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New Pairwise Separation Axioms in Bitopological Spaces

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ABSTRACT

Several pairwise concepts for bitopological spaces (BTS) have been studied by many researchers. In this paper we introduce new pairwise separation axioms p'-Ti (I = 0, 1, 2, 3, 4) and p'-Ri (I = 0, 1) in bitopological spaces, then we study their properties and their relations with the standard separation axioms in BTS.

***Keywords*:** bitopological spaces, separation axioms

AMS Subject Classification (2000): 54A05, 54D05, 54D10, 54D30, 54E55

1. INTRODUCTION

The concept of bitopological spaces (BTS for short) was introduced by Kelly [1] in 1963; where he considered a bitopological space (X,τ,σ) as a set X equipped with two topologies τ and σ. In this paper Kelly defined pairwise separation axioms in bitopological spaces as pairwise Hausdorff, pairwise regular and pairwise normal axioms, and he studied their properties. General details on BTS can be found in [2-9]. In 1966, Murdeshwar and Naimpally [10] offered the notions of pairwise T_0, pairwise T_1, pairwise R_0 and pairwise R_1 bitopological

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spaces. More details on the properties of p-R_0 and p-R_1 in bitopological spaces can be found in both [11] and [12]. Pairwise compact BTS was introduced by swart [13] in 1971, after that in 1973, Reilly [14] defined pairwise Lindelöf BTS and he investigated its properties. See [15-19].

In this paper, we study geeral concepts of bitopological spaces, then we define new pairwise separation axioms in bitopological spaces using the notion of τσ-open sets, and discuss their properties and derive some relations between the separation axioms and the new pairwise separation axioms in BTS which we define.

We organize our work as follows: firstly, we give a brief introduction to the notions of bitopological spaces BTS, then we introduce τσ-open sets and τσ-closed sets in the bitopological space (X, τ, σ) which are due to Lellis and Ravi [20], when we use them to introduce the notion of τσ-closure of a subset of BTS. Some properties of τσ-closure are different from the standard closure in topological space, as: the τσ-closure of τσ-closed set is equal the τσ-closed set but not conversely. Secondly, we mention the concepts of separation axioms in bitopological spaces, as T_i (i=0, 1, 2, 3, 4) and R_i (i=0, 1) spaces, where (X,τ,σ) is T_i (or R_i) if both τ and σ are T_i (or R_i).

The properties of the separation axioms in BTS are similar to the separation axioms in topological spaces. Finally, we define a new pairwise axioms in BTS as; p'-T_i space (i=0, 1, 2, 3, 4), p'-regular space, p'-normal space, and p'-R_i (i=0, 1) space. Note that our definition of p'-T_0 space is identical with p-T_0 space which due to Murdeshwar and Naimpally [10], but the other axioms as: p'-T_i spaces (i=1, 2, 3, 4) and p'-R_i (i=0, 1) spaces are different from Kelly and Murdeshwar's definitions [1, 10]. We concentrate to derive the properties of these new pairwise separation axioms, and how they relate to the separation axioms in BTS.

2. BITOPOLOGICAL SPACES

In this section we give a brief introduction to the notions and concepts of bitopological space BTS that we need in the sequel.

Definition 2.1. [1] Let X be a non-empty set and let τ, σ be two topologies on X, then (X,τ,σ) is called a bitopological space (BTS for short).

Definition 2.2. [20] A subset V of a bitopological space (X,τ,σ) is called τσ-open set if $V \in \tau \cup \sigma$. A subset F of X is called τσ-closed set if $F^c = X/F$ is τσ-open set.

Remark: In bitopological space (X,τ,σ), the subset F of X is τσ-closed if $F \in \mathcal{F}_\tau \cup \mathcal{F}_\sigma$ where $\mathcal{F}_\tau$ is the collection of all closed sets in (X,τ), and $\mathcal{F}_\sigma$ is the collection of all closed sets in (X,σ).

Example 2.1. Let X = {a,b,c,d,e}, τ = {X,φ,{a},{c,d},{a,c,d},{b,c,d,e}}, σ = {X,φ,{a},{c},{a,c}}. Then (X,τ,σ) is a bitopological space, and {c,d},{a,c} are τσ-open sets, while {a,b,d,e}, {a,b,e} are τσ-closed sets, but {a,b}, {a,d} are not τσ-open sets and are not τσ-closed sets.

Definition 2.3. [20] Let (X,τ,σ) be a bitopological space and let $B \subseteq X$, then the τσ-closure of B is denoted by $\overline{B}^{\tau\sigma}$ and define as $\overline{B}^{\tau\sigma} = \overline{B} = \cap \{F : F \text{ is } \tau\sigma\text{-closed set}, B \subseteq F\}$.

Theorem 2.1. If (X,τ,σ) is a bitopological space and A, B are subsets of X then:

(1) $\overline{A}^{\tau\sigma}=\overline{A}^{\tau}\cap\overline{A}^{\sigma}$.

(2) $\overline{A}^{\tau\sigma}\subseteq\overline{A}^{\tau}$ and $\overline{A}^{\tau\sigma}\subseteq\overline{A}^{\sigