

Theorem 1.43[51]

If P^* is a refinement of P, then

$$S^+(P^*, f, g) \leq S^+ (P, f, g).$$

Proof

Assume that P^*contains just one point more than P.

Let this be c and $x_{i-1} \leq c \leq x_i$.

Let $M_i^' = \sup \{ f(x) / x \in [x_{i-1}, c] \}$

and $M_i^" = \sup \{ f(x) / x \in [c, x_i] \}$,

then $M_i^' \leq M_i$, and $M_i^" \leq M_i$.

consider

$$S^+(P^*, f, g) = \sum_{\substack{k=1\\k\neq i}}^{n} M_k \Delta_k g + M_i^' [g(c) - g(x_{i-1})] + M_i^"[g(x_i) - g(c)]$$

$$\leq \sum_{\substack{k=1\\k\neq i}}^{n} M_K \Delta_k g + M_i [g(c) - g(x_{i-1})] + M_i[g(x_i) - g(c)]$$

$$\leq \sum_{\substack{k=1\\k\neq i}}^{n} M_K \Delta_k g + M_i [g(x_i) - g(x_{i-1})]$$

$$= \sum_{\substack{k=1\\k\neq i}}^{n} M_K \Delta_k g + M_i\Delta_i g$$

$$= S^+(P, f, g).$$

Theorem 1.44 [51]

$$\underline{\int_a^b} fdg \leq \overline{\int_a^b} fdg$$

Proof:

Let $P_1$and P_2 be any partitions of $[a, b]$.

Let $P^* = P_1 \cup P_2$

Then P^* is the common refinement of P_1 as well as P_2.

Therefore by theorem 1.43

$$\mathrm{S}^+ (P^*, f, g) \leq \mathrm{S}^+ (P_1, f, g) \qquad 12.1$$

And $S_- (P^*, f, g) \geq S_- (P_2, f, g)$ 13.1

Also we know that

$$S_- (P^*, f, g) \leq \mathrm{S}^+ (P^*, f, g) \qquad 14.1$$

From (12.1), (13.1) and (14.1), we get

$$S_-(P_2, f, g) \leq S_- (P^*, f, g) \leq S^+ (P^*, f, g) \leq S^+ (P_1, f, g)$$

Therefore for any two partitions $P_1$and P_2 of $[a, b]$, we have

$$S_-(P_2, f, g) \leq \mathrm{S}^+(P_1, f, g).$$

Keeping P_2 fixed and varying P_1 over all partitions of $[a, b]$,

$$S_-(P_2, f, g) \leq \inf \mathrm{S}^+(P, f, g).$$

Now this is true for all partitions P_2 of $[a, b]$.

Therefore

Sup $S_-(P, f, g) \leq \inf S^+(P, f, g)$, so

$$\underline{\int_a^b} f\mathrm{d}g \leq \overline{\int_a^b} f\mathrm{d}g.$$

Theorem1.45 [51]

$f \in R(g)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exist a partition P of $[a, b]$ such that,

$$\mathrm{S}^+ (P, f, g) - S_- (P, f, g) < \varepsilon.$$

Proof:

If $f \in R (g)$ on $[a, b]$, then

$$\underline{\int_a^b} f\, dg = \overline{\int_a^b f dg}\,, \qquad 15.1$$

when $\overline{\int_a^b} f\, dg =$ inf $S^+(P,\ f,\ g)$,

and $\underline{\int_a^b} f dg$ = sup $S_-(P,\ f,\ g)$.

Therefore, by definition of infinimum and supremum,

For given ε > 0, there exists a partition P_1 of $[a, b]$ such that

$$S^+ (P_{1,}\ f, g) < \overline{\int_a^b} fdg + \varepsilon/2 \qquad 16.1$$

And a partition P_2 of $[a, b]$ such that

$$S_- (P_2, f,\ g) > \underline{\int_a^b} fdg - \varepsilon/2 \qquad 17.1$$

Let $P = P_1 \cup P_2$

Then by theorem 1.43

$$S^+ (P,\ f,\ g) \leq S^+ (P_1, f,\ g) \qquad 18.1$$

and $$S_-(P, f,\ g) \geq S_- (P_2, f,\ g) \qquad 19.1$$

From (15.1), (16.1), (17.1), (18.1) and (19.1), we get

$$S^+ (P, f, g) \leq S^+ (P_1, f, g) < \overline{\int_a^b} fdg + \varepsilon/2$$

$$< \underline{\int_a^b} fdg + \varepsilon/2 < S_- (P_2, f, g) + \varepsilon/2 + \varepsilon/2$$

$$< S_- (P_2, f, g) + \varepsilon < S_- (P, f, g) + \varepsilon$$

Therefore there exists a partition P of $[a, b]$ such that

$$S^+ (P,\ f,\ g) - S_-(P, f,\ \ g) < \varepsilon \qquad 20.1$$

Then for every partition P of $[a,\ b]$, we have

$$S_-(P,\ f,\ g) \leq \underline{\int_a^b} fdg \leq \overline{\int_a^b} fdg \leq S^+ (P, f,\ g) \qquad 21.1$$

From (20.1) and (21.1), we get that

$$0 \leq \overline{\int_a^b} fdg - \underline{\int_a^b} fdg \leq S^+(P, f, g) - S_-(P, f, g) < \varepsilon.$$

This is true for ever $\varepsilon > 0$.

Therefore $\overline{\int_a^b} fdg = \underline{\int_a^b} fdg$

Hence $f \in \mathrm{R}(g)$ on $[a, b]$.

Corollary 1.46 [10]

Let f be bounded and g be monotone increasing on $[a, b]$,then $f \in R(g)$ on $[a, b]$

if and only if for every $\varepsilon > 0$ there exists P_ε such that if P is a refinement of P_ε then,

$$\sum_{i=1}^{n} (\mathrm{M_i} - \mathrm{m_i})\Delta g_\mathrm{i} < \varepsilon,$$

Where M_i= sup $\{f(x): x \in [x_{i-1}, x_i]\}$ and m_i = inf $\{ f(x) : x \in [x_{i-1}, x_i]\}$.

Theorem 1.47 [51]

If $f, h \in R(g)$ on $[a, b]$, then

(a) $|f|$, $f^2 \in \mathrm{R}(g)$ on $[a, b]$.

(b) $fh \in R(g)$ on $[a, b]$.

Proof

Let $\varepsilon > 0$,

since $f \in R(g)$, then there exists P_ε such that if P is a refinement of P_ε then

$$\sum_{i=1}^{n} (\mathrm{M_i} - \mathrm{m_i})\Delta g_\mathrm{i} < \varepsilon, \text{ where } P = \{x_0, x_1, x_2, \dots, x_n\} \supset P_\varepsilon$$

We note

$$\mathrm{M_i} - \mathrm{m_i} = \sup \{ f(x) : x \in [x_{i-1}, x_i]\} - \inf \{ f(y) : y \in [x_{i-1}, x_i]\}$$

$$= \sup \{ f(x) : x \in [x_{i-1}, x_i]\} + \sup \{(-f(y)) : y \in [x_{i-1}, x_i]\}$$

$$= \sup \{ f(x) - f(y) : x, y \in [x_{i-1}, x_i]\}, \qquad 22.1$$

And

$$\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$$

So by (1.46) and (22.1), we have

$$\sum_{i=1}^{n} \{\sup \{ |f(x)| - |f(y)| : x, y \in [x_{i-1}, x_i]\}\} \Delta g_i < \varepsilon,$$

So $|f| \in R(g)$.

Now observe that $|f(x)| \leq K$ for $x \in [a, b]$, and

$$M_i(f^2) = \sup \{f^2(x)/ x \in [x_{i-1}, x_i]\}$$

$$= [M_i(|f|)]^2$$

$$m_i(f^2) = [m_i(|f|)]^2$$

$$M_i(f^2) - m_i(f^2) = [M_i(|f|)]^2 - [m_i(|f|)]^2$$

$$= [M_i(|f|) + m_i(|f|)] [M_i(|f|) - m_i(|f|)]$$

$$\leq 2k [M_i(|f|) - m_i(|f|)] < 2k (\varepsilon/2k) < \varepsilon.$$

So by Corollary (1.46) $f^2 \in R(g)$.

(b) Since $f, h \in R(g)$ on $[a, b]$,

By theorem 1.35,

$f + h \in R(g)$ and $f - h \in R(g)$ on $[a, b]$,

Therefore by part (a),

$(f + h)^2 \in R(g)$ on $[a, b]$, and $(f - h)^2 \in R(g)$ on $[a, b]$ and

$$f h = (1/4) [(f + h)^2 - (f - h)^2] \in R(g),$$

Hence $fh \in R(g)$ on $[a, b]$.

Theorem 1.48 [51]

If f is continuous and g is increasing on $[a, b]$, then $f \in R(g)$ on $[a, b]$.

Proof:

Let ε>0.

Choose $\eta > 0$ such that $\eta < \frac{\varepsilon}{[g(\mathrm{b}) - g(\mathrm{a})]}$.

Since f is continuous on $[a, b]$, [$[a, b]$ is compact]. then f is uniformly continuous on$[a, b]$.

Therefore for this $\eta > 0$, there exists $\delta > 0$, such hat

$$|f(x) - f(y)| < \eta \quad \text{whenever} \quad x, y \in [a, b] \text{ with } |x - y| < \delta.$$

If P is any partition of $[a, b]$ such that $\Delta_i x < \delta$, Then

$$M_i - m_i = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \leq \eta, \quad \text{for} \quad i = 1, 2 \ldots n.$$

Therefore

$$S^+(P, f, g) - S_-(P, f, g) = \sum_{i=1}^{n} M_i \Delta_i g - \sum_{i=1}^{n} m_i \Delta_i g$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta_i g$$

$$\leq \eta \sum_{i=1}^{n} \Delta_i g$$

$$\leq \eta [g(b) - g(a)] < \varepsilon.$$

Hence, from 1.45 $f \in R(g)$.

Proposition 1.49 [47]

If f *is* continuous and g is of bounded variation, then $f \in R(g)$.

(from [1.18], [1.21] and [1.36]).

Theorem 1.50 [51]

If f is bounded on $[a,\ b]$, and f has only finitely many points of discontinuity on $[a,b]$, and g is continuous at every point at which f is discontinues, Then

$f \in R(g)$.

Proof

Let ε >0,
Put $|f\ (x)| \leq M$ for $x\, \epsilon\, [a,b]$, and let E = $\{x\ :\ f\ (x)$ is discontinues $\}$
so g is continuous at E = $\{\ x_1,\ x_2, \dots, x_m\}$ and since E is compact then g is uniformly continuous at E ,
therefore we can cover E by finitely many disjoint intervals $[u_{\mathrm{j}},\ v_{\mathrm{j}}] \subset [a,b]$ where $1 \leq j \leq m$, and

$$\sum_{i=1}^{n}(g(v_{\mathrm{i}}) - g(u_{\mathrm{i}})) < \frac{\varepsilon}{4M},$$

And for any $x_j \in$ E there exist $[u_{\mathrm{j}},\ v_{\mathrm{j}}] \ni x_j$.
Now, let $k = [a,\ b] \setminus (u_{\mathrm{j}},\ v_{\mathrm{j}})$ for $j = 1,2,3, \dots, m$, then k is closed subset of compact set it's compact,
Hence f uniformly continuous on k, there exist $\delta > 0$ such hat

$$|f(s) - f(t)| < \frac{\varepsilon}{2(g(b) - g(a))} \quad \text{whenever } t, s \in k \text{ with } |s - t| < \delta.$$

Now, let $= \{x_{0,}\ x_1,\ x_2, \dots, x_n\}$ be a partition of $[a,\ b]$, such that

$u_{\mathrm{j}\,,} v_j \in P$ for all j and no point of any $(u_{\mathrm{j}},\ v_{\mathrm{j}})$ occurs in P.

If $x_{\mathrm{i}-1}$ is not one of the u_{j}, then $|\ x_i - x_{i-1}| < \delta$,

We note that

$$-M \leq m_i \leq M_i \leq M,$$

So $M_i \leq M$ and $-m_i \leq M$ therefore $M_i - m_{\mathrm{i}} \leq 2M$, for all i

And $M_i - m_{\mathrm{i}} \leq \dfrac{\varepsilon}{2(g(b) - g(a))}$ unless x_{i-1} is one of the u_j [by uniformly continuity of f].

Then $S^+\ (P, f, g) - S_-(P, f, g) = \sum_{i=1}^{n}(M_i - m_i)\,\Delta_{\mathrm{i}} g$

$$\leq \big(g(b) - g(a)\big)\frac{\varepsilon}{2(g(b)- g(a))} + 2M\left(\frac{\varepsilon}{4M}\right) = \varepsilon.$$

So by (1.45) $f \in R(g)$.

Notation 1.51

The Riemann-Stieltjes Integral may not exist if f has a single point of discontinuity, and g is also discontinuous at the same point.

Example 1.52

If f, g: [0,1] →R dented by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < ½ \\ 2 & \text{for } ½ \leq x \leq 1, \end{cases} \quad g(x) = \begin{cases} 0 & \text{for } 0 \leq x < ½ \\ 1 & \text{for } ½ \leq x \leq 1, \end{cases}$$

let $P = \{x_1, x_2, x_3, \dots, x_n\}$ be any partition of [0, 1], then

If $x_k \in P$ such that $x_{k-1} \leq \frac{1}{2} \leq x_k$ and $x_{k-1} \leq \xi_k \leq x_k$, then

$[g(x_k) - g(x_{k-1}) = 1 - 0$, but $f(\xi_k)$ will be 1 or 2, depending on whether $\xi_k < \frac{1}{2}$ or $\xi_k \geq \frac{1}{2}$, since these two choices may be made regardless of the mesh of the partition, then $\int_0^1 f dg$ dose not exist.

Theorem 1.53 [51]

If $f_1, f_2 \in R(g)$ and g monotonic on $[a, b]$ and $f_1(x) \leq f_2(x)$ on $[a, b]$, Then

$$\int_a^b f_1 dg \leq \int_a^b f_2\, dg.$$

Proof

Let P be any partition of $[a, b]$.

Since $f_1(x) \leq f_2(x)$,

$$\sup \{ f_1(x) \ / \ x \in [x_{k-1}, x_k] \} \ \leq \ \sup \{ f_2(x) \ / \ x \in [x_{k-1}, x_k] \}$$

Therefore,

$$S^+(P, f_1, g) \leq S^+(P, f_2, g), \text{ then}$$

$$\text{Inf } S^{+}(P, f_1, g) \leq \inf S^{+}(P, f_2, g)$$

So

$$\overline{\int_a^b} f\, dg_1 \quad \leq \overline{\int_a^b} f\, dg_2,$$

[Since $f_1 \in R(g)$ *on* $[a, b]$, $f_2 \in R(g)$ *on* $[a, b]$,

[But we know $\int_a^b f_1 dg = \overline{\int_a^b} f\, dg_1$ and $\int_a^b f_2\, dg = \overline{\int_a^b} f\, dg_2$].

Hence $$\int_a^b f_1 dg \leq \int_a^b f_2 dg.$$

Proposition 1.54 [9]

If $f \in R(g)$ on $[a, b]$, then

$$\left|\int_a^b f dg\right| \leq \int_a^b |f| dg .$$

Proof:

By (1.47) we have $|f| \in \mathrm{R}(g)$.

Now for all $x \in [a, b]$ then $f(x) \leq |f(x)|$,

So by theorem (1.52)

$$\int_a^b f dg \leq \int_a^b |f(x)|\, dg$$

And $-\int_a^b f dg = \int_a^b -f dg \leq \int_a^b |-f(x)|\, dg = \int_a^b |f(x)|\, dg$

So

$$-\int_a^b |f(x)|\, dg \leq \int_a^b f d\, g \leq \int_a^b |f(x)|\, dg.$$

Then

$$\left|\int_a^b f d\, g\right| \leq \int_a^b |f| dg.$$

CHAPTER 2

Some Integral Inequalities

,

2.1 Inequalities for function of bounded variation

Lemma 2.1 [50]

Let $f: [a,b] \to$ R be a continuous function on $[a,b]$, and g is of bounded variation on $[a, b]$, then

$$\left| \int_a^b f(t) d\, g(t) \right| \leq \underset{a \leq t \leq b}{\text{Max}} \left| f(t) \right| \bigvee_a^b g. \qquad (1.2)$$

Proof:

Let $\Delta_n : a = x_0{}^{(n)} < x_1{}^{(n)} < \ldots\ldots < x_{n-1}{}^{(n)} < x_n{}^{(n)} = b$

be a sequence of partitions of $[a,b]$, such that $\text{V}(\Delta_n) \to 0$ as $n \to \infty$,

where $\text{V}(\Delta_n) = \underset{a \leq t \leq b}{Max} \{h_{\text{i}}{}^{(\text{n})}\}$, with $h_i{}^{(n)} = x_{i+1}{}^{(n)} - x_i{}^{(n)}$,

and if $\xi_i{}^{(n)} \in [x_i{}^{(n)}, x_{i+1}{}^{(n)}]$ for $i \in \{0, 1, \ldots, n-1\}$, then

$$\left| \int_{\text{a}}^{\text{b}} f(\text{t})\, dg(\text{t}) \right| = \left| \lim_{\text{V}(\Delta_n) \to 0} \sum_{\text{i}=1}^{\text{n}-1} f\,(\xi_i{}^{(n)})\, [g\,(x_{i+1}{}^{(n)}) - g\,(x_i{}^{(n)})] \right|$$

$$\leq \lim_{\text{V}(\Delta_n) \to 0} \sum_{\text{i}=0}^{\text{n}-1} \left| f(\xi_i{}^{(n)}) \right| \left| g\,(x_{i+1}{}^{(n)}) - g\,(x_i{}^{(n)}) \right|$$

$$\leq \underset{a \leq t \leq b}{Max} \left| f \right| \bigvee_a^b g$$

Where $\bigvee_a^b g = \sup \sum_{\text{i}=1}^{\text{n}} \left| g\,(x_{i+1}{}^{(n)}) - g\,(x_i{}^{(n)}) \right|$.

Theorem 2.2 [31]

Let $f : [a,b] \to R$ be a function of bounded variation, then

$$\left| \int_{\text{a}}^{\text{b}} f(t)dt - [\,(x-a)\, f(a) + (b - x)\, f(b)] \right|$$

$$\leq [\tfrac{1}{2}\,(b - a) + | x - \tfrac{a+b}{2} \big|] \bigvee_a^b f, \qquad (2.2)$$

for $x, t \in [a,b]$.

Proof

Using the integration by parts formula for Riemann-Stieltjes integral,

$$\int_a^b (x-t)df(t) = (x-t)\,f(t)\,\Big|_a^b + \int_a^b f(t)\,dt$$

$$= (x-b)\,f(b) - (x-a)\,f(a) + \int_a^b f(t)\,dt \qquad (3.2)$$

(by lemma 2.1)

$$\left|\int_a^b f(t)dt - [(x-a)\,f(a) + (b-x)f(b)]\right| = \left|\int_a^b (x-t)\,df(t)\right|$$

$$\leq \sup_{a\leq t\leq b} |x-t| \bigvee_a^b f,$$

$$\sup_{a\leq t\leq b} |x-t| = \text{Max}\{x-a, b-x\},$$

From proposition 1.2

$$\text{Max}\{x-a, b-x\} = \frac{1}{2}[b-a] + \left|x - \frac{a+b}{2}\right| \qquad (4.2)$$

Then by (3.2), (4.2)

$$\left|\int_a^b f(t)\,dt - [(x-a)\,f(a) + (b-x)\,f(b)]\right|$$

$$\leq \left[\frac{1}{2}[b-a] + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b f$$

To prove that $\frac{1}{2}$ is the best possible suppose that (2.2) holds with constant C >0.

$$\left|\int_a^b f dt - [f(b)(b-x) + f(a)(x-a)]\right|$$

$$\leq \left[c\,[b-a] + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b f, \quad x \in [a, b]$$

If let $x = \frac{a+b}{2}$, we get

$$\left|\int_a^b f dt - \frac{f(a)+f(b)}{2}\,(b-a)\right| \leq c\,(b-a) \bigvee_a^b f.$$

Consider the function $f : [a, b] \to R$ by

$$f(x) = \begin{cases} 0 & \text{if } x = a, \\ 1 & \text{if } x \in (a, b), \\ 0 & \text{if } x = b, \end{cases}$$

then f is of bounded variation and let $P = \{a = x_0,\ x_1,\ x_{2,}\ \dots, x_n = b\}$, any partition of $[a, b]$, then

upper and lower Darboux - Stieltjes integral is defined as

$$\overline{\int_a^b} f\,dx = \inf\ S^{+}(P, f, x) = \inf \sum_{i=2}^{n-1} M_{\mathrm{i}}\, \Delta_{\mathrm{i}} x = \sum_{i=2}^{n-1} (1)\, \Delta_{\mathrm{i}} x = b - a$$

$$\underline{\int_a^b} f dx = \sup S^{-}(P,\ f, x) = \sup \sum_{i=2}^{n-1} m_{\mathrm{i}}\, \Delta_{\mathrm{i}} x = \sum_{i=2}^{n-1} (1)\, \Delta_{\mathrm{i}} x = b - a$$

So $\int_{\mathrm{a}}^{\mathrm{b}} f(x) dx = b - a$, and

$$\sum_{i=0}^{n} | f(x_{i+1}) - f(x_i) | = |1 - 0| + |1 - 1| + \dots + |1 - 1| + |0 - 1|$$

$$= 1 + 0 + 0 + \dots + 0 + 1 = 2$$

Then

$$\bigvee_a^b f = 2$$

Hence, from inequality (2.2) applied for this particular mapping we have

$$(b - a) \leq 2\, C\, (b - a)$$

Which we get $c \leq \frac{1}{2}$ and from this showing that $\frac{1}{2}$ is the best constant in (2.2).

Corollary 2.3

If we choose $x = \frac{a+b}{2}$ in (2.2), we obtain for the trapezoid formula for function of bounded variation;

$$\left| \int_{\mathrm{a}}^{\mathrm{b}} f dt - \frac{f(a) + f(b)}{2} (b - a) \right| \leq \frac{1}{2} (b - a) \bigvee_a^b f.$$

Corollary 2.4 [20]

Let $f: [a,\ b] \ \rightarrow\ R$ be a monotonic nondecreasing, Then

$$\left| \int_a^b f\,(t)dt - [(x-a)\,f(a) + (b-x)\,f(b)] \right|$$

$$\leq (b-x)\,f(b)\ - (x-a)\,f(a)\ + \int_a^b sgn\,(xt)\,f dt$$

$$\leq (x-a)\,[\,f(x) - \ f(a)]\ + (b-x)\,[\,f(a) - \ f(x)]$$

$$\leq\ [\tfrac{1}{2}(b - a) + \left| x - \tfrac{a+b}{2} \right|][f(b) - f(a)],\quad \text{for}\quad x \in [a, b]$$

Proof:

Applying the inequality $\left|\int_a^b f dg\right| \leq \int_a^b \left| f \right| dg$, then

$$\left| \int_a^b (x-t\,)df(t) \right|\ \leq \int_a^b \left| x-t\ \right| df(t)$$

$$= \int_a^x (x-t)df + \int_x^b (t-x)\ df$$

$$= (x-t)\ f(t)\ \Big|_a^x + \int_a^x f(t)dt + (t-x)\ f(t)\Big|_t^b + \int_x^b f(t)dt$$

$$= -\ (x-a)\ f\ (a) + \int_a^x f(t)dt + (b-x)\ f\ (b) - \int_x^b f(t)dt$$

$$=\ (b-x)f(b) - \ (x-a)f(a) + \int_a^b sgn\,(x-t)f(t)dt,$$

f is monotonic nondecreasing on $[a,b]$, then f is bounded variation and

$$f(a)\ \leq\ f(t)\quad \text{for}\quad t\ \in\ [a,b].$$

So

$$\int_a^x f(a)\,dt\quad \leq\ \int_a^x f\,(t)\ dt$$

Implies to $\qquad (x - a)\ f(a) \geq\ - \int_a^x f\ (t)dt$

And if $\quad b\ \geq\ t\quad$ for $\quad t\ \in\ [a,\ b],\quad$ then

$$f(b)\ \geq\ f(t)$$

$$\int_x^b f\,(b)dt \;\geq\; \int_x^b f\,(t)\,dt$$

$$(b - x)\,f(b) \;\geq\; \int_x^b f\,(t)\,dt$$

Therefor

$$\int_a^b sgn\,(x-t)f(t)dt = \int_a^x f dt - \int_x^b f dt$$

$$\leq (x - a)\,f(x) + \;(x - b)\,f(x).$$

Then

$$(b - x)\,f(b) - (x - a)\,f(a) \;+ \int_a^b sgn\,(x - t)\,f dt$$

$$\leq \left(b - x\right)f(b) - \left(x - a\right)f(a) + \left(x - a\right)f(x)$$

$$+(x - b)\,f(x)$$

$$= (x - a)\,[f(x) - f(a)] + (b - x)\,[f(b) - f(x)],$$

But

$f(a) \;\leq\; f(x) \;\leq\; f(b)$ for all $x \in [a, b]$,

so

$$\left(x - a\right)[f\,(x) - f\,(a)] + (b-x)\,[f\,(b) - f(x)]$$

$$\leq \text{ Max }\{x - a, b - x\}\left[f(x) - f(a) + \; f(b) - f(x)\right]$$

$$= [\,\tfrac{1}{2}\,[\,b - a] + \left|x - \tfrac{a+b}{2}\right|\,]\left[f(b) - f(a)\right].$$

Corollary 2.5

If we choose $x = \frac{a+b}{2}$ in 2.4, then

$$\left|\,\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\;(b-a)\right| \leq \frac{1}{2}\;(b - a)\,[f(b) - f(a)].$$

Theorem 2.6 [19]

[Ostrowski for mapping of bounded variation]

Let $f : [a, b] \rightarrow R$ be a mapping of bounded variation on $[a, b]$, Then

$$\left| \int_a^b f(t)\,dt - (b-a) f(x) \right| \leq \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b f \qquad (5.2)$$

(The constant $^1/_2$ is the best possible one)

Proof:-

by the integration by parts for Riemann - Stieltjes integrals , we have

$$\int_a^x (t-a)\,df(t) = f(x)(x-a) - \int_a^x f(t)dt \qquad (6.2)$$

and

$$\int_x^b (t-b)\,df(t) = f(x)(b-x) - \int_x^b f(t)dt \qquad (7.2)$$

By add the above two equalities then

$$(b-a) f(x) - \int_a^b f(t)dt = \int_a^b K(x, t)\,df(t)$$

Where

$$K(x, t) = \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in [x, b] \end{cases}, \qquad \text{for } t, x \in [a, b].$$

And we know $\left| K(x, t)\,df(t) \right| \leq \sup_{a \leq t \leq b} \left| K(x,t) \right| \bigvee_a^b f,$

$$= \text{Max}\,\{(x-a), (b-x)\} \bigvee_a^b f$$

$$= \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b f$$

Therefor

$$\left| \int_a^b f(t)\,dt - (b-a) f(x) \right| \leq \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b f.$$

Now to prove that $\frac{1}{2}$ is the best possible constant assume that the inequality (5.2) holds with a constant c > 0 that is,

$$\left| \int_a^b f(t)\, dt - (b-a) f(x) \right| \leq \left[c\,(b-a) + \left| x - \frac{\boldsymbol{a+b}}{\boldsymbol{2}} \right| \right] \bigvee_a^b f \qquad (8.2)$$

Consider the mapping $f : [a,b] \to R$ given by

$$f(x) = \begin{cases} 0 \text{ if } x \neq \frac{a+b}{2} \\ 1 \text{ if } x = \frac{a+b}{2} \end{cases}$$

then

$$\bigvee_a^b f = \sup \{ \sum_{i=1}^n \left| f(x_i) - f(x_{i+1}) \right| \} = |-1| + |1| = 2$$

And $\int_a^b f(t)dt = 0$

If we letting $x = \frac{a+b}{2}$ in (2.8) we get,

$$| 0 - 1(b-a)| \leq 2\left[c\left(b-a\right) + 0 \right]$$

$$1\,(b-a)\,| \leq 2\,[\,c\,(b-a)$$

$$1 \leq c\,2$$

$$\frac{1}{2} \leq c$$

Hence c = $\frac{1}{2}$ is the best possible constant

Corollary 2.7 [29]

(1) If we choose $x = \frac{a+b}{2}$ we get the following inequality which is well known in the literature as the midpoint inequality

$$\left| \int_a^b f dt - (b-a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{2}(b-a) \bigvee_a^b f.$$

(2) If f is a monotonic mapping on $[a, b]$, then

$$\left|\int_a^b f\,(\mathrm{t})\,dt-(b-a)\,f(x)\right|\le\left[\tfrac{1}{2}(b-a)+\left|x-\tfrac{a+b}{2}\right|\right]\left|f(b)-f(a)\right|.$$

Example 2.8

If $f:[a,\ b]\ \rightarrow\ R$ be a monotonic nondecreasing mapping on $[a\ ,b]$.

Let $P=\ \{x_\circ,x_1\ ,\dots\ ,x_n\}$ be a division of the interval $[a,b]$, and if

$\xi_i\in[\,x_i,\ x_{i+1}]$ for $i\ =\ 1,2,3,\dots,n$, then

$$\int_{\mathrm{a}}^{\mathrm{b}} f(x)dx=\mathcal{R}_n\,(f,\ I_n,\ \xi)+\mathcal{W}_n(f,\ I_n,\xi)$$

Where

$$\mathcal{R}_n\,(f,\ I_n,\xi)\ =\sum_{\mathrm{i=0}}^{\mathrm{n-1}}\ f(\xi_i)\ h_i$$

And there mainder satisfies the estimation

$$\left|\ \mathcal{W}_n(f,\ I_n,\xi)\ \right|\le\nu\,(h)\Big(f(b)-\ f(a)\Big),\ \text{Where}\ \ \nu\,(h)=\ \underset{0\le i\le n}{\mathrm{Max}}\{h_i\}.$$

Proof:

Apply (2.9) on the interval $[\,x_i,\ x_{i+1}]$, to get

$$\left|\int_{x_i}^{x^{i+1}} f(x)\,dx-f(\xi_i)\,h_i\right|\ \le\ [½h_i+\left|\xi_i\ \ -\tfrac{x_{i\,+}x_{i+1}}{2}\right|]\,(f\ (x_{i+1})-\ f\ (x_i))$$

Summing over i from 0 to $n\ -1$

$$\left|\ \mathcal{W}_n(f,\ I_n,\xi)\ \ \right|\le\ \sum_{\mathrm{i=0}}^{\mathrm{n-1}}\ \left|\int_{x_i}^{x^{i+1}} f(x)dx-f(\xi_i\,)\,h_i\right|$$

$$\le\ \sum_{\mathrm{i=0}}^{\mathrm{n-1}}\ [½h_i+\left|\xi_i\ \ -\tfrac{x_{i\,+}x_{i+1}}{2}\right|]\,(f\ (x_{i+1})-\ f\ (x_i))$$

$$\le\underset{o\le i\le n}{\mathrm{Max}}\ \{[½h_i\ +\left|\xi_i\ \ -\tfrac{x_{i\,+}x_{i+1}}{2}\right|]\sum_{\mathrm{i=0}}^{\mathrm{n-1}}(f\ (x_{i+1})-\ f\ (x_i))$$

But,

$\underset{o\le i\le n}{\mathrm{Max}}$ ½ h_i= ½ ν(h), and if $x_i\le\xi_i\le\ x_{i+1}$, then

$$x_i - \frac{x_i + x_{i+1}}{2} \leq \xi_i - \frac{x_i + x_{i+1}}{2} \leq x_{1+i} - \frac{x_i + x_{i+1}}{2}$$

$$\frac{x_i - x_{i+1}}{2} \leq \xi_i - \frac{\mathrm{x_i + x_{i+1}}}{2} \leq \frac{x_{i+1} - x_i}{2}$$

$$- \tfrac{1}{2}\,(x_{i+1} - x_i\,) \leq \xi_i - \frac{x_i + x_{i+1}}{2} \leq \tfrac{1}{2}\,(x_{i+1} - x_i)$$

$$- (\tfrac{1}{2} h_i) \leq \xi_i - \frac{x_i + x_{i+1}}{2} \leq \tfrac{1}{2}\, h_i$$

Then by triangle inequality

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq \tfrac{1}{2}\, h_i$$

So

$$\left| \mathcal{W}_n\,(f,\, I_n, \xi) \right| \leq [\tfrac{1}{2}\, \nu\,(h) + \operatorname*{Max}_{o \leq i \leq n} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right|]\,(f(b) - f(a))$$

$$= \nu\,(h)\left(f\,(b) - f\,(a)\right).$$

2.2 Inequalities for Lipschitzian functions

Proposition 2.9 [28]

If $\boldsymbol{f}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{g}$ is lipschitzian with $\boldsymbol{L} > 0$, Then

$$\left| \int_a^b f dg \right| \leq L \int_a^b |f| dt \qquad (9.2)$$

Proof

Let $P = \{ x_0, x_1, \dots, x_n\}$ is a partitions of $[a, b]$ and $\xi_i \in [x_i, x_{i+1}]$,

if $V(P) = \underset{o \leq i \leq n-1}{\text{Max}} h_i$ where $h_i = x_{i+1} - x_i$ then

$$\left| \int_a^b f dg \right| = \left| \lim_{V(P) \to 0} \sum_{i=1}^{n-1} f(\xi_i) \; [g(x_{i+1}) - g(x_i)] \right|$$

$$\leq \lim_{V(P) \to 0} \sum_{1=1}^{n-1} |f(\xi_i)| \, |g(x_{i+1}) - g(x_i)|$$

$$\leq \lim_{V(P) \to 0} \sum_{1=1}^{n-1} |f(\xi_i)| \; L \, |x_{i+1} - x_i|$$

$$= L \lim_{V(P) \to 0} \sum_{1=1}^{n-1} |f(\xi_i)| \, |x_{i+1} - x_i| = L \int_a^b |f| \, \text{dt}.$$

Theorem 2.10 [20]

If f:$[a, b] \to R$ is Lipschitzian with $L > 0$, then

$$\left| \int_a^b f(t) dt - [\, f(a)\,(x-a) + f(b)\,(b-x)] \right|$$

$$\leq L \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right], \quad \text{for } x, t \in [a, b]. \qquad (10.2)$$

(The constant $\frac{1}{4}$ is the best possible one).

Proof:

From 2.9,

$$\int_a^b (x-t)\,df(t) \;=\; \int_a^b f(t)dt \,-(x-a)f(a)\,+(b-x)\,f(b)$$

$$\left|\int_a^b (x-t)\,df(t)\right| \leq \quad L\int_a^b |x-t|\,dt$$

$$= L\,[\int_a^x (x-t)dt + \int_x^b (t-x)dt\,]$$

$$= L\,[\frac{(x-a)^2+(b-x)^2}{2}\,]$$

$$= L\,[\,\frac{x^2-2ax+a^2}{2} + \frac{b^2-2bx+x^2}{2}\,]$$

$$= L\,[\,x^2 - a\,x + \frac{a^2}{2} + \frac{b^2}{2} - b\,x + \frac{x^2}{2}\,]$$

$$= L\,[\,x^2 - a\,x - b\,x + \frac{b^2}{2} + \frac{a^2}{2}\,]$$

$$= L\,[\,x^2 - a\,x - b\,x + \frac{b^2}{4} + \frac{b^2}{4} + \frac{a^2}{4} + \frac{a^2}{4} - \frac{a\,b}{2} + \frac{a\,b}{2}\,]$$

$$= L\,[(\frac{b^2}{4} - \frac{a\,b}{2} + \frac{b^2}{4}) + (\,x^2 - a\,x - b\,x + \frac{a^2}{4} + \frac{b^2}{4} + \frac{a\,b}{2}\,)$$

$$= L\,[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2}\,)^2]$$

$$= L\,[\,\frac{1}{4}\,(b-a)^2 + (x - \frac{a+b}{2})^2].$$

To prove that $\frac{1}{4}$ is the best possible constant assume that the inequality (10.2) holds with a constant c > 0 that is,

$$\left|\int_a^b f(t)dt - [\,f(a)\,(x-a) + f(b)\,(b-x)]\right|$$

$$\leq L\ [\ c(b-a)^2 + (x - \frac{a+b}{2})^2], \quad \text{For} \quad x \in [a,b]$$

If $x = \frac{a+b}{2}$, we get

$$\left| \int_a^b f dt - \frac{f(a)+f(b)}{2}\ (b-a) \right| \leq c\, L\ (b-a)^2 \qquad (11.2)$$

Consider the function $f : [a,b] \to R$ by

$$f(t) = \begin{cases} t+1 & \text{if} \quad t \in [-1,\ 0], \\ 1-t & \text{if} \quad t \in (0,\ 1] \end{cases}$$

Now, to show that f is Lipschitzian with $L = 1$,

If $x, y \in [0,1]$, then

$$|f(x) - f(y)| = |(1-x) - (1-y)| = |y-x| = |x-y|,$$

If $x, y \in [-1, 0)$, then

$$|f(x) - f(y)| = |x-y|,$$

If $x \in [0,1]$ and $y \in [-1,\ 0)$, then

$$|f(x) - f(y)| = \big|(1-x) - (y+1) = |-(x+y)|\big| = |x+y|,$$

but $y \leq 0$ then $y \leq -y$, so

$$x + y \leq x - y \geq 0,$$

And $x \geq 0$ so $-x \leq x$, then

$$-x + y \leq x + y$$

$$-(x-y) \leq x + y$$

Therefor

$$-(x-y) \leq x + y \leq x - y, \quad \text{Then}$$

$$|x+y| \leq |x-y|,$$

So $$|f(x)-f(y)| = |x+y| \leq |x-y|,$$

Hence f is Lipschitzian with $L = 1$.

We have $f(1) = f(-1) = 0$, from (11.2)

$$\left| \int_{-1}^{1} f(t)dt - \frac{f(1)+f(-1)}{2}\,(1-(-1)) \right| \leq c\,(1)\,(1-(-1))^2,$$

$$\int_{-1}^{1} f(t)dt = \int_{-1}^{0}(t+1)dt + \int_{0}^{1}(1-t)dt$$

$$= \left(\frac{t^2}{2}+t\right)\Big|_{-1}^{0} + \left(t-\frac{t^2}{2}\right)\Big|_{0}^{1} = \frac{1}{2}+\frac{1}{2} = 1.$$

Then

$$\left| 1-(0)\,(2) \right| \leq 4c \qquad c \geq \frac{1}{4}$$

Hence $c = \frac{1}{4}$ is the best possible constant.

Remark 2.11 [20]

If we choose $x = \frac{a+b}{2}$, then the trapezoid formula for Lipschitzian function, as

$$\left| \int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{4}(b-a)^2 L.$$

Theorem 2.12 [28] [Ostrowski for Lipschitzian function]

If $f : [a,b] \to R$ be an L– Lipschitzian function on $[a,b]$ then

$$\left| \int_a^b f dt - (b-a)f(x) \right| \leq L\left[\frac{(b-a)^2}{4} + \left(x-\frac{a+b}{2}\right)^2\right] \text{ for } x \in [a,b] \qquad (12.2)$$

And the Constant $\frac{1}{4}$ is the best possible one.

Proof:

Consider the function

$$K\ (x,\ t) = \begin{cases} t-a & \text{if } t \in [a\ ,\ x] \\ t-b & \text{if } t \in [x\ ,\ b] \end{cases},$$

$$\int_a^b K(x,\ t)\ df(t) = \int_a^x (t-a)\ df(t) + \int_x^b (t-b)\ df(t)$$

But,

$$\int_a^x (t-a)\ df(t) = f\ (x)\ (x - a) - \int_a^x f(t)\ dt$$

And

$$\int_x^b (t-b)\ df(t) = f(x)\ (b - x) - \int_x^b f(t)\ dt$$

So

$$\int_a^b K\ (x,\ t)\ df\ (t) = (b - a)\ f(x)\ (b - x) - \int_a^b f(t)\ d\ t$$

And

$$\left| \int_a^b K(x,\ t)\ df(t) \right| = \left| (b - a)\ f(x) - \int_a^b f\ (t)\ d\ t \right|$$

$$\leq L \left[\int_a^x |t-a|\,dt + \int_x^b |t-b|\ dt \right]$$

$$= L\left[\int_a^x |t-a|\,dt + \int_x^b |t-b|\ dt \right]$$

$$= L\ \left[\frac{(x-a)^2}{2} + \frac{(b-x)^2}{2}\right]$$

Then

$$\left| \int_a^b f dt - (b - a) f(x) \right| \leq \quad L\ \left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2\right].$$

To show the sharpness of the inequality with the constant$\frac{1}{4}$.

Consider the mapping, $f\colon [a,\ b] \to R, \quad f(x) = x$

Then f is lipschitzian with $L \geq 1$, ($|f(x) - f(y)| = |x - y| \leq L|x - y|$),

So

$$\left|x - \frac{a+b}{2}\right| \leq c(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2, \text{ for } x \in [a, b]$$

If $x = a$, we get

$$\frac{b-a}{2} \leq \left(c + \frac{1}{4}\right)(b - a)$$

$$\frac{1}{2} \leq c + \frac{1}{4}$$

Then, $c \geq \frac{1}{4}$,

Therefor $c = \frac{1}{4}$.

Corollary 2.13

Let f; $[a\ ,b] \rightarrow R$ be as theorem (2.12) then by letting x = $\frac{a+b}{2}$, we obtain on the midpoint inequality;

$$\left| \int_a^b f\mathrm{dt} - (b - a) f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4} L\ (b - a)^2.$$

2.3 Inequalities for differentiable and twice differentiable functions.

Lemma 2.14 (Gruss type inequality) [33]

(i) Let h , g $[a,b] \to$ R be two integrable mappings so that

$\varphi \le h(x) \le \phi$ and $n \le g(x) \le m$ for $x \in [a,\ b]$,where φ, ϕ, m, n are real numbers, then

$$\left| \frac{1}{b-a}\int_{\mathrm{a}}^{\mathrm{b}} hg\, dx - \frac{1}{b-a}\int_{\mathrm{a}}^{\mathrm{b}} h\, dx \,.\, \frac{1}{b-a}\int_{\mathrm{a}}^{\mathrm{b}} g\, dx \right|$$

$$\le \frac{1}{4}(\phi - \varphi)(m - n).$$

(ii) $$\left| \frac{1}{b-a}\int_{a}^{\mathrm{b}} h(x)g(x)dx - \frac{1}{b-a}\int_{a}^{\mathrm{b}} h(x)\, dx \,.\frac{1}{b-a}\int_{a}^{\mathrm{b}} g(x)dx \right|$$

$$\le \frac{1}{b-a}\int_{a}^{\mathrm{b}} \left| \left(h(x) - \frac{1}{b-a}\int_{a}^{\mathrm{b}} h(y)\, dy\right) - \left(g(x) - \frac{1}{b-a}\int_{a}^{\mathrm{b}} g(y)\, dy\right) \right| dx.$$

Theorem 2.15 [20]

Let $f : [a,b] \to R$ be differentiable function on $[a,\ b]$ have the first derivative $f : [a,b] \to R$ bounded on $[a,b]$. Then,

$$\left|\int_{a}^{b} f(x)\, dx - \frac{f(a) + f(b)}{2}(b - a)\right| \le \frac{(b-a)^2}{4} \underset{a \le t \le b}{Max} \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right|$$

Proof

By modification of the integral and integration by parts gives that:

Let $g = \left(x - \frac{a+b}{2}\right)$

$$\int_a^b \left(x - \frac{a+b}{2}\right) f' d\,x = \int_a^b \left(x - \frac{a+b}{2}\right) df = \int_a^b g df$$

$$= f(b)g\,(b) - f\,(a)g\,(a) - \int_a^b f dg$$

$$= f(b)\left(\frac{b-a}{2}\right) - f(a)\left(\frac{a-b}{2}\right) - \int_a^b f dx$$

$$= \frac{f(a) + f(b)}{2}(b - a) - \int_a^b f dx$$

So

$$\int_a^b \left(x - \frac{a+b}{2}\right) f' d\, x = \frac{f(a) + f(b)}{2}(b - a) - \int_a^b f dx, \qquad (13.2)$$

Now applying the inequality in lemma (2.14) we find that

$$\left| \frac{1}{b-a}\int_a^{\mathrm{b}} \left(x - \frac{a+b}{2}\right) f'(x) dx - \frac{1}{b-a}\int_a^{\mathrm{b}} (x - \frac{a+b}{2})\, dx \,. \frac{1}{b-a}\int_a^{\mathrm{b}} f'(x) dx \right|$$

$$\leq \frac{1}{b-a}\int_a^{\mathrm{b}} \left| \left(\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^{\mathrm{b}} (y - \frac{a+b}{2}) dy\right) \right.$$

$$\left. - (f'(x) - \frac{1}{b-a}\int_a^{\mathrm{b}} (f'(y)\, dy) \right| dx.$$

As

$$\int_a^b \left(x - \frac{a+b}{2}\right) dx = \left(\frac{x^2}{2} - \left(\frac{a+b}{2}\right) x\right) \Big|_a^b = \left[\frac{b^2}{2} - \left(\frac{ab+b^2}{2}\right)\right] - \left[\frac{a^2}{2} - \frac{ab+a^2}{2}\right]$$

$$= \frac{b^2-a^2}{2} + \frac{a^2-b^2}{2} = 0,$$ then

$$\left| \int_a^b \left(x - \frac{a+b}{2}\right) f' d\, x\right| \leq \int_a^b \left| \left(x - \frac{a+b}{2}\right)\left(f'(x) - \frac{f(b)-f(a)}{b-a}\right) \right| dx.$$

$$\leq \underset{a\leq x\leq b}{\mathrm{Max}} \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| \int_a^b \left| \left(x - \frac{a+b}{2}\right) \right| dx$$

$$= \frac{(b-a)^2}{4} \underset{a\leq x\leq b}{\mathrm{Max}} \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right|,$$

By (13.2) we can say:

$$\left| \int_a^b f(x)\, dx - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{(b-a)^2}{4} \underset{a\leq x\leq b}{\mathrm{Max}} \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right|.$$

Corollary 2.16 [33]

If f' is integrable on $[a, b]$, then

$$\left| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{b-a}{2} \int_a^b \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| dx.$$

Remark 2.17 [31]

If $f \in C^{(1)}[a,b]$, then

$$\left| \int_a^b f dt - \frac{b-a}{2} \left(f(a) + f(b) \right) \right| \leq {}^1\!/_2 (b-a) \|f'\|_1, \quad \text{for } x \in [a, b],$$

where $\| \ \|_1$ is the L_1 – norm, namely $\|f'\|_1 = \int_a^b |f'| \ dt$.

Proof:

$$\int_a^b (x-t)df(t) = \int_a^b (x-t)f'dt$$

$$\leq (x-t) \int_a^b \left| f'(t) \right| dt$$

$$= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1$$

If $x = \frac{a+b}{2}$

$$\text{then} \left| \int_a^b f dt - \frac{f(a)+f(b)}{2} \right| \leq \frac{b-a}{2} \|f'\|_1.$$

Theorem 2.18 [5]

Let $f : [a,b] \to R$ be differentiable function on $[a, b]$, have the bounded first derivative on (a, b). Then,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2}}{(b-a)^2} \right](b-a) \right] \|f'\|_\infty$$

Where $f' : (a,b) \to R$ is bounded, and $\|f'\|_\infty = \sup \left| f'(x) \right| < \infty$

Proof

Using integration by parts, and define $K\ (x,\ t)$ by

$$K\ (x,\ t) = \begin{cases} t-a & \text{if } t \in [a,\ x] \\ t-b & \text{if } t \in [x\ ,\ b] \end{cases}$$

$$\int_a^b K\ (x,\ t)\ df\ = \int_a^x (t-a)\ d\,f\ \ + \int_x^b (t-b)\ df.$$

by theorem (1.38)

$$\int_a^x (t-a)\ d\,f\ + \int_a^x f dt = f\ (x)\ (x - a)\ +\ f\ (a)\ (a - a) = f\ (x)\ (x - a)$$

So, $\int_a^x (t-a)\ d\,f\ = f\ (x)\ (x - a) - \int_a^x f dt$

And similarly for

$$\int_{\mathrm{x}}^{\mathrm{b}} (t-b)\ d\,f\ = f\ (x)\ (b - x) - \int_x^b f dt$$

Therefore

$$\int_{\mathrm{a}}^{\mathrm{b}} K\ (x,\ t)\ df\ = f\ (x)\ (b - a) - \int_{\mathrm{a}}^{\mathrm{b}} f\ (t)\ dt, \text{ then}$$

$$\left| f\ (x)\ (b - a) - \int_a^b f\ (t)\ dt \right| = \left| \int_{\mathrm{a}}^{\mathrm{b}} K\ (x,\ t)\ df\ \right| = \left| \int_{\mathrm{a}}^{\mathrm{b}} K\ (x,\ t)\ f' dt \right|$$

Hence

$$\left| f\ (x) - \frac{1}{b-a} \int_{\mathrm{a}}^{\mathrm{b}} f\ (t)\ dt \right| \leq \frac{1}{b-a} \left[\int_a^b \left| K(x\ ,t) \right| \left| f' \right| dt\ \right].$$

$$\leq \frac{M}{b-a} \left[\int_a^x \left| x-a \right| dt + \int_x^b \left| x-b \right| dt \right]$$

Where $\|f'\|_\infty$= sup $\left|\ f'\,(x) \right|$ $=M$

$$\left| f\ (x) - \frac{1}{b-a} \int_{\mathrm{a}}^{\mathrm{b}} f\ (t)\ dt \right| \leq \frac{M}{b-a} \left[\int_a^x (x-a) dt + \int_x^b (b-x) dt \right]$$

$$= \frac{M}{b-a} \left[\frac{(x-a)^2}{2} + \frac{(b-x)^2}{2} \right]$$

by proof (2.10), we have

$$[\frac{(x-a)^2+(b-x)^2}{2(b-a)}] = [\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}](b-a)$$

Hence,

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt| \leq [\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}][b-a]\|f'\|_\infty$$

Remark 2.19 [19]

If f is continuous on $[a,b]$, and differentiable on (a,b), then

$$|f(x) - \frac{1}{\text{b}-\text{a}}\int_{\text{a}}^{\text{b}} f(t)dt| \leq [\frac{1}{2} + \frac{|x-\frac{a+b}{2}|}{b-a}]\|f'\|_1, \text{ for}$$

Where $\|f'\|_1 = \int_{\text{a}}^{\text{b}} |f'(t)|dt$

Proof:

Using the integration by parts formula for

$\int_a^x (t-a)\,f'(t)\,dt$ and $\int_x^b (t-b)\,f'(\text{t})\,dt$

So, $\int_a^x (t-a)\,f'(t)\,dt = \int_a^x (t-a)\,df(t)$

$$= (x-a)\,f(x) - (a-a)\,f(a) - \int_a^x f(t)\,\text{dt}$$

$$= (x-a)\,f(x) - \int_a^x f(t)\,dt$$

Similarly for $\int_x^b (t-b)\,f'(\text{t})\,dt$, then

$$\int_x^b (t-b)\,f'(t)\,dt = (b-x)f(x) - \int_x^b f(t)\,dt$$

If we add the above two equalities, we obtain

$$(b-a)f(x) - \int_a^b f(t)\,dt = \int_a^x (t-a)\,f'(t)\,dt + \int_x^b (t-b)\,f'(t)\,dt$$

$$= \int_a^b K(x,\ t)\,f'(t)\,dt$$

Where

$$K(x,\ t) = \begin{cases} t-a & \text{if } t \in [\,a\,, x\,] \\ t-b & \text{if } \ t \ \in [x\,, b\,] \end{cases},$$

So

$$\int_a^b K\,(x,\ t)\,f'(t)\,dt \ \leq \ \sup_{a\leq t\leq b} \big|\ K(x,\ t)\big| \ \int_a^b \ \big|f'\,(t)\big|\ dt$$

$$= \text{Max}\ \{x-a,\, b-x\}\ \|f'\|_1$$

$$= [\ \tfrac{b-a}{2} + \big|x - \tfrac{a-b}{2}\big|\ ]\ \|f'\|_1$$

Hence

$$\big|\,(\,b - a)\ f\ (x) - \int_a^b f\ (t)\ dt \leq [\ \frac{b-a}{2} + \big|x - \frac{a-b}{2}\big|\ ]\ \|f'\|_1.$$

Or $\big|f(x) - \frac{1}{\text{b}-\text{a}}\int_a^b f\,(t)dt\,\big| \leq \ [\ \frac{1}{2} + \frac{\big|x - \frac{a+b}{2}}{b-a}\big|\ ]\ \|f'\|_1$

Theorem 2.20 (the perturbed Ostrowski inequality) [32]

Let $f\ :[a\,,b] \ \rightarrow\ R$ be continuous on $[a, b]$, differentiable on (a, b) and whose the first derivative bounded on $(a\,,\,b)$, and $\|f'\|_{\infty} = \ \underset{a\leq tb}{\text{Max}}|f'\,(x)|$, then

$$\left|\int_a^b f\ (t)\ dt - \left[\,f\ (x)\ (1-\lambda) + \ \frac{f\ (a)+f(b)}{2}\lambda\right](b-a)\right|$$

$$\leq [\,\tfrac{1}{4}\ (b-a)^2[\ \lambda^2 + (\lambda-1)^2] + \left(x - \tfrac{a+b}{2}\right)^{\,2}\ ]\|f'\|_{\infty}$$

For all $\lambda \in [0,\ 1]$, and $x \in [a + \lambda\ \left(\frac{b\ -a}{2}\right),\ b - \lambda\frac{(b-a)}{2}]$

Proof

Let us define the mapping : $[a, b]^2 \rightarrow$ R given by

$$K(x,\ t)= \begin{cases} t-\left[a-\lambda\,(\frac{b-a}{2})\right], & t\in[a\,,x] \\ t-\left[b-\lambda\,(\ \frac{b-a}{2})\right]\,, & t\in(x\,,b] \end{cases}$$

Then by integrating by parts, we have

$$\int_a^b K(x,\ t)f'(t)\,dt$$

$$=\int_a^x (t-[a+\lambda\,(\frac{b-a}{2}))f'(t)\,dt+\ \int_x^b (t-[b-\lambda\,(\frac{b-a}{2})])\,f'dt$$

$$=(b-a)\,\lambda\frac{(f(a)+f(b)}{2}+\ (b-a)(1-\lambda)f\ (x)-\ \int_a^b f\ (t)dt$$

On the other hand

$$\left|\int_a^b K(x,\ t)f'(t)dt\right|\le\int_a^b|K(x,\ t)|\ |f'(t)|\,dt$$

$$\le\|f'\|_\infty\int_a^b|K\ (x,\ t)\ dt$$

$$=\|f'\|_\infty\ [\int_a^x\left|t-\left(a+\lambda\,.\frac{b-a}{2}\right)\right|dt+\int_x^b\left|t-\left(b-\lambda\,.\frac{b-a}{2}\right)\right|dt]\ =\ \|f'\|_\infty\,L$$

Now, to find L let us observe that

$$\int_b^r|t-q|\,dt=\ \int_p^q(q-t)dt+\ \int_q^r(t-q)\,dt\ =\ \frac{1}{2}\ [(q-p)^2+(r-q)^2$$

by proof 2.10 we have;

$$\frac{1}{2}\ [(q-p)^2+(r-q)^2\ ]\ =\ \frac{1}{4}\ (p-r)^2\ +\ (q-\frac{r+p}{2})^2,\quad \text{for } p\ \le q\ \le r$$

Then

$$\int_a^x\left|\ t-\left(a+\lambda\,.\ \frac{b-a}{2}\right)\right|dt$$

$$=\ \frac{1}{4}\ (x-a)^2+\ \left[\left(a+\lambda\,.\frac{b-a}{2}\right)-\ \frac{a+x}{2}\right]^2$$

And similarly for

$$\int_x^b \left|t-(b-\lambda\,.\frac{b-a}{2})\right|\,dt = \frac{1}{4}\,(b-x)^2+\left[\left(b-\lambda\,.\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2 \text{,so}$$

$$L=\frac{1}{2}\,\frac{(x-a)^2+(b-x)^2}{2}+\left(\lambda\,.\frac{b-a}{2}-\frac{x-a}{2}\right)^2+\left(\frac{b-x}{2}-\lambda\,(\frac{b-a}{2})\right)^2$$

$$=\frac{(b-a)^2}{4}\,[\lambda^2+(\lambda-1)^2]+\,(x-\frac{a+b}{2})^2.$$

Notation 2.21

(a) If we let $\lambda=0$ then we get Ostrowski's integrality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty$$

(b) If we choose $\lambda=1$ and $x=\frac{a+b}{2}$ then we get the trapezoid inequality:

$$\left|\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\,(b-a)\right|\le\frac{1}{4}\,(b-a)^2\|f'\|_\infty\,.$$

Theorem 2.22 [57]

Let $f\,:[a,b]\to$ R is a differentiable function on (a,b) such that $\gamma\le f'(t)\le\mu$ for $t,x\in[a\,,b]$, for some constants $\gamma\,,\mu\;\in R$, then

$$\left|(a-b)\left[\frac{\lambda}{2}\left(f(a)+f(b)\right)+(1-\lambda)f(x)-\frac{\mu-\gamma}{2}\,(1-\lambda)\left(x-\frac{a+b}{2}\right)\right]-\int_a^b f\,dt\right|$$

$$\le\frac{\mu-\gamma}{2}\left[\frac{(b-a)^2}{4}(\lambda^2+(1-\lambda)^2)+\left(x-\frac{a+b}{2}\right)^2\right],$$

Where $a+\lambda\left(\frac{b-a}{2}\right)\le x\le b-\lambda\left(\frac{b-a}{2}\right)$ and $\lambda=[0,\;1]$.

Proof:

Let us define the mapping

$$K(x\,,t)=\begin{cases} t-\left(a+\lambda\frac{b-a}{2}\right), t\in[a\,,x] \\ t-\left(b-\lambda\frac{b-a}{2}\right)\,, t\in(x\,,b] \end{cases}$$

Then

$$\int_a^b K(x,t)f'(t)dt=(b-a)\,[\tfrac{\lambda}{2}\,(f(a)+f(b)+(1-\lambda)f(x)]-\int_a^b f\,dt$$

We also

$$\int_a^b K\,(x\,,\ t)\,dt=(1-\lambda)(b-a)\left(x-\frac{a+b}{2}\right).$$

Let $c=\frac{\mu+\gamma}{2}$, then

$$\int_a^b K\,(x\,,t)|f'(t)-c|\,dt=(b-a)\,[\tfrac{\lambda}{2}\left(f(a)+f\,(b)+\ (1-\lambda)\,f\,(x)-\right.$$
$$\left.c\,(1-\lambda)\,\left(x-\ \frac{a+b}{2}\right)\right]-\ \int_a^b f\,(t)\,dt$$

And we know that

$$\left|\int_a^b K\,(x\,,\ t)[f'(t)-c|\,]\,dt\right|\ \leq\sup_{a\leq t\leq b}\ |f'(t)-c\,|\ \int_a^b|K(x,\ t)|dt \qquad (14.2)$$

$$\int_a^b K(x,\ t)dt=\ \frac{(b-a)^2}{4}\,[\lambda^2+\ (1-\lambda)^2]+\ (x-\frac{a+b}{2})^2\,,$$

and we have $\gamma\ \leq f'\leq\ \mu$ therefor

$$\gamma-\frac{\mu+\gamma}{2}\leq f'-\frac{\mu+\gamma}{2}\leq\ \mu-\ \frac{\mu+\gamma}{2}$$

$$-\left(\frac{\mu-\gamma}{2}\right)\leq f^{'}-\frac{\mu+\gamma}{2}\leq\frac{\mu-\gamma}{2},$$ then

$|f'(t)-c|\ \leq\ \frac{\mu-\gamma}{2}$, and Max $|f'(t)-c|\ \leq\ \frac{\mu-\gamma}{2}$ (15.2)

From (14.2) and (15.2), if follows that

$$\left|\int_a^b K(x,\ t)\left[f'(t)-\frac{\mu+\gamma}{2}\right]dt\right| \le \frac{\mu-\gamma}{2}\ [\frac{(b-a)^2}{4}\left(\lambda^2+\ (1-\ \lambda^2)+\right.$$
$$\left(x-\ \frac{a+b}{2}\right)^2\Big],$$

Then we get

$$\left|(a-b)\ \left[\frac{\lambda}{2}\left(f\ (a)+\ f\ (b)\right)+\ (1-\lambda)f\ (x)-\frac{\mu+\gamma}{2}\ (1-\lambda)\left(x-\ \frac{a+b}{2}\right)\right]-\right.$$
$$\left.\int_a^b f\ dt\right| \le \frac{\mu-\ \gamma}{2}\left[\frac{(b-a)^2}{4}\ (\lambda^2+\ (1-\lambda)^2)+\left(x-\ \frac{a+b}{2}\right)^2\right].$$

Theorem 2.23 [20]

Let $\boldsymbol{f}$: $[a, b] \to R$ be a twice differentiable mapping on $(\boldsymbol{a},\ \boldsymbol{b})$, then

$$\left|\int_a^b f dx - \frac{b-a}{2}\ [\ f(a)+f(b)]\le\right.$$
$$\begin{cases} \frac{\|f''\|_\infty}{12} \quad (b-a)^3 \quad \text{if } f'' \in L_\infty[a,b] \\ \frac{\|\ f''\|_1}{8} \quad (b-a)^2 \quad \text{if } f'' \ \in L_1\ [a,b] \end{cases}$$

Where $\|f'\|_{\infty}$ = sup $\left|f''(t)\right|$, and $\|\ f''\|_1 = \int_a^b |f''(\text{t})|\ dt.$

Proof:

From integrating by parts;

$$\int_a^b (x-a)\ (b-x)\ \ f''dx\ = [\ (x-a)\ (b-x)\ f'(x)\ \Big|_a^b$$
$$-\int_a^b [(a+b)-\ 2x]\ \ f''dx$$
$$= \int_a^b\ [2\ x-(a+b)\ \Big|_a^b - 2\int_a^b f\ (x)\ d\ x\ ,$$

So

$$\int_a^b f\ (x)dx\ =\ \frac{b-a}{2}\ [\ f(a)\ +\ f(b)]\ -\ \frac{1}{2}\int_a^b (x-a)\ (b-x)\ f''dx$$

Therefore

$$\left| \int_a^b f dx - \frac{b-a}{2} \left[f(a) + f(b) \right] \right|$$

$$\leq \frac{1}{2} \int_a^b (x-a)(b-x) \left| f''(x) \right| dx \qquad (16.2)$$

Let us observe that

$$\int_a^b (x-a)(b-x) \left| f''(x) \right| dx$$

$$\leq \sup_{a \leq t \leq b} \left| f''(t) \right| \int_a^b (x-a)(b-x)\, dx, \text{ but}$$

$$\int_a^b (x-a)(b-x) = \int_a^b (bx - x^2 - ab + ax)\, dx$$

$$= \left[\frac{x^2 b}{2} - \frac{x^3}{3} - ab\, x + \frac{ax^2}{2} \right]$$

$$= \left[\frac{b^3}{2} - \frac{b^3}{3} - 2ab + \frac{ab^2}{2} \right] - \left[\frac{a^2 b}{2} - \frac{a^3}{2} - a^2 b + \frac{a^3}{2} \right]$$

$$= \left[\frac{b^3}{6} - \frac{ab^2}{2} \right] - \left[\frac{a^3}{6} - \frac{b\, a^2}{2} \right]$$

$$= \frac{[b^3 - 3ab^2 + 3a^2 b - a^3]}{6}$$

$$= \frac{(b-a)^3}{6}.$$

So

$$\left| \int_a^b f dx - \frac{f(a) + f(b)}{2} (b-a) \right| \leq \frac{(b-a)^3}{12} \left\| f'' \right\|_\infty .$$

Now, from (16.2) and lemma (2.1) we can say that

$$\int_a^b (x-a)(b-x) \left| f''(x) \right| dx \leq \operatorname*{Max}_{a \leq t \leq b} (x-a)(b-x) \int_a^b |f''(t)|\, dt$$

$$= \operatorname*{Max}_{a \leq t \leq b} (x-a)(b-x) \| f'' \|_1$$

Let $h(x) = (x-a)(b-x)$, then

$$h'(x) = (-1)(x-a) + (b-x) = -2x + a + b$$

If $h'(x) = 0$ then $x = \frac{a+b}{2}$ and if $x \in \left(a, \frac{a+b}{2}\right)$ then $h'(x) \geq 0$ and if $x \in \left(\frac{a+b}{2}, b\right)$ then $h'(x) \leq 0$,

Therefor

$$\operatorname*{Max}_{a \leq t \leq b} h(x) = \left(\frac{a+b}{2} - a\right)\left(b - \frac{a+b}{2}\right) = \frac{(b-a)^2}{4},$$

$$\left| \int_a^b f dx - \frac{\mathrm{b-a}}{2}[f(a) + f(b)] \right| \leq \frac{1}{2}\int_a^b (x-a)(b-x)\left|f''(x)\right| dx$$

$$\leq \frac{1}{2}\left[\frac{(b-a)^2}{4} \|f''\|_1\right] = \frac{(b-a)^2}{8} \|f''\|_1].$$

Remark 2.24 (Hermite – Hadamard inequality) [46]

If f is a convex ($f'' \geq 0$) on $[a, b]$, the midpoint Rule is the approximation

$$\int_a^b f dt \cong f\left(\frac{a+b}{2}\right)[b-a],$$

And the trapezoid Rule is the approximation

$$\int_{\mathrm{a}}^{\mathrm{b}} f dt \cong \frac{f(\mathrm{a}) + f(\mathrm{b})}{2}[b-a],$$

There is a very useful relationship between these rules as follows,

$$f\left(\frac{a+b}{2}\right)[b-a] \leq \int_a^b f dt \leq \frac{f(\mathrm{a}) + f(\mathrm{b})}{2}[b-a]$$

Then by corollary (2.3) and corollary (2.7), we can say;

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f dt \leq \frac{1}{2} \bigvee_a^b f, \text{ and}$$

$$0 \leq \frac{1}{b-a}\int_a^b f dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \bigvee_a^b f,$$

2. 4 inequalities for absolutely continuous functions

Theorem 2.25 [27]

Let $f : [a,b] \to R$ be an absolutely continuous function on $[a,b]$, then

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f\ (t)dt\right| \leq \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,$$

for $x, t \in [a, \frac{a+b}{2}]$.

Proof:

Let us define the mapping

$$K(x,\ t) = \begin{cases} t-a, & t \in [a,x] \\ t-\frac{a+b}{2}, & t \in (x, a+b-x] \\ t-b, & t \in (a+b-x, b] \end{cases}$$

for $x \in [a, \frac{a+b}{2}]$

Integrating by parts

$$\int_a^x (t-a)df\ (t) = f(x)\ (x-a) - \int_a^x f\ (t)dt, \qquad 17.2$$

and

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t) = f\ (a+b-x)\left(\frac{a+b}{2}-x\right) - f(x)\ (x- -\frac{a+b}{2})$$

$$- \int_x^{a+b-x} f\ (t)dt, \qquad 18.2$$

$$\int_{a+b-x}^b (t-b)\ df(t) = f\ (a+b-x)\ (x-a) - \int_{a+b-x}^b f\ (t)dt \qquad 19.2$$

By add the above three equalities, we obtain

$$\int_a^b K(x,\ t)df\ (t) = \frac{1}{b-a}\int_a^b K(x,\ t)f'\ (t)\ dt$$

$$= \frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f\,dt,$$

And by lemma (2.1)

$$\left|\int_a^b K(x,\ t)f^{'}(t)\ dt\right| \ \leq\ \underset{a\leq t\leq b}{\text{Max}}\ |f'|\ \int_a^b |K(x,\ t)|dt$$

But we have from proof 2.10, we proved that

$$\frac{(x-a)^2+(b-x)^2}{2} = \frac{1}{4}(\text{b - }a)^2 + (x-\frac{a+b}{2})^2,$$

Or $$\frac{(x-a)^2+(b-x)^2}{2(b-a)} = [\frac{1}{4}+\frac{(x-\frac{a+b}{2}}{(b-a)^2}](b-a).$$

By using the last fact, we can say

$$\frac{4(x-a)^2+(a+b-2x)^2}{4(b-a)} = \frac{(x-a)^2+(\frac{a+b}{2}-x)^2}{(b-a)} = 2[\frac{(b-a)^2}{16}+\left(x-\frac{3a+b}{4}\right)^2[b-a]$$

$$= (b-a)\,[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2].$$

Then

$$\frac{1}{b-a}\int_a^b |K(x,t)|dt = \frac{4(x-a)^2+(a+b-2x)^2}{4(b-a)} = (b-a)\,[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2]$$

Therefor

$$\underset{a\leq t\leq b}{\text{Max}}\ |f'|\frac{1}{b-a}\int_a^b |p(x,t)|dt \ \leq [\frac{1}{8}+2\,(\frac{x-\frac{3a+b}{4}}{b-a})^2](b-a)\|f'\|_\infty .$$

Theorem 2.26 [3]

Let $f:[a,\ b]\ \rightarrow\ R$ be an absolutely continuous functions on $[a\ ,b]$ whose derivative is bounded on $[a\ ,b]$, then

$$\left|(b-a)\left[(1-\lambda)\frac{f\,(x)+f(a+b-x)}{2}+\ \lambda\left(\frac{f(a)+f(b)}{2}\right)\right]-\int_a^b f dt\right|$$

$$\leq [\frac{(b-a)^2}{8}(2\lambda^2+\left(1-\lambda\right)^2+2(x-\frac{(3-\lambda)a+(1+\lambda)b}{4})^2\right]\|f'\|_\infty$$

Where $\lambda \in [0,\ 1]$ and $x \in \left[a+\lambda\frac{b-a}{2}\ ,\frac{a+b}{2}\right]$.

Proof

Using the integration by parts

$$\int_a^x \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right) df = \left(x-a-\lambda\frac{b-a}{2}\right) f(x)$$

$$+\lambda\frac{b-a}{2} f(a) - \int_a^x f(t)dt$$

$$\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right) df(t) = \left(\frac{a+b}{2}-x\right)(f(x)+f\ (a+b-x)$$

$$-\int_x^{a+b-x} f(t)dt,$$

And

$$\int_{a+b-x}^{b} \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right) df(t) =$$

$$\lambda\left(\frac{b-a}{2}\right) f(b)\ +\left(x-a-\lambda\frac{b-a}{2}\right) f(a+b-x)$$

$$-\int_{a+b=x}^{b} f(t)dt$$

Adding the above inequalities, we get

$$\int_a^b k(x\ ,t) f'(t)dt = (b-a)\left[\lambda\frac{f(a)+f(\mu)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x}{2}\right]$$

$$-\ \int_a^b (t)\ dt,$$

Where

$$k(x\ ,t) = \begin{cases} t-\left(a+\lambda\frac{b-a}{2}\right), & t \in [a,x], \\ t-\frac{a+b}{2}, & t \in (x, a+b-x], \\ t-\left(b-\lambda\frac{b-a}{2}\right), & t \in (a+b-x, b], \end{cases}$$

For all $\lambda \in [0,1]$ and $a+\lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$. since , f' is bounded , so

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)\frac{f(x)+f(a+b-\lambda)}{2}\right]-\int_a^b f(t)dt\right|$$

$$\le \int_a^b |k(x,t)||f'(t)||dt \le |\|f'\|_\infty \int_a^b |k|dt,$$

Now we using the fact

$$\int_p^r |t-q|dt = \int_p^q (q-t)dt + \int_q^r (t-q)dt = \frac{(q-p)^2+(r-q)^2}{2}$$

$$= \frac{1}{4}(p-r)^2 + (q-\frac{r+p}{2})^2 \qquad (20.2)$$

For $p \le q \le r$, then

$$\int_a^x \left|t-\left(a+\lambda\frac{b-a}{2}\right)\right| dt$$

$$= \frac{1}{4}(x-a)^2 + \left(\lambda\frac{b-a}{2}-\frac{x-a}{2}\right)^2,$$

$$\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right| dt = \left(x-\frac{a+b}{2}\right)^2$$,and .

$$\int_{a+b-x}^b \left|t-\left(b-\lambda\frac{b-a}{2}\right)\right| dt - \frac{1}{4}(x-a)^2 + \left(\frac{x-a}{2}-\lambda\frac{b-a}{2}\right)^2.$$

So , we obtain

$$\int_a^b |k(x,t)|dt = \frac{(x-a)^2+\left((x-a)-\lambda(a-b)\right)^2}{2} + \left(x-\frac{b+a}{2}\right)^2$$

$$= \underbrace{\frac{1}{4}\lambda^2(b-a)^2 + \left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)^2}_{\text{by (20.2)}} + \left(x-\frac{a+b}{2}\right)^2$$

$$= \frac{\lambda^2}{4}(b-a)^2 + \underbrace{\frac{(1-\lambda)^2}{8}(b-a)^2 + 2\left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2}_{\text{by (20.2)}}$$

$$= \frac{(b-a)^2}{8}(2\lambda^2+(1-\lambda)^2 + 2\left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2$$

Hence

$$\left| (b-a)\left[(1-\lambda)\frac{f\,(x)+f(a+b-x)}{2} + \lambda\left(\frac{f(a)+f(b)}{2}\right)\right] - \int_a^b f dt\right|$$

$$\leq [\frac{(b-a)^2}{8}(2\lambda^2 + \left(1-\lambda\right)^2 + 2(x - \frac{(3-\lambda)a+(1+\lambda)b}{4})^{\,2}\right] \|f'\|_\infty$$

Corollary 2.27 [3]

(a) If choose $\lambda = 0$, then we have

$$\left|(b-a)\frac{f(x)+f(a+b-x}{2} - \int_a^b f(t)dt\right| \leq$$

$$\left[\frac{(b-a)^2}{8} + 2(x - \frac{3a+b}{4})^2\right] \|f'\|_\infty$$

(b) If $\lambda = 1,\ x = \frac{a+b}{2},$ then we have

$$\left|(b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)dt\right| \leq \frac{1}{4}(b-a)^2\|f'\|_\infty$$

Lemma 2.28 [24]

Let $f : [a,\ b] \rightarrow R$ be an absolutely continuous on $[a, b]$ and $x \in [a,\ b]$ then for any $\lambda_1(x)$ and $\lambda_2\,(x)$ real functions on $[a, b]$, we have

$$f(x) + \frac{1}{2(b-a)}\ [(b-a)^2\lambda_2(x) - (x-a)^2\lambda_1(x)] - \frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{1}{b-a}\int_a^x (t-a)[f'(t) - \lambda_1\,(x)]\,dt + \frac{1}{b-a}\int_x^b (t-b)[f'(t)\ \ -\lambda_2(x)]\,dt,$$

Proof

To find $\int_a^x (t-a)[f'(t) - \lambda_1(x)]dx$ and $\int_x^b (t-b)[f'(t) - \lambda_2(x)]\,dx,$

We can utilizing the integration by parts formula

So

$$\int_a^x (t-a)[f'(t) - \lambda_1(x)]dt$$

$$= (t-a)[f(t)-\lambda_1(x)t]\ \Big|_{a}^{x} - \int_a^x [f(t)-\lambda_1(x)t]\,dt$$

$$= (x-a)[f(x)-\lambda_1(x)x(x-a) - \int_a^x f(t)dt + \frac{1}{2}\lambda_1(x)(x^2a^2)$$

$$= (x-a)f(x) - \int_a^x f(t)dt - \frac{1}{2}(x-a)^2\lambda_1(x) \qquad 21.2$$

And for $\int_x^b (t-b)[f'(t)-\lambda_2(x)]dt$

$$= (t-b)[f(t)-\lambda_2(x)t]\ \Big|_{x}^{b} - \int_a^x [f(t)-\lambda_2(t)t]\,dt$$

$$= (b-x)[f(x)-\lambda_2(x)x] - \int_x^b f(t)dt - (b-x)\,\lambda_2(x)x$$

$$+\frac{1}{2}\,\lambda_2(x)(b^2x^2)$$

$$= (b-x)f(x) - \int_x^b f(t)dt + \frac{1}{2}(b-x)^2\,\lambda_2(x) \qquad 22.2$$

So, by add the identifies (21.2), (22.2) and divide by $(b-a)$, we have

$$\frac{1}{b-a}\Big[\int_a^x (t-a)[f'(t)-\lambda_1(x)\Big]dt + \int_x^b (t-b)[f'(x)-\lambda_2(x)]dt$$

$$= f(x) + \frac{1}{2(b-a)}[(b-x)^2\lambda_2(x) - (x-a)^2\,\lambda_2(x)] - \frac{1}{b-a}\int_a^b f\,dt\,.$$

Remark 2.29 [24]

The last identify has many particular cases of interest.

(i) If choose $\lambda_1 = \lambda_2 = \lambda$ then we have

$$f(x) + \left(\frac{a+b}{2} - x\right)\lambda - \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{b-a}\int_a^x (t-a)[f'(t)-\lambda(x)]dt$$

$$+\frac{1}{b-a}\int_x^b (t-b)[f'(t)-\lambda(x)]dt$$

In particular if $\lambda \in R,\ x = \frac{a+b}{2}$ then, we have the midpoint

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)[f'(t)-\lambda]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}(t-b)[f'-\lambda]dt$$

(ii) If $\lambda_1 = \lambda_2 = 0$ then we get,

$$f(x)-\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{b-a}\int_a^x(t-a)f'dt+ \frac{1}{b-a}\int_x^b(t-b)f'dt$$

(iii) If $x \in (a,\ b)$ is a point of differentiability for the absolutely continuous function $f\colon [a,\ b] \rightarrow \mathrm{R}$ then,

$$f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{1}{b-a}\int_a^x(t-a)[f'(t)-f'(x)]dt+\frac{1}{b-a}\int_x^b(t-b)[f'(t)-f'(x)]dt\,.$$

Theorem 2.30 [25]

Let $f : [a,\ b] \rightarrow R$ be a differentiable function, and f' is of bounded variation on $[a, b]$, then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right.$$

$$+\ \frac{1}{4(b-a)}[(b-x)^2f'(b)-(x-a)^2f'(a)]$$

$$\leq \frac{1}{4}(b-a)[\left(\frac{x-a}{b-a}\right)^2 \mathrm{V}_a^x f' + \left(\frac{b-x}{b-a}\right)^2 \mathrm{V}_x^b f']$$

$$\leq \frac{1}{4}(b-a)\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\mathrm{V}_a^b f',$$

for $x \in [a\ ,b]$.

Proof

Let $\lambda_1\,(x) = \frac{f^1(a)+f^1(x)}{2}$, $\lambda_2 = \frac{f^1(x)+f^1(b)}{2}$

In lemma (2.28) we get the modulus

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f^2(x)$$

$$+ \frac{1}{4(b-a)}[(b-x)^2\, f^1(b) - (x-a)^2 f^2(a)]$$

$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f'(a)+f'(x)}{2}\right] dt$$

$$+ \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f'(x)+f'(b)}{2}\right] dt, \qquad \text{for } x \in [a\,, b].$$

So know,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x)\right.$$

$$\left. + \frac{1}{4(b-a)}[(b-x)^2 f'(b) - (x-a)^2 f'(a)]\right|$$

$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{f'(a)+f'(x)}{2}\right] dt\right|$$

$$+ \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{f'(x)+f'(b)}{2}\right] dt\right|$$

$$\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt$$

$$+ \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt. \qquad (23.2)$$

But $f'\colon (a,\ b) \to R$ is of bounded variation on $[a,\ x]$ and $[x,\ b]$ so

$$\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| = \frac{|f'(t)-f'(a)+f'(t)-f'(x)|}{2}$$

$$\leq \frac{1}{2}[|f'(t) - f'(a)| + |f'(x) - f'(t)|]$$

$$\leq \frac{1}{2}\mathrm{V}_a^x(f\,'), \text{ for } t \in [a,\ x]$$

Similarly for

$$\int_a^x (t-a)\left|f'(t)-\frac{f'(a)+f'(x)}{2}\right|dt$$

$$\leq \frac{1}{2}\bigvee_a^x(f')\int_a^x (t-a)dt$$

$$= \frac{1}{2}\bigvee_a^x(f')\left[\left(\frac{t^2}{2}-at\right)\Big|_a^x\right]$$

$$= \frac{1}{2}\bigvee_a^x(f')\left(\frac{x^2}{2}-ax\right)-\left(\frac{a^2}{2}-a^2\right)$$

$$= \frac{1}{2}\bigvee_a^x(f')\left(\frac{x^2}{2}-ax+\frac{a^2}{2}\right)$$

$= \frac{1}{4}\bigvee_a^x f'\,(x-a)^2,$ then from (23.2) we get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right.$$

$$\left.+\frac{1}{4(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]\right|$$

$$\leq \frac{1}{b-a}\left[\frac{1}{4}(x-a)^2\bigvee_a^x f'+\frac{1}{4}(x-a)^2\bigvee_x^b f'\right]$$

$$= \frac{(b-a)}{4}\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x f'+\left(\frac{b-x}{b-a}\right)^2\bigvee_x^b f'\right]$$

$$\leq \frac{(b-a)}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b f',\ for\ x\in[a,\ b].$$

Theorem2.31 [26]

Let $f: I \to R$ be differentiable function on I and $[a\,,b] \subset I^\circ$. If the derivative $f': I^\circ \to R$ is of bounded variation on $[a,\ b]$, then for any $x, t \in [a,\ b]$

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')dt+\int_x^b (b-t)\bigvee_t^b(f')dt\right]$$

$$\leq \frac{1}{b-a}\begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x(f'), \\ (x-a)\int_a^x (\bigvee_a^t(f'))dt \end{cases}$$

$$+\frac{1}{b-a}\begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b(f'), \\ (b-x)\int_x^b \left(\bigvee_t^b(f')\right)dt. \end{cases} \tag{24.2}$$

Proof

By lemma (2.28)

If we assume that the lateral derivatives $f'(a)$ and $f'(b)$ exist and are finite, then for $\lambda_1(x) = f'(a)$ and $\lambda_2(x) = f'(b)$, we have

$$\left|f(x) + \frac{1}{2(b-a)}[(b-x)^2 f'(b) - (x-a)^2 f'(a)] - \frac{1}{b-a}\int_a^b f(t)dt\,\right|$$

$$\leq \frac{1}{b-a}\int_a^x (t-a)|f'(t) - f'(a)|\,dt + \frac{1}{b-a}\int_x^b (b-t)|f'(b) - f'(t)|dt,$$

For any $x \in [a,\ b]$.

Since the derivative f': Ỉ → R is of bounded variation on $[a,\ b]$, then

$$|f'(t) - f'(a)| \leq \bigvee_a^t(f') \text{ for any } t \in [a,\ x]$$

And

$$|f'(b) - f'(t)|\ \leq \bigvee_t^b(f') \text{ for any } t \in [x,\ b].$$

Therefore

$$\int_a^x (t-a)|f'(t) - f'(a)|dt \leq \int_a^x (t-a) \bigvee_a^t(f')\, \mathrm{dt}$$

And

$$\int_x^b (b-t)|f'(b) - f'(t)|dt \leq \int_x^b (b-t) \bigvee_t^b(f')dt, \text{ for any } x \in [a\,, b].$$

Adding these two inequalities and dividing by $b-a$ we get the first inequality, and Using Holder, s integral inequality we have

$$\int_a^x (t-a)\bigvee_a^t(f')\,\mathrm{dt} \le \begin{cases} \bigvee_a^x(f')\int_a^x (t-a)dt, \\ (x-a)\int_a^x (\bigvee_a^t(f'))dt\,. \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x-a)^2\bigvee_a^x(f'), \\ (x-a)\int_a^x (\bigvee_a^t(f'))dt\,. \end{cases}$$

And

$$\int_x^b (b-t)\bigvee_t^b(f')dt \le \begin{cases} \frac{1}{2}(b-x)^2\bigvee_x^b(f'), \\ (b-x)\int_x^b (\bigvee_x^b(f'))dt\,. \end{cases}$$

Remark 2.32 [26]

From the first branch in (24.2) we have the sequence of inequalities

$$\left|f(x)+\frac{1}{2(b-a)}[(b-x)^2f'(b)-(x-a)^2f'(a)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\le \frac{1}{b-a}[\int_a^x (t-a)\bigvee_a^t(f')dt+\int_x^b (b-t)\bigvee_t^b(f')dt]$$

$$\le \frac{1}{2}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x(f')+\left(\frac{b-x}{b-a}\right)\bigvee_x^b(f')\right]$$

$$\le\ \frac{1}{2}(b-a)\begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)\right]\left[\frac{1}{2}\bigvee_a^b(f')+\frac{1}{2}\left|\bigvee_a^x(f')\right|\right], \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f'), \end{cases} \tag{25.2}$$

from the second branch in (24.2) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2f'(b)-(x-a)^{2f'}(a)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\le \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')dt+\int_x^b (b-t)\bigvee_t^b(f')dt\right]$$

$$\le \left(\frac{x-a}{b-a}\right)\int_a^x (\bigvee_a^t(f'))\,dt+\left(\frac{b-x}{b-a}\right)\int_x^b \left(\bigvee_t^b(f')\right)dt$$

$$\leq \begin{cases} \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\int_a^x (\bigvee_a^t(f'))\,dt+\int_x^b \left(\bigvee_t^b(f')\right)dt\ \right] \\ \max\left\{\int_a^x (\bigvee_a^t(f'))\,dt, \int_x^b \left(\bigvee_t^b(f')\right)dt\right\} \end{cases} \qquad \square$$

Corollary 2.33 [26]

We observe that, if we take $x=\frac{a+b}{2}$ in (25.2) then we get the perturbed midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'^{(a)}\right]-\ \frac{1}{b-a}\int_a^b f(t)dt\ \right|$$

$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\bigvee_a^t(f')dt+\int_{\frac{a+b}{2}}^b(b-t)\bigvee_t^b(f')dt\right]\leq \frac{1}{8}(b-a)\bigvee_a^b(f').$$

2.5 Inequalities for n-time differentiable functions.

Lemma 2.34 [18]

Let $f : [a, b] \rightarrow R$ be a mapping such that the derivation $f^{(n-1)}$, $n \geq 1$ is absolutely continuous on $[a, b]$, then

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)^1} [(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b - x)^{k+1} f^{(k)}(b)] + \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\,dt,$$

For all $x, t \in [a, b]$.

Proof:

The proof is by mathematical induction,

For n = *1* we have for prove that

$$\int_a^b f(t)dt = (x-a)\ f\ (a) + (b-x)\ f(b) + \int_a^b (x-t)\ f^{(1)}(t)\,dt,$$

Which is clearly by integration by parts formula applied for $\int_a^b (x-t)\,df$,

$$\int_a^b (x-t)\ f^{(1)}(t)\,dt = \int_a^b (x-t)\,df = (x-t)\ f(t) \Big|_a^b + \int_a^b f dt$$

$$= (x-a) f\ (a) + (b-x)\ f(b) + \int_a^b f(t)\ dt$$

So

$$\int_a^b f(t)dt = (x-a)\ f\ (a) + (b-x) + \int_a^b (x-t) f^{(1)}\,dt.$$

Assume that it's holds for (n) and let us prove it for ((n+1)) that is we wish to show that

$$\int_a^b f(t)dt = \sum_{k=0}^{n} \frac{1}{(k+1)!} [(x-a)^{k+1} f^k(a) + (-1)^k (b-x)^{(k+1)} f^{(k)}(b)] + \frac{1}{(n+1)!} \int_a^b (x-t)^{k+1} f^{(n+1)}(t)\,dt.$$

Now let g(t) = $(x-t)^{\mathrm{k}}\, f^{(\mathrm{k})}$ (t),

(Which is absolutely continuous on [a, b]),

$$\int_{\mathrm{a}}^{\mathrm{b}}(x-t)^{\mathrm{k+1}}\, f^{(\mathrm{k+1})}(\mathrm{t})\, dt = (x-a)\,(x-a)^{\mathrm{k}}\ f^{(\mathrm{k})}(a) + (b-x)\,(x-b)^{\mathrm{k}}$$

$$f^{\,(\mathrm{k})}(b) + \int_{\mathrm{a}}^{\mathrm{b}}(x-t)\,\frac{d}{dt}\,[(x-t)^{\mathrm{k}}\ f^{\,(\mathrm{k})}(\mathrm{t})]dt$$

$$= \int_{\mathrm{a}}^{\mathrm{b}}(x-t)[\text{-n}\,(x-t)^{\,(\mathrm{k\text{-}1})}\, f^{\,(\mathrm{n})}(\mathrm{t}) + (x-t)^{\,\mathrm{k}}\, f^{\,(\mathrm{n+1})}(\mathrm{t})]dt$$

$$+\,(x-a)^{\mathrm{n+1}}\, f^{\,(\mathrm{n})}(\mathrm{a}) + (\text{-}1)^{\mathrm{n}}\,(b-x)^{\mathrm{n+1}} f^{\,(\mathrm{n})}(\mathrm{b})$$

$$= \text{-}n\int_{a}^{b}(x-t)^{\mathrm{n}}\, f^{\,(\mathrm{n})}(\mathrm{t})\, dt + \int_{\mathrm{a}}^{\mathrm{b}}(x-t)^{\mathrm{n+1}}\, f^{\,(\mathrm{n+1})}(\mathrm{t})\, dt$$

$$+\,(x-a)^{\mathrm{n+1}}\, f^{\,(\mathrm{n})}(a) + (\text{-}1)^{\mathrm{n}}\,(b-x)^{\,n+1} f^{\,(n)}(b) \qquad (26.2)$$

From (26.2) we can get

$$\int_{a}^{b}(x-t)^{\mathrm{n}}\, f^{(n)}(\mathrm{t})\, dt = \frac{1}{(n+1)}\ \int_{a}^{b}(x-t)^{\mathrm{k+1}} f^{(\mathrm{k+1})}\ (\mathrm{t})\ dt$$

$$+\,\frac{1}{(n+1)}\,[(x-a)]\, f^{(n)}\ (a) + \frac{1}{(n+1)}\,[(x-a)]^{\mathrm{n+1}}\ \ f^{\,(n)}(a)$$

$$+\,(\text{-}1)^{n}\,(b-x)^{\mathrm{n+1}}\, f^{(n)}(\mathrm{b})]$$

by using the induction hypothesis

$$\int_{\mathrm{a}}^{\mathrm{b}} f(\mathrm{t})\ dt = \sum_{\mathrm{k=0}}^{\mathrm{n-1}}\frac{1}{(\mathrm{k+1})!}\,[(x\text{–}\,a)^{\mathrm{k+1}}\ \ f^{\,(\mathrm{k})}\,(a) + (\text{-}1)^{\mathrm{k}}\,(b\ -\ x)^{\mathrm{k+1}} f^{\,(\mathrm{k})}\,(b)] + \frac{1}{n!}$$

$$[\ \frac{1}{(n+1)}\,[(x-t)]^{\mathrm{n+1}}\ \ f^{\,(\mathrm{n+1})}\ (t)\ dt$$

$$+\ \ \frac{1}{(n+1)}\,[\ (x-a)^{\mathrm{n+1}}\ \ f^{\,(\mathrm{n})}(a) + (b-x)^{\mathrm{n+1}}\ \ f^{\,(\mathrm{n})}\,(b)]]$$

$$= \sum_{\mathrm{k=0}}^{\mathrm{n}}\frac{1}{(\mathrm{k+1})!}\ [(x-a)^{\mathrm{k+1}}\, f^{(k)}(a) + (\text{-}1)^{\mathrm{k}}\,(b-x)^{(\mathrm{k+1})}\, f^{\,(\mathrm{k})}(b)]$$

$$+\,\frac{1}{(\mathrm{n+1})!}\ \int_{\mathrm{a}}^{\mathrm{b}}(x-t)^{\mathrm{k+1}}\, f^{(\mathrm{n+1})}\ (t)\ dt. \qquad \square$$

Theorem 2.35 [18]

Let $f : [a,\ b] \to$ R be a mapping so that derivative $f^{(n-1)}$ is absolutely continuous then

$$\Big| \int_a^b f\ (t)\ dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)^1} [(x- a)^{k+1}\ f^{(k)}(a) + (-1)^k (b\ -\ x)^{k+1} f^{(k)}(b)\ ]$$

$$\leq \quad \frac{\|f^{(n)}\|_\infty}{(n+1)!} [\ (x-a)^{n+1} + (b-x)^{n+1}] \quad \text{if} \quad f^{(n)} \in L_\infty[a, b]$$

Proof:

From lemma [2.34] we have

$$\Big| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} [(x- a)^{k+1}\ f^{(k)}(a) + (-1)^k (b\ -\ x)^{k+1}\ f^{(k)}(b)] \Big|$$

$$\leq \frac{1}{n!} \int_a^b |x-t|^n \left| f^{(n)} \right| dt \leq \Big[\frac{1}{n!} \int_a^b |x-t|^n\ dt\Big] \Big\| f^{(n)} \Big\|_\infty$$

$$= \frac{\|f^{(n)}\|_\infty}{n!} \Big[\ \int_a^x (x-t)^n\, dt + \int_x^b (t-x)^n\, dt\ \Big]$$

$$= \frac{\|f^{(n)}\|_\infty}{n!} \Big[\ \frac{(x-a)^{n+1} + (b-x)^{n+1}}{n+1} \Big]$$

$$= \frac{\|f^{(n)}\|_\infty}{(n+1)!} \big[\big(x-a)^{n+1} + (b-x)^{n+1}\ \big].$$

To prove the second inequality we have

$$\frac{1}{n!} \int_a^b |x-t|^n \left|\ f^{(n)} \right| dt\ \leq \frac{1}{n!} \sup_{a \leq t \leq b} |x-t|^n \int_a^b \left| f^{(n)}(t) \right| dt$$

$$= \frac{1}{n!} [sup\ |x-t|\,]^n\ \|f^n\|_1$$

$$= \frac{1}{n!} \big[\sup\ (x-a, b-x)\ \big]^n\ \ \big\|f^{(n)}\big\|_1$$

$$= \frac{1}{n!}\ [{}^1\!/_2\, (b-\ a) + \big|x- \frac{a+b}{2}|]^n\ \ \big\|f^{(n)}\big\|_1$$

Hence

$$\left| \int_{\mathrm{a}}^{\mathrm{b}} f\ (\mathrm{t})\ dt - \sum_{\mathrm{k=0}}^{\mathrm{n-1}} \frac{1}{(\mathrm{k+1})!} [(x - a)^{\mathrm{k+1}}\ f^{(\mathrm{k})}(a) + (-1)^{\mathrm{k}} (b - x)^{\mathrm{k+1}} f^{(\mathrm{k})}(b)] \right|$$

$$\leq \frac{\|f^{n}\|}{\mathrm{n}!} [1/_2 (a - b) + \left| x - \frac{a+b}{2} \right|]^{n}$$

Lemma 2.36 [17]

Let $f : [a,\ b] \rightarrow R$ be a mapping such that $f^{\ (\mathrm{n-1})}$ is absolutely continuous on $[a, b]$. Then

$$\int_{\mathrm{a}}^{\mathrm{b}} f\ (t)\ dt = \sum_{\mathrm{k=0}}^{\mathrm{n-1}} [\frac{(b-x)^{k+1} + (-1)^{k} (x-a)^{k+1}}{(k+1)\ !}]\ f^{\ (k)}(x)$$

$$+ (-1)^{\mathrm{n}} \int_{a}^{b} \mathrm{k}_{n} (x, t)\ f^{(\mathrm{n})}(t)\ dt \qquad (27.2)$$

Where the kernel K_n: $[a,\ b]^2 \rightarrow R$ is given by

When $x \in [a,\ b]$, n is a natural number, $n \geq 1$

Proof:

We use proof by Mathematical Induction.

For $n = 1$

$$\int_{a}^{b} f\ (t)\ dt = (b - a)\ f\ (x) - \int_{a}^{b} K_1(x, t)\ f^{(1)}(t)\ dt$$

Where

$$\mathrm{K}_1\ (x,\ t) = \begin{cases} t - a & \text{if } t \in [a,\ x] \\ t - b & \text{if } t \in [x,\ b] \end{cases}$$

$$\int_{a}^{b} K_1(\mathrm{x, t})\ f^{(1)}(t)\ dt = \int_{a}^{x} (t - a)\ f\ '(t)\ dt + \int_{x}^{b} (t - b) f\ '(t)\ dt$$

$$= \int_{a}^{x} (t - a)\ df + \int_{x}^{b} (t - a)\ df$$

$$= (t - a) f(t) \Big|_{a}^{x} - \int_a^x f\, dt + (t - b) f(t) \Big|_{x}^{b} - \int_x^b f\,(t)dt$$

$$= (x - a) f\,(x) + (b - x) f\,(x) - \int_a^b f(\mathrm{t})\ \mathrm{dt}$$

$$= (b - a) f\,(x) - \int_a^b f\,(t)\, dt$$

So

$$\int_a^b f\,(\mathrm{t})\, dt = (b - a) f(x) - \int_a^b K_1\,(x\,, t)\ f^{(1)} dt$$

Now assume that (27.2) holds for n and let us prove it for (n+1)

That is prove the equality

$$\int_a^b f\,(\mathrm{t})\, dt = \sum_{\mathrm{k=0}}^{\mathrm{n}} \left[\frac{(b-x)^{\mathrm{k+1}} + (-1)^{\mathrm{k}}\,(x-a)^{\mathrm{k+1}}}{(\mathrm{k+1})\,!}\right] f^{\,(\mathrm{k})}(x)$$

$$+ (-1)^{n+1} \int_a^b k_{n+1}\,(x\,, t)\ f^{(\mathrm{n+1})}(t)\, dt\,.$$

Using

$$K_{\mathrm{n+1}}\,(x,\ t) = \begin{cases} \dfrac{(t-a)^{(n+1)}}{(n+1)!} & \text{if } t \in [a\,, x] \\ \dfrac{(t-b)^{n}}{n!} & \text{if } t \in [x\,, b] \end{cases}$$

And

$$\int_a^b k_{n+1}\,(x, t)\ f^{(\mathrm{n+1})}\,(t)\, dt = \int_a^x \frac{(t-a)^{n+1}}{(n+1)!} f^{(\mathrm{n+1})}\,(t)\, dt$$

$$+ \int_x^b \frac{(t-b)^{n+1}}{(n+1)!} f^{(\mathrm{n+1})}\,(\mathrm{t})\, dt$$

So, using the integrating by parts for

$$\int_a^x \frac{(t-a)^{(n+1)}}{(n+1)!} f^{(\mathrm{n+1})}\,(t)\, dt = \int_a^x \frac{(t-a)^{n+1}}{(n+1)!}\ df^{(\mathrm{n})}(\mathrm{t})$$

And

$$\int_x^b \frac{(t-a)^{n+1}}{(n+1)!} f^{(\mathrm{n+1})}(\mathrm{t})dt = \int_x^b \frac{(t-a)^{n+1}}{(n+1)!} df^{(n)}(t)$$

Now, put $g = \frac{(t-a)^{n+1}}{(n+1)!}$ and $h = f^{(n)}(t)$

$$\int_a^x gdh = h\ (x)\ g\ (x) - h(a)\ g\ (a) - \int_a^x g\ dh$$

$$= h(x)\ g(x) - \int_a^x hg'\, dt$$

So

$$h(x)\ g(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n)}(x) \text{ and } \int_a^x hg'\, dt = \int_a^x \frac{(n+1)(t-a)^n}{(n+1)!} f^{\ (\mathrm{n})}\ (\mathrm{t})\ dt$$

$$\int_a^x \frac{(t-a)^{n+1}}{(n+1)!} \mathrm{d}\ f^{(n)}(t) \quad = \quad \frac{(x-a)^{n+1}}{(n+1)!} f^{(\mathrm{n})}(x) - \frac{1}{\mathrm{n!}} \int_{\mathrm{a}}^{\mathrm{x}} (\mathrm{t-a})^{\mathrm{n}} f^{\mathrm{n}}\ (\mathrm{t})\ \mathrm{dt}$$

Similarly a bout

$$\int_x^b \frac{(t-b)^{n+1}}{(n+1)!} f^{(\mathrm{n+1})}(t)\ dt = \int_x^b \frac{(t-b)^{n+1}}{(n+1)!}\ df^{(\mathrm{n})}(\mathrm{t}) = \frac{-\ (x-b)^{n+1}}{(n+1)!} f^{\ (\mathrm{n})}(x)$$

$$+ \left(\frac{-1}{n!}\right) \int_x^b (t-a)^n\, f^{\ (\mathrm{n})}(t)\ dt$$

$$\int_x^b (t-b)^{n+1} f^{\ (\mathrm{n+1})}(t)\ dt = \frac{(-1)^{n+2}(b-x)^{n+1}}{(n+1)!} f^{(\mathrm{n})}\ (x)$$

$$-\frac{1}{\mathrm{n!}} \int_x^b (t-b)^n f^{(\mathrm{n})}\ (t)dt \qquad (28.2)$$

Note that

$$(\ -1)^{n+2}\ (b - x)^{n+1} = (-1)^{\mathrm{2n+3}}\ (x - b)^{(\mathrm{n+1})} = -\ (x - b)^{\mathrm{n+1}}$$

From (27.2) and (28.2) we have

$$\int_a^b k_{n+1}\ (x\ ,\ t)\ f^{\ (\mathrm{n+1})}(\mathrm{t}) = \frac{(x-a)^{n+1} f^n + (-1)^{n+2}(b-x)^{n+1} f^n(x)}{(n+1)!}$$

$$-[\int_a^x \frac{(t-a)^n}{n!}\ f^{(n)}\ dt + \int_x^b \frac{(t-a)^n}{n!} f^{(n)}(t)\ dt$$

$$= \frac{(x-a)^{n+1}+(-1)^{n+2}(b-x)^{n+1}}{(n+1)!}\ f^{(n)}(x)$$

$$-\int_a^b k_n\ (x\ ,t)\ f^{(n)}(t)\ dt$$

So

$$\int_a^b k_n\ (x\ ,t)\ f^{(n)}(t)\ dt = \frac{(x-a)^{n+1}+(-1)^{n+2}(b-x)^{n+1}}{(n+1)!} f^{(n)}(x)$$

$$-\int_a^b k_{n+1}\ (x\ ,t)\ f^{(n+1)}(t)\ dt$$

by mathematical induction hypothesis we have

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} [\frac{(b-x)^{k+1}+(-1)^k\ (x-a)^{k+1}}{(k+1)\ !}]\ f^{(k)}(x)$$

$$+\frac{(b-x)^{n+1}+(-1)^n\ (x-a)^{n+1}}{(k+1)\ !}\ f^{(n)}(x)$$

$$-(-1)^n \int_a^b k_{n+1}\ (x\ ,t)\ f^{(n+1)}(t)\ dt$$

$$= \sum_{k=0}^{n} [\frac{(b-x)^{k+1}+(-1)^k\ (x-a)^{k+1}}{(k+1)\ !}]\ f^{(k)}\ (x)$$

$$+\ (-1)^{n+1} \int_a^b k_{n+1}\ (x\ ,t)\ f^{(n+1)}(t)\ dt.$$

Corollary 2.37 [19]

Let $f\ ;\ [a,\ b]\ \rightarrow\ R$ such that $f^{(n-1)}$ is absolutely continuous on $[a,\ b]$ then

$$\int_a^b f\ (t)\ dt = \sum_{k=0}^{n-1} [\frac{1+(-1)^k}{(k+1)!}]\ \frac{(b-a)^{k+1}}{2^{k+1}}\ f^{(k)}\ (\frac{a+b}{2})$$

$$+\ (-1)^n \int_a^b M_n\ (t)\ f^{(n)}(t)\ dt$$

Where

$$M_n\ (t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t\ \in [a, \frac{a+b}{2}] \\ \frac{(t-b)^n}{n!} & \text{if } t\ \in (\frac{a+b}{2}, b] \end{cases}$$

Proof:

From lemma [2.36] by choosing $x = \frac{a+b}{2}$

Corollary 2.38 [19]

Let $f : [a,\ b] \rightarrow R$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$.

Then

$$\int_a^b f\ (t)\ dt = \sum_{k=0}^{n-1} \ (\frac{b-a}{(k+1)!})^{k+1} \left[\frac{f^{(k)}(a)+(-1)^k f^k(b)}{2} \right]$$

$$+ \ \frac{1}{n!}\int_a^b \frac{(b-t)^n + (-1)^n(t-a)^n}{2} \ ]\ f^{(n)}(t)\ dt$$

for $t, x\ \in\ [a,\ b]$.

Proof:

Let $x = a$ and $x = b$ in (2.26) then summing the resulting identifies and dividing by 2 ,

So where $x = a$ we have inequality

$$\int_a^b f\ (t)\ dt = \sum_{k=0}^{n-1} \ \left[\ \frac{(b-a\ )^{k+1}}{(k+1)!}\ f^{(k)}(a) \right] + \int_a^b \frac{(t-b)^n}{n!} f^{\ (n)}(t)\ dt$$

$$= \sum_{k=0}^{n-1} \ \left[\frac{(b-a\ )^{k+1}}{(k+1)!}\ f^{\ (k)}(a) \right] + \ \frac{1}{n!}\ (b-t)^n f^{\ (n)}(t)\ dt \qquad (29.2)$$

Now $x = b$

$$\sum_{k=0}^{n-1}[\frac{(-1)^k(b-a)^{k+1}}{(k+1)!} f^{(k)}(b)] + (-1)^n \int_a^b \frac{(t-a)^n}{n!} f^{(n)}(t)\,dt \qquad (30.2)$$

Then from (29.2) and (30.2)

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} (\frac{b-a}{(k+1)!})^{k+1}[\frac{f^{(k)}+(-1)^k f^k(b)}{2}]$$

$$+ \frac{1}{n!}\int_a^b \frac{(b-t)^n+(-1)^n(t-a)^n}{2}]\ f^{(n)}(t)dt$$

Theorem 2.39 [17]

Let $f : [a, b] \to R$ be a mapping such that $f^{(n-1)}$ is a absolutely continuous on $[a, b]$. Then

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} [\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}] f^{(k)}(x) \right|$$

$$\le \begin{cases} \frac{\|f^{(n)}\|\infty}{(n+1)!} \quad [(x-a)^{n+1}+(b-x)^{n+1}] \text{ if } f^{(n)} \in L_\infty [a,b], \\ \frac{\|f^{(n)}\|1}{n!} \quad [½(b-a)+\left|x-\frac{a+b}{2}\right|]^n \text{ if } f^{(n)} \in L_1 [a,b], \end{cases}$$

Where

$$\|f^{(n)}\|_\infty = \sup_{a\le t\le b} |f^{(n)}(t)| < \infty$$

And

$$\|f^{(n)}\|_1 = \int_a^b |f^n(t)|\,d(t)$$

Proof:

By Lemma (2.36), and observe that $|f^{(n)}| \le \|f^{(n)}\|_\infty$

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} [\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}] f^{(k)}(x) \right|$$

$$= \left| \int_a^b k_n\,(x,\ t)\, f^{(n)}(t)\, dt \right|$$

$$\leq \int_a^b \left| k_n\,(x,\ t) \right| \left| f^{(n)}(t) \right| dt$$

$$\leq \left\| f^{(n)} \right\|_{\infty} \int_a^b \left| k_n\,(x\,, t) \right|$$

$$= \| k_n\,(x,\ t) \|_1 \,.\ \left\| f^{(n)} \right\|_{\infty}$$

$$= \left\| f^{(n)} \right\|_{\infty} \left[\int_a^x \frac{|t-a|^n}{n!} \mathrm{dt} + \int_x^b \frac{|t-b|^n}{n!} \mathrm{dt} \right]$$

$$= \left\| f^{(n)} \right\|_{\infty} \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right]$$

$$= \left\| f^{(n)} \right\|_{\infty} \left[\frac{(t-a)}{(n+1)!}^{n+1} \Big|_a^x + \frac{(b-t)}{n+1!}^{n+1} \Big|_x^b \right]$$

$$= \frac{\left\| f^{(n)} \right\|_{\infty}}{(n+1)!} \left[(x - a)^{n+1} + (b - x)^{n+1} \right]$$

and clearly that

$$\left| \int_a^b k_n\,(x,\ t)\, f^{(n)}(t)\, dt \right| \leq \int_a^b \left| f^n\,(t) \right| \left| k_n(x\,, t) \right| dt$$

$$\leq \left[\int_a^b \left| f^n\,(t) \right| \right] \| k_n\,(x\,, t) \|_{\infty}$$

$$= \left\| f^{(n)} \right\|_1 \,.\ \sup_{a \leq t \leq b} \left| k_n(x\,, t) \right|$$

$$= \left\| f^{(n)} \right\|_1 .\ \mathrm{Max} \left\{ \frac{(x-a)^n}{n!} \,,\ \frac{(b-x)^n}{n} \right] \right\}$$

$$= \frac{\left\| f^{(n)} \right\|_1}{n!} \,.\ \mathrm{Max} \left\{ (x-a)^n\,, (b-x)^n \right\}$$

$$= \frac{\left\| f^{(n)} \right\|_1}{\mathrm{n!}} \left[\mathrm{Max} \left\{ x-a\,, b-x \right\} \right]^n$$

$$= \frac{\|f^{(n)}\|_1}{n!} \quad [\,\frac{b-a}{2} + \mid x - \frac{a+b}{2} \mid]^{n}\,.$$

Notation 2.40

Can easily notice that the Ostrowski inequality

$$\mid \frac{1}{b-a}\int_{a}^{b} f\ (t)\ dt - f(x)\ \mid \leq [\tfrac{1}{4} + \frac{(x-\frac{(a+b)^2}{2})}{(b-a)^2}\,]\ (b-a)\|f'\|_{\infty}$$

We obtain from (2.39) by put ($n = 1$) and as a simple last calculation we shows that

$$\frac{1}{2}\,[(x-a)^2 + (b-x)^2] = [\frac{1}{4}(b-a)^2 + (x\ -\frac{a+b}{2}\ )^2\ ].$$

Chapter 3

Inequalities for the Riemann - Stieltjes integral Of product integrators.

3.1 Inequalities of Ostrowski and trapezoid type for the Riemann-Stieltjes integral

In this section we point out some recent results by the authors in [50], [15], [11] and [46] concerning certain inequalities of trapezoid type, Ostrowski type and for Riemann-Stieltjes integrals,

The section is structured as follows:

The first part deals with the estimation of the magnitude of the difference,

$$\frac{f(a)+f(b)}{2}\,(\,g(b)-g(a)\,)\;-\int_{a}^{b} f(\mathrm{t})dg(\mathrm{t}),$$

Where f is of $p-H-$ Holder type and g is of bounded variation, and vice versa.

The second part provides an error analysis for the quantity

$$f(x)\big(g(b)-g(a)\big)-\int_{a}^{b} f(t)dg(t),$$

This is commonly known in the literature as an Ostrowski type inequality, for the same classes of mappings.

Definition 3.1 [50]

The function $f:[a,\ b]\ \to\ R,$ be a $p-H-$ Holder type, if it satisfies the condition, $|f(x)-f(y)|\ \le H|x-y|^{p},$ for $,y\in[a,\ b]$, and $\mathrm{H}>0,$ $p\in(0,\ 1]$ are given.

Theorem 3.2 [15]

Let $f:\ [a,\ b]\ \to\ R,$ be a $p-H-$ Holder type mapping and $g:[a,\ b]\ \to\ R$ is a mapping of bounded variation on $[a,b]$, then

$$\left|\,\frac{f(a)+f(b)}{2}\,(\,g(b)-g(a)\,)\;-\int_{a}^{b} f(\mathrm{t})dg(\mathrm{t})\,\right|$$

$$\leq \frac{1}{2^p} H (b-a)^p \vee_a^b g, \text{For } t, x \in [a, b]$$

Proof

Using the property in lemma 2.1 we have

$$\left| \frac{f(a)+f(b)}{2} (g(b) - g(a)) - \int_a^b f(t) dg \right|$$

$$= \left| \int_a^b \left(\frac{f(a)+f(b)}{2} - f(t)\right) dg(t) \right|$$

$$\leq \sup_{a \leq t \leq b} \left| \frac{f(a)+f(b)}{2} - f(t) \right| \vee_a^b(g).$$

As f is of p – H – Holder type, then

$$\left| \frac{f(a)+f(b)}{2} - f(t) \right| = \left| \frac{f(a) - f(t) + f(b) - f(t)}{2} \right|$$

$$\leq 1/2 \left[\left| f(a) - f(t) \right| + \left| f(b) - f(t) \right| \right]$$

$$\leq 1/2 \, H \left[(t-a)^p + (b-t)^p\right]$$

Now consider the mapping

$$h(t) = (t-a)^p + (b-t)^p, t \in [a, b], p \in (0, 1]$$

Then

$$h'(t) = p(t-a)^{p-1} - p(b-t)^{p-1} = 0 \quad \text{iff} \quad t = \frac{a+b}{2}$$

And $h'(t) \geq 0$ on $[a, \frac{a+b}{2}]$, $h'(t) < 0$ on $(\frac{a+b}{2}, b]$

Which shows that maximum is realized at t = $\frac{a+b}{2}$, and

$$\sup_{a \leq t \leq b} h(t) = h\left(\frac{a+b}{2}\right) = 2^{(1-p)} (b-a)^p$$

so

$$\sup_{a\leq t\leq b} \left| \left(\frac{f(\mathrm{a})+f(\mathrm{b})}{2} - f(\mathrm{t}) \right| \leq H \left(\frac{b-a}{2}\right)^{\mathrm{p}}$$

Hence

$$\left| \frac{f(\mathrm{a})+f(\mathrm{b})}{2} .\ (g(b) - g(a)) - \int_a^b f\, dg \ \right| \leq \frac{1}{2^p} H\, (b-a)^{\,p} \bigvee_a^b(g)$$

Corollary3.3 [50]

Let $f\ : [a,\ b] \ \rightarrow\ R$ be a p – H- Holder type mapping, and g: $[a,\ b] \ \rightarrow\ R$ be a monotonic mapping on $[a, b]$. Then

$$\left| \frac{f(\mathrm{a})+f(\mathrm{b})}{2} \left(g(b) - g(a) \right) - \int_a^b f\,(t) dg\,(t) \right| \leq \frac{1}{2^p} H\,(b-a)^p\ |g\,(b) - g(a)|\,.$$

[Since g is monotonic so it is of bounded variation and $\bigvee_a^b(g) = |\,g\,(b) - g(a)\ |$]

Corollary 3.4 [11]

Let f be a p- H – Holder mapping and g be a Lipschitzian mapping with

L > 0. Then

$$\left| \frac{f(\mathrm{a})+f(\mathrm{b})}{2} \big(g(b) - g(a)\big) - \int_a^b f dg \ \right| \leq \frac{1}{2^p} H\, L\, (b-a)^{\,p+1}$$

(We know that $\bigvee_a^b(g) \leq L\,[b-a]$ where g is Lipschitzian mapping).

Theorem 3.5 [50]

Let f: $[a,\ b] \rightarrow R$ be $a\ p - H -$ Holder type mapping, Where $H > 0$ and $p \in (0,1]$ are given, and g: $[a,\ b] \rightarrow R$ is a mapping of bounded variation on $[a, b]$.

Then we have the Ostrowski inequality,

$$\left| f(x)\big(g(b) - g(a)\big) - \int_a^b f(t) dg(t) \right|$$

$$\leq H \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^p \bigvee_a^b(g) \qquad (1.3)$$

For all $x \in [a,b]$, furthermore, the constant $\frac{1}{2}$ is the best possible, for all $p \in (0,1]$

Proof

Using the property in lemma (2.1) we have

$$\left|f(x)\big(g(b)-g(a)\big)-\int_a^b f(t)dg(t)\right| = \left|\int_a^b \big(f(x)-f(t)\big)dg(t)\right|$$

$$\leq \sup_{t\in[a,b]} |f(x)-f(t)| \bigvee_a^b(g).$$

As f is of $p-H-$ Holder type, we have

$$\sup_{t\in[a,b]} |f(x)-g(t)| \leq \sup_{t\in[a,b]} [H|x-t|^p]$$

$$= H \max\{(x-a)^p, (b-x)^p\}$$

$$= H[max\{x-a, b-x\}]^p$$

$$= H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p$$

To prove the sharpness of the constant $\frac{1}{2}$ for any $p \in (0,1]$, assume that (1.3) holds with a constant $c > 0$, that is

$$\left|f(x)\big(g(b)-g(a)\big)-\int_a^b f(t)dg(t)\right|$$

$$\leq H\left[c(b-a)+\left|x-\frac{a+b}{2}\right|\right]^p \bigvee_a^b(g) \tag{2.3}$$

For $f\ be\ p-H-$ Holder type mappings on $[a,b]$ and g of bounded variation on the same interval.

Choose $f(x)=x^p$ $(p\in(0,1])$, $x\in[0,1]$ and $g:[0,1]\to[0,\infty]$ given by

$$g(x)=\begin{cases} 0 & \text{if } x\in[0,1) \\ 1 & \text{if } x=1 \end{cases}$$

As

$$|f(x)-f(y)| = |x^p-y^p| \leq |x-y|^p$$

For all $x, y \in [0,1]. p \in (0,1]$. it follows that f is of $p - H -$ Holder type with the constant 1.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\int_0^1 f(t)dg(t) = f(t)g(t)_0^1 - \int_0^1 g(t)df(t)$$

$$= 1 - 0 = 1$$

And $\bigvee_a^b(g) = 1,$ so

$$|x^p - 1| \leq \left[C + \left|x - \tfrac{1}{2}\right|\right]^p, \text{ for all } \in [0,1].$$

For $x = 0$, we get $1 \leq \left(C + \frac{1}{2}\right)^P$, which implies that $C \geq \frac{1}{2}$.

Remark 3.6 [46]

If f is a convex function on ($f'' \geq 0$), and g is increasing on $[a, b]$ then

by turning to Riemann–Stieltjes integrals The Hermite – Hadamard inequality is not true in general.

$$f\left(\frac{a+b}{2}\right)[g(b) - g(a)] \leq \int_a^b f dg \leq \frac{f(a) + f(b)}{2}[g(b) - g(a)].$$

Example: 3.7

Let $[a, b] = [0, 1]$ and

$f(t) = t^2, \ g(t) = \sqrt{t}$

So left – hand inequality does not hold in general

And if $g(t) = t^{5/2,}$ then

The right – hand inequality does not hold in general to see this, we need shows

$$f\left(\frac{a+b}{2}\right)[g(b) - g(a)] > \int_a^b f dg$$

By the modification of the integral, we have $g' = \frac{1}{2\sqrt{t}}$

So $\int_0^1 f\,\mathrm{d}g = \int_0^1 f\,g'dt = \int_0^1 t^2\left(\frac{1}{2\sqrt{t}}\right)\mathrm{d}t = \frac{1}{2}\int_0^1 t^{\frac{3}{2}}\,\mathrm{d}t$

$$= \frac{1}{2}\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\Big|_0^1\right] = \frac{1}{5}$$

And $f\left(\frac{1+0}{2}\right)[\sqrt{1}+\sqrt{0}] = \frac{1}{4} > \frac{1}{5}$

Thus left – hand inequality does not hold.

Now if $g(t) = t^{5/2}$ so $g' = \frac{5}{2}t^{\frac{3}{2}}$

$$\int_0^1 f\,g'\mathrm{d}t = \int_0^1 t^2\left(\frac{5}{2}t^{\frac{3}{2}}\right)dt$$

$$= \frac{5}{2}\int_0^1 t^{\frac{7}{2}}\,\mathrm{d}t = \frac{5}{2}\left[\frac{t^{\frac{9}{2}}}{\frac{9}{2}}\Big|_0^1\right] = \frac{5}{9}$$

And $\frac{f(1)+f(0)}{2}\,[g(1) - g(0)] = \frac{1}{2} < \frac{5}{9}$

So the right – hand inequality does not hold.

3-2 Inequalities for the Riemann - Stieltjes integral of Product integrators.

In this section we show that if $\boldsymbol{f}, \boldsymbol{g} : [\boldsymbol{a},\ \boldsymbol{b}] \to \boldsymbol{R}$ are two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f} \boldsymbol{d} \boldsymbol{g}$ exists, then for any continuous functions $h : [\boldsymbol{a}, \boldsymbol{b}] \to \boldsymbol{R}$, the Riemann-Stieltjes integral $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{h}\, \boldsymbol{d}(\boldsymbol{f}\boldsymbol{g})$ exists and using this result we then provide sharp upper bounds for the quantity

$$\left|\int_a^b hd(fg)\right|,$$

And apply them for trapezoid and Ostrowski type inequalities.

Lemma3. 8 [22]

If f, g be two functions of bounded variation on $[a,\ b]$, and $\int_{\mathrm{a}}^{\mathrm{b}} f dg$ exists, then for any $\in\ [\ a,\ b\ ]$,

$$L(x) = \int_a^x f\ (x)\ dg\ (t) \text{ of bounded variation and}$$

$$\vee_a^b L \leq \|f\|_\infty \vee_a^b g$$

Proof:

We know the integral $\int_a^x f\ d\ g$ exists for all $x\ \in\ [\ a,\ b\ ]$

Let

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

a division for the Interval $[a, b]$, then

$$\sum_{i=0}^{n-1} \mid L(x_{i+1}) - \ L(x_i) \mid\ = \sum \mid \int_a^{xi+1} f\ d\ g\ - \int_a^{xi} f\ d\ g \mid$$

$$= \sum_{i=0}^{n-1} \mid \int_{xi}^{a} f\ d\ g\ \ + \int_a^{xi+1} f\ d\ g \mid$$

$$= \sum_{i=0}^{n-1} \left| \int_{xi}^{xi+1} f \, d \, g \right|$$

by Lemma (2.1) , then

$$\int_{xi}^{xi+1} f \, dg \ \leq sup_{xi \ t \leq xi+1} \left| f \ (t) \right| \ \bigvee_{xi}^{xi+1} g$$

Therefor

$$\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left| L \ (x_{i+1}) - L(x_i) \right| \ \leq \sum_{i=0}^{n-1} (\sup_{\mathrm{xi \leq t \leq xi+1}} \left| \left| f(t) \right| \ \bigvee_{xi}^{xi+1} g \right)$$

$$\leq \sup_{\mathrm{xi \leq t \leq xi+1}} \left| f(t) \right| \sum_{i=0}^{n-1} \ \bigvee_{xi}^{xi+1}(g)$$

$$= \sup_{\mathrm{a \leq t \leq b}} \left| \ \ \left| f(t) \right| \bigvee_{a}^{b}(g) \right.$$

but $f \ g$ are of bounded variation on $[a \, , b]$

So $= \sup_{\mathrm{a \leq t \leq b}} \left| \ \ \left| f(\mathrm{t}) \right| \bigvee_{a}^{b} g \right. < \infty$

Therefor $\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \left| L \ (x_{i+1}) - L(x_i) \right| < \infty$

Hence $L \ (x)$ is of bounded variation on $[a, \ b]$. □

Theorem 3.9 [22]

Let h: $[a, \ b] \ \rightarrow \ R$ is continuous and f, g be two function of bounded variation on $[a, \ b]$ and $\int_a^b f dg$ exists. Then

$$\int_a^b \ h \ d(fg) \text{ exists, and}$$

$$\int_a^b \ hd(fg) = \int_a^b (hf)dg + \int_a^b (hg)df \quad (3.3)$$

Proof

Let $x \ \in \ [a, \ b]$ then by the integration by parts theorem

$$\int_a^x g(t)\, df(t) \text{ exists}$$

And

$$f\ (x)\ g(x)\ = f\ (a)\ g(a) + \int_a^x f(t)\ dg\ (t) + \int_a^x g(t)\ df(t) \qquad (4.3)$$

We can using (3.3) to say

$$d(f\ (x)\ g(x)\ ) = d\ (f\ (a)\ g(a)\ ) + d \int_a^x f\ (t)\ dg\ (t)\ )$$

$$+ d \int_a^x g\ (t)\ d\ f\ (t)$$

$$h\ (\ x)d\ \big(\ f\ (\ x)g\ (\ x)\big) = \ h\ (x)d\ \left(\int_a^x f\ (t)\ dg\ (t)\ \right)$$

$$+ h\ (a)d\ \ (\ \int_a^x (g\ (t)\ df\ (t)\ )$$

Therefor

$$\int_a^x h\ (x)d\ (\ f(x)\ g(x) = \int_a^x h\ (x)d\ (\ \int_a^x f\ (t)dg\ (t)\ )$$

$$+\int_a^b h\ (x)d\ (\ \int_a^x g\ (t)df\ ) \qquad (5.3)$$

by last lemma

$$\int_a^x fdg \text{ and } \int_a^x gdf \text{ are of bounded variation on } [a, b]$$

Therefor

$$\int_a^b h(x)d\ (\ \int_a^x f(t)dg(t)\ ) \text{ and } \int_a^b h(x)d\ \left(\int_a^x g(t)df(t)\ \right) \text{ exist.}$$

And

$$\int_a^b h(x)d\ (\ \int_a^x f(t)dg(t)\ ) = \int_a^b h(x)f(x)dg(x) \qquad (6.3)$$

$$\int_a^b h(x)d\left(\int_a^x g(t)df(t) \right) = \int_a^b h(x)g(x)df(x) \qquad (7.3)$$

So by (4.3), (5.3) and (6.3)

$$\int_a^b h(x)d\ \big(f(x)\ g(x)\big) = \int_a^b h(x)\ f(x)\ dg(x)$$

$$+\int_a^b h(x)g\ \ (x)\ df(x) \quad \text{for } x \in\ [a,\ b].$$

Notation 3.10

If $f : [a,\ b] \to R$ is a functions of bounded variation $\int_a^b f df$ exists

$h : [a,\ b] \to R$ Continuous then $\int_a^b f df^2 = 2\int_a^b f df$

and if f' exists then

$$\int_a^b h df^2 = 2 \int_a^b f h f' dt$$

Theorem 3.11 [22]

Let $f, g : [a,\ b] \to R$ be two functions of bounded variation such that $\int_a^b f dg$ exists. If h: $[a,\ b] \to R$ is continuous. Then

$$\left| \int_a^b hd(fg) \right| \leq \|fh\|_\infty \mathrm{V}_a^b(g) + \|hg\|_\infty \ \mathrm{V}_a^b(f) \qquad (7.3)$$

$$\leq \|h\|_\infty \left[\ \|f\|_\infty \ \mathrm{V}_a^b(g) + \|g\|_\infty \ \mathrm{V}_a^b(f)\ \right]. \qquad (8.3)$$

Both the above inequalities are sharp

Proof

From (3.3) and lemma (2.1) We have

$$\left| \int_a^b hd(fg) \right| \leq \left| \int_a^b hf dg \right| + \left| \int_a^b hg df \right|$$

$$\leq \|hf\| \mathrm{V}_a^b(g) + \|hg\|_\infty \mathrm{V}_a^b(f)$$

$$\leq \|h\|_\infty \ \|f\|_\infty \mathrm{V}_a^b(g) + \|h\|_\infty \ \|g\|_\infty \mathrm{V}_a^b(f)$$

$$= \|h\|_\infty \ [\ \|f\|_\infty \mathrm{V}_a^b(g) + \|g\|_\infty \mathrm{V}_a^b(f)\ ]$$

Now, to prove the sharpness of (8.3)

let the functions $f, g : [a, b] \to R$ giving by

$$f(t) = \begin{cases} 0 & \text{if} \quad t = a \\ 1 & \text{if} \quad t \in (a, b] \end{cases},$$

and

$$g(t) = \begin{cases} 1 & \text{if} \quad t \in [a, b) \\ 0 & \text{if} \quad t = b \end{cases}$$

The functions f and g are of bounded variation,

$\mathrm{V}_a^b f = \sup \left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) : \{x_i : 1 \le i \le n\}\right.$ is a partition of $[a, b]\}$,

and

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = |1 - 0| + |1 - 1| + \ldots + |1 - 1| = 1$$

So

$$\mathrm{V}_a^b(f) = 1$$

and

$\mathrm{V}_a^b g = \sup \{ \sum_{j=1}^{m} |g(x_j) - g(x_{j-1})| : \{x_j : 1 \le j \le m\}$ is a partition of $[a, b]\}$,

and

$$\sum_{j=1}^{m} |g(x_j) - g(x_{j-1})| = |1 - 1| + |1 - 1| + \cdots + |1 - 1| + |0 - 1| = 1$$

Then

$$\mathrm{V}_a^b(g) = 1 \text{ and } \|g\|_\infty = \|f\|_\infty = 1$$

From

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)| = \sup \{0, 1\} = 1$$

$$\|g\|_\infty = \sup_{t\in[a,b]} |g(t)| = \sup \{0, 1\} = 1,$$

and we have

$$f(t)\ g(t) = \begin{cases} 0 \text{ if } t \in \{a, b\} \\ 1 \text{ if } t \notin \{a, b\} \end{cases}$$

then fg is of bounded variation and for continuous function h

then $\int_a^b hd\,(fg)$ exists

to show $f\,g$ of bounded variation

$$V_a^b(fg) = \sup \{ \sum_{i=0}^{n} \left|(fg)(x_i) - (fg)(x_{i-1})\right| : \{x_i: 0 \le i \le n\} \text{ is a partition of } [a, b] \}$$

$$V_a^b(fg) = \sup \{ |1-0| + |1-1| + \ldots + |0-1| \} = 2$$

We know by the integration by parts

$$\int_a^b hd\,(fg) = f(b)\,g(b)\,h(b) - f\,(a)\,g(a)\,h(a) - \int_a^b fg\,dh$$

$$= -\int_a^b fg\,d\,h \qquad (9.3)$$

To find $\int_a^b fgdh$ consider the following sequence of divisions and intermediate points:

$\Delta_n : a = {x_0}^{(n)} < {\xi_0}^{(n)} < {x_1}^{(n)} < \ldots.. < {x_{n-1}}^{(n)} < {\xi_{n-1}}^{(n)} < {x_n}^{(n)} = b,$

Such that $V(\Delta_n) \to 0$ as $n \to \infty$ where $V(\Delta_n) = \max_{0\le i\le n-1} ({x_{i+1}}^{(n)} - {x_i}^{(n)})$

and if ${\xi_i}^{(n)} \in [{x_i}^{(n)}, {x_{i+1}}^{(n)}]$ for $i \in \{0, 1, \ldots, n-1\}$ then

$$\int_a^b fg\mathrm{d}h = \lim_{V(\Delta_n)\to 0} \sum_{i=1}^{n-1} (f\,g)({\xi_i}^{(n)})\,[h\,({x_{i+1}}^{(n)}) - h\,({x_i}^{(n)})]$$

$$\le \lim_{V(\Delta_n)\to 0} \sum_{i=0}^{n-1} (h\,({x_{i+1}}^{(n)}) - h\,({x_i}^{(n)}))$$

$$= h(b) - h(a),$$

From (9.3)

$$\int_a^b h d\,(fg) \; = -\int_a^b fg\, d\,h \; = \; h(a) - h(b),$$

we also have

$$h(t)f(t) \; = \begin{cases} 0 & \text{if } \; t = a \\ h(t) & \text{if } \; t \in (a\,, b]' \end{cases}$$

and

$$h(t)g\,(\mathrm{t}) = \begin{cases} h(t) & \text{if } \; t \in [a, b) \\ 0 & \text{if } \; t = b \end{cases}$$

then $\|hg\|_\infty = \|hf\|_\infty = \|h\|_\infty,$

by inequality (2.9)

$$|h(b) - h(a)| \; \leq 2\, \|h\|_\infty \tag{10.3}$$

Now, we need show that (10.3) is sharp, so

Let $h(t) = t - \frac{a+b}{2}$, $t \in [a,\; b]$, then

$$|h(b) - h(a)| = b - a, \quad \|h\|_\infty = \frac{b-a}{2}$$

Then $b - a = 2(\frac{b-a}{2}),$

Therefor (8.3) is sharp.

3.3 The Ostrowski and Trapezoid inequalities with product Integrators.

Proposition 3.12 [22]

Let $f, g: [a,\ b] \to R$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)dg(t)$ exists. Then for any $x \in [a,\ b]$we have

$$\left|f(b)g(b)(b-x)+f(a)g(a)(x-a)-\int_a^b f(t)g(t)dt\right|$$

$$\leq\ sup_{t\in[a\ ,b]}|(t-x)g(t)|\, \mathrm{V}_a^b(f)+sup_{t\in[a\ ,b]}|(t-x)f(t)|\, \mathrm{V}_a^b(g)$$

$$\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][\|g\|_\infty \mathrm{V}_a^b(f)+\|f\|_\infty \mathrm{V}_a^b(g)]\,. \qquad (11.3)$$

In particular, we have

$$\left|\frac{f(b)g(b)+f(a)g(a)}{2}(b-a)-\int_a^b f(t)g(t)dt\right|$$

$$\leq sup_{t\in[a,b]}\left|\left(t-\frac{a+b}{2}\right)g(t)\right|\mathrm{V}_a^b(f)+sup_{t\in[a\ ,b]}\left|\left(t-\frac{a+b}{2}\right)f(t)\right|\mathrm{V}_a^b(g)$$

$$\leq\frac{1}{2}(b-a)[\|g\|_\infty \mathrm{V}_a^b(f)+\|f\|_\infty \mathrm{V}_a^b(g)], \qquad (12.3)$$

The inequalities (11.3), (12.3) are sharp.

Proof

We use the following identity

$$F(b)(b-x)+F(a)(x-a)-\int_a^b F(t)dt=\int_a^b (t-x)\, dF(t) \qquad (13.3)$$

That holds for any function of bounded variation $F: [a,b] \to R$ and any $x \in [a,b]$.

If we write the equality (13.3) for $F = fg$ we get

$$f(b)g(b)(b-x)+f(a)g(a)(x-a)-\int_a^b f(t)g(t)dt \qquad (14.3)$$

$$=\int_a^b(t-x)d\big(f(t)g(t)\big),\ \text{for any } x\in[a,\ b]$$

If we use theorems (3.9) and (3.11) for the function $h(t) = t - x, t \in [a,\ b]$, then we have the inequality

$$\left|\int_a^b (t-x)d\big(f(t)g(t)\big)\right|$$

$$\leq \sup_{t\in[a,b]} |(t-x)g(t)| \bigvee_a^b(f) + sup_{t\in[a\ ,b]}|(t-x)g(f)| \bigvee_a^b(g)$$

$$\leq \sup_{t\in[a,b]} |t-x|[\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)\ ]$$

$$= max\{x-a, b-x\}[\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)]$$

$$= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)] \qquad (15.3)$$

The inequality (12.3) follows from (11.3) for $x = \frac{a+b}{2}$.

Consider the functions $f, g: [a, b] \to R$ defined by

$$f(t) = \begin{cases} 0 \text{ if } t = a \\ 1 \text{ if } t\ \in (a,\ b], \end{cases} \qquad g(t) = \begin{cases} 1 \text{ if } t \in [a,\ b) \\ 0 \text{ if } t = b. \end{cases}$$

We observe that f and g are of bounded variation and

$$\bigvee_a^b(f) = \bigvee_a^b(g) = 1$$

Take the sequence of divisions and intermediate points

$$dn: a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < \xi_{n-1}^{n} < x_n^{(n)} = b$$

Such that $\triangle\ (dn) :=\ max_{t\in[0,\dots,n-1]}\left\{x_{i+1}^{(n)} - x_i^{(n)}\right\} \to 0\ as\ n \to \infty$

By the definition of the Riemann-Stieltjes integral $\int_a^b f(t)dg(t)$ we have

$$\int_a^b f(t)dg(t) = \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-2} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right]$$

$$+\ \lim_{n\to\infty} f\left(\xi_{n-1}^{(n)}\right)\left[g(b) - g\left(x_{n-1}^{(n)}\right)\right] = 0 - 1 = -1.$$

Which shows that this integral exists? Observe that

$$\left(t-\frac{a+b}{2}\right)f(t) \;=\begin{cases} \quad 0 & \text{if} \quad t=a \\ t-\frac{a+b}{2} & \text{if } \; t\in(a,\ b], \end{cases}$$

$$\left(t-\frac{a+b}{2}\right)g(t)=\begin{cases} t-\frac{a+b}{2} & \text{if } t\in[a,\ b) \\ 0 & \text{if } \; t=b. \end{cases}$$

Then

$$sup_{t\in[a,b]}\left|\left(t-\frac{a+b}{2}\right)g(t)\right|=\frac{b-a}{2}$$

And

$$sup_{t\in[a,b]}\left|\left(t-\frac{a+b}{2}\right)f(t)\right|=\frac{b-a}{2}.$$

We also have

$$\frac{f(b)g(b)+f(a)g(a)}{2}(b-a)-\int_a^b f(t)g(t)dt=-(b-a).$$

$$|-(b-a)| = \; b-a = \;\; \frac{b-a}{2}+ \;\; \frac{b-a}{2}.$$

So (12.3) is sharp.

Corollary 3.13 [22]

Assume that f,g: $[a,\ b]\rightarrow R$ are monotonic nondecreasing on $[a\ ,b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)dg(t)$ exists. Then for any $x\in[a,\ b]$ we have

$$\left|f(b)g(b)(b-x)+f(a)g(a)(x-a)-\int_a^b f(t)g(t)dt\right|$$

$$\le\; \int_a^b|t-x||g(t)|df(t)+\int_a^b|t-x||f(t)|dg(t)$$

$$\le\; \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left(\int_a^b|g(t)|\,df(t)+\int_a^b|f(t)|dg(t)\right)$$

In particular, we have

$$\left|\frac{f(b)g(b)+f(a)g(a)}{2}(b-a)-\int_a^b f(t)g(t)dt\right|$$

$$\leq \int_a^b \left|t-\frac{a+b}{2}\right| |g(t)|df(t)+\int_a^b \left|t-\frac{a+b}{2}\right| |f(t)|dg(t)$$

$$\leq \frac{1}{2}(b-a)\left(\int_a^b|g(t)|df(t)+\int_a^b|f(t)|dg(t)\right).$$

Corollary 3.14

If f is Lipschitzian with $L \geq 0$, g is Lipschitzian with $K \geq 0$,

and $h:[a, b] \rightarrow R$ is continuous

Then

$$\left| \int_a^b hd(fg)\right| \leq K \int_a^b \left|h f\right| dt + L \int_a^b \left|h g\right| dt$$

$$\leq M \int_a^b \left|h\right| [\left|f\right| + \left|g\right|] dt$$

Where $M = Max\{K, L\}$.

Remark 3.15 [22]

If f, g are continuous at $[a, b]$ and h is Lipschitzian with $M > 0$

Then

$$\left|\int_a^b hd(fg) - I_{b,a}\right| \leq M \int_a^b (fg)\ dh \leq M \|fg\|_\infty$$

Where

$$I_{b,a} = h(b)f(b)g(b) - h(a)f(a)g(a).$$

Proposition 3.16 [22]

Let $f, g: [a, b] \to R$ be two functions of bounded variation and such that for $x \in [a,\ b\ ]$ the Riemann-Stieltjes integrals $\int_a^b f(t)dg(t)$, then

$$\left|f(x)g(x)(b-a) - \int_a^b f(t)g(t)dt\right|$$

$$\leq (x-a) \sup_{t\in[a,x]} \{|f(t)|\} \bigvee_a^x(g) + (x-a) \sup_{t\in[a,x]} \{|g(t)|\} \bigvee_a^x(f)$$

$$+(b-x) \sup_{t\in[a,x]} \{|f(t)|\} \bigvee_x^b(g) + (b-x) \sup_{t\in[a,x]} \{|g(t)|\} \bigvee_x^b(f)$$

$$\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] [\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)]. \qquad (16.3)$$

In particular if the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)dg(t)$ and $\int_{\frac{a+b}{2}}^b f(t)dg(t)$ exist. Then we have

$$\left|f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)g(t)dt\right|$$

$$\leq \frac{b-a}{2} \left[\sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|f(t)|\} \bigvee_a^{\frac{a+b}{2}}(g) + \sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|g(t)|\} \bigvee_a^{\frac{a+b}{2}}(f) \right.$$

$$\left. \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|f(t)|\} \bigvee_a^{\frac{a+b}{2}}(g) + \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|g(t)|\} \bigvee_a^{\frac{a+b}{2}}(f) \right]$$

$$\leq \frac{1}{2}(b-a)[\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)] \qquad (17.3)$$

The inequalities are sharp.

Proof

We use the following identity

$$F\ (x)(b-a) - \int_a^b F\ (t)dt = \int_a^x (t-a)dF(t) + \int_x^b (t-b)dF(t)$$

That holds for any function of bounded variation $F : [a, b] \to R$ and any $x \in [a, b]$.

If we write the equality for $F = fg$ we get

$$f(x)g(x)(b-a) - \int_a^b f(t)g(t)dt$$

$$= \int_a^x (t-a)d\big(f(t)g(t)\big) + \int_x^b (t-b)d\big(f(t)g(t)\big)$$

For any function $f, g: [a,\ b] \to R$ of bounded variation and any $x \in [a,\ b]$.

Taking above modulus:

$$\left|f(x)g(x)(b-a) - \int_a^b f(t)g(t)dt\right|$$

$$\leq \left|\int_a^x (t-a)d\big(f(t)g(t)\big)\right| + \left|\int_x^b (t-b)d\big(f(t)g(t)\big)\right|$$

$$\leq \sup_{t\in[a,x]}\{(t-a)|f(t)|\} \vee_a^x(g) + \sup_{t\in[a,x]}\{(t-a)|g(t)|\} \vee_a^x(f)$$

$$+ \sup_{t\in[x,b]}\{(b-t)|f(t)|\} \vee_x^b(g) + \sup_{t\in[x,b]}\{(b-t)|g(t)|\} \vee_x^b(f)$$

$$\leq (x-a) \sup_{t\in[a,x]}\{|f(t)|\} \vee_a^x(g) + (x-a) \sup_{t\in[a,x]}\{|g(t)|\} \vee_a^x(f)$$

$$+(b-x) \sup_{t\in[x,b]}\{|f(t)|\} \vee_x^b(g) + (b-x) \sup_{t\in[x,b]}\{|g(t)|\} \vee_x^b(f)$$

$$\leq \max\{x-a, b-x\} \sup_{t\in[a,b]}\{|f(t)|\} \vee_a^b(g)$$

$$+ \max\{x-a, b-x\} \sup_{t\in[a,b]}\{|g(t)|\} \vee_a^b(f)$$

$$= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] [\|g\|_\infty \vee_a^b(f) + \|f\|_\infty \vee_a^b(g)],$$

Consider now the functions $f, g: [a,\ b] \to R$ defined by

$$f\ (t)\ = \begin{cases} 0 \text{ if } t \in \left[a,\ \frac{a+b}{2}\right) \\ 1 \text{ if } t \in \left[\frac{a+b}{2},\ b\right] \end{cases} \quad g(t)\ = \begin{cases} 1 \text{ if } t \in \left[a,\ \frac{a+b}{2}\right] \\ 0 \text{ if } t\ \in \left(\frac{a+b}{2},\ b\right] \end{cases}$$

We observe that f and g are of bounded variation and

$$\bigvee_a^b(f) = \bigvee_a^b(g) = 1$$

The Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)dg(t)$ and $\int_{\frac{a+b}{2}}^a f(t)dg(t)$ exist since one function is continuous while the other is of bounded variation on those intervals.

We observe that for these functions we have

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) (b-a) - \int_a^b f(t)g(t)\,dt = b-a,$$

$$\sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|f(t)|\} \bigvee_a^{\frac{a+b}{2}}(g) + \sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|g(t)|\} \bigvee_a^{\frac{a+b}{2}}(f)$$

$$+ \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|f(t)|\} \bigvee_{\frac{a+b}{2}}^b(g) + \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|g(t)|\} \bigvee_{\frac{a+b}{2}}^b(f) = 2$$

and

$$\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g) = 2$$

Therefor

$$|\ f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) (b-a) - \int_a^b f(t)g(t)\,dt| = b-a = \frac{1}{2}(b-a)(2)$$

$$= \frac{1}{2}(b-a)[\|g\|_\infty \bigvee_a^b(f) + \|f\|_\infty \bigvee_a^b(g)]$$

So (17.3) is sharp.

References المراجع

[1] A. M. Acu, A. Babos, and F. Sofonea, The mean value theorems and inequalities of Ostrowski type, Sci. Stud. Res. Ser. Math. Inform. 21 (2011), no. 1, 5–16.

[2] A. M. Acu and F. Sofonea, On an inequality of Ostrowski type, J. Sci. Arts (2011), no. 3(16), 281–287.

[3] M. W. Alomari, A companion of Ostrowski's inequality with applications, Transylv. J. Math. Mech. 3 (2011), no. 1, 9–14.

[4] M. W. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, Appl. Math. Lett. 23 (2010), no. 9, 1071–1076.

[5] G. A. Anastassiou, Ostrowski type inequalities, Proc. Amer. Math. Soc. 123 (1995), no.12, 3775–3781.

[6] G. A. Anastassiou, Ostrowski inequalities for cosine and sine operator functions, Mat. Vesnik, 64 (2012), no. 4, 336–346.

[7] G. A. Anastassiou , Multivariate right fractional Ostrowski inequalities, J. Appl. Math. Inform. 30 (2012), no . 3-4, 445–454.

[8] D. N. Arnald, Aconcise Introduction to Numerical Analysis, © Douglas. N. A., 1999.

[9] D. Anevski, Riemann-Stieltjes integrals, Lund university press, 2012.

[10] R. G. Bartle, The Elements of Real Analysis, Second Edition, John Wiley & Sons Inc., 1976.

[11] N. S. Barnett, W. S. Cheung, S. S. Dragomir, and A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, Comput. Math. Appl. 57 (2009), no. 2, 195–201

[12] N. S. Barnett, S. S. Dragomir, and I. Gomm, A companion for the Ostrowski and then generalised trapezoid inequalities, Math. Comput. Modelling, 50 (2009), no. 1-2, 179–187.

[13] R. F. Bass, Real analysis for graduate students, © copyright 2011 Richard ,F., Bass.

[14] R. L. Burden and J. D. Faires, Numerical Analysis, BROOKS / COLE, Cengage learning, 2010.

[15] C. Buse, M. U. Boldea, S. S. Dragomir, and L. Braescu, A Generalisation Of the trapezoidal rule for the Riemann- Stieltjes integral and Applications, Math. Comput. Modelling, 60 (2000), no.3 123- 342.

[16] P. Cerone, W. S. Cheung, and S. S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrator of bounded variation, Comput. Math. Appl. 54 (2007), no. 2, 183–191.

[17] P. Cerone, J. Roumeliotis, and S. S. Dragomir, some Ostrowski type inequalities for n-time differentiable functions and applications, preprint RGMIA Res. Rep. coll. Vol. 1, no. 1, 1998.

[18] P. Cerone, S. S. Dragomir, J. Roumeliotis and J. Sunde, A new generalization of the Trapezoid formula for n-time differentiable mappings and applications, Demonstration Math. 33(2000), no. 4, 719-736.

[19] P. Cerone, and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Handbook of analytic-computational methods in applied mathematics, 135–200, Chap-man & Hall/CRC, Boca Raton, FL, 2000.

[20] P. Cerone, and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view, Handbook of analytic-computational methods in applied mathematics, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[21] X. L. Cheung and J. Sun, A note on the perturbed trapezoid inequality, J. Inequal. Pure Appl. Math. 3 (2002), no. 2, Article 29, 7 pp. (electronic).

[22] S. S. Dragomir, inequalities for the Riemann- Stieltjes inequalities of Product integrators with Applications, J. Korean Math. Soc. 51(2014), No. 4, pp. 791-815

[23] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of The upper and lower bounds of The first derivation, Tamkang J. Math. 34 (2003), no. 3, 213-222.

[24] S . S . Dragomir, some perturbed ostrowski type inequality for absolutely Continuous functions (I), preprint RGMIA Res. Rep. coll. 16 (2013), Art. 94.

[25] S . S . Dragomir, some perturbed ostrowski type inequality for absolutely Continuous functions (II), preprint RGMIA Res. Rep. coll. 16 (2013), Art. 95.

[26] S . S . Dragomir, some perturbed ostrowski type inequality for absolutely Continuous functions (III), TJMM 7 (2015), no. 1, 31- 43.

[27] S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely Continuous functions and applications, Bull. Korean Math. Soc. 42 (2005), no. 2, 213–230.

[28] S. S. Dragomir, The Ostrowski's integral inequality for lipschitzian Mappings and applications, Bull. Austral. Math. Soc. 60 (1999), no. 3, 495–508.

[29] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, Kragujevac J. Math. 22 (2000), 13–19.

[30] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, Math. Inequal. Appl. 4 (2001), no. 1, 59–66.

[31] S. S. Dragomir, On the trapezoid quadrature formula and applications, Kragujevac J. Math. 23 (2001), 25–36.

[32] S. S. Dragomir, P. Cerone and J. Roumeliatis, Anew generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and Applications in numerical integration and for special means , Appl. Math. Lett. 13(1), 19 – 25, (2000).

[33] S. S. Dragomir, and A. Mcandrew, On trapezoid inequality via a Grüss type result and applications, Tamkang J. Math. 31 (2000), no. 3, 193–201.

[34] S. S. Dragomir, J. Pecarić, and S. Wang, The unified treatment of trapezoid, Simpson, and Ostrowski type inequality for monotonic mappings and applications, Math. Comput. Modelling 31 (2000), no. 6-7, 61–70.

[35] J. Feldman, functions of bounded variation, © joel, f. 2008.

[36] R. A. Gordon, Real Analysis : A first course 2nd edition. Boston, Person Education Inc. 2002.

[37] H. Gunawan, A note on Dragomir - McAndrew's trapezoid inequalities, Tamkang J.Math. 33 (2002), no. 3, 241–244.

[38] A. Iserles, the peano kernel theorem, Copyright © 1999 University of Cambridge.

[39] M. M. Jamei, and S. S. Dragomir, A new generalization of the Ostrowski Inequality and applications, Filomat 25 (2011), no. 1, 115–123.

[40] Z. Liu, Some inequalities of perturbed trapezoid type, J. Inequal. Pure Appl. Math. 7 (2006), no. 2, Article 47, 9 pp.

[41] Z. Liu, Some Ostrowski type inequalities and applications, Vietnam J. Math. 37 (2009), no. 1, 15–22.

[42] Z. Liu, Some companions of an Ostrowski type inequality and applications, J. Inequal. Pure Appl. Math. 10 (2009), no. 2, Article 52, 12 pp.

[43] Z. Liu, A note on Ostrowski type inequalities related to some s-convex functions in the second sense, Bull. Korean Math. Soc. 49 (2012), no. 4, 775–785.

[44] W. J. Liu, Q. L. Xue, and J. W. Dong, New generalization of perturbed trapezoid, midpoint inequalities and applications, Int. J. Pure Appl. Math. 41 (2007), no. 6, 761–768.

[45] W. J. Liu, and J. Park, A Generalization of the companion of ostrowski –like Inequality and applications, App. Math. Inf. Sci.7 (2013), no. 1, 273-278.

[46] P. R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral, J. Math. Anal. Appl. 344 (2008), no. 2, 921–926.

[47] M. H. Protter, Basic Elements of Real Analysis, 1998, Springer Verlag , New York , Inc.

[48] B. G. Pachpatte, A note on a trapezoid type integral inequality, Bull. Greek Math. Soc. 49 (2004), 85–90.

[49] B. G. Pachpatte , New inequalities of Ostrowski and trapezoid type for n-time differentiable functions , Bull. Korean Math. Soc. 41 (2004), no. 4, 633–639.

[50] T. M. Rassias and S. S. Dragomir, Ostrowski type inequalities and Applications in numerical integration, Melbourne university press , 2000.

[51] W. Rudin, principles of mathematical Analysis, third edition, McGraw-Hill, Inc., 1976.

[52] M. Z. Sarikaya, On the Ostrowski type integral inequality, Acta. Math. Univ. Comenian. 79 (2010), no. 1, 129–134.

[53] L. R. Scott, Numerical Analysis, Princeton university press, 2011.

[54] S. C. Schumacher, closer and closer : Introducing Real Analysis 1st edition. Boston, Jones and Bant lett publishers, Inc. 2008.

[55] E. M. Stein and R. Shakarchi, Real analysis, princetion university press, 2007.

[56] B. Thomson, Real analysis , 2nd Edition (2008), www. Classical Real Analysis . com.

[57] N. Ujevic, A generalization of Ostrowski's inequality and applications in numerical integration , Appl. Math . lett . 17 (2004) 133 – 137.

[58] N. Ujevic ', Perturbed trapezoid and mid-point inequalities and applications, Soochow J. Math. 29 (2003), no. 3, 249–257.

[59] N. Ujevic', On perturbed mid-point and trapezoid inequalities and applications, Kyungpook Math. J. 43 (2003), no. 3, 327–334.

[60] N. Ujevic', Error inequalities for a generalized trapezoid rule, Appl. Math. Lett. 19 (2006), no. 1, 32–37.

[61] A. Xiao, Morry space in Harmonic analysis, Ark. Mat. 51(2012), 201- 230.

[62] http: // Math world. Wolfram. Com.

[63] http: // rigmia. Vu. Edu. au / SSDragomir web. Htm1.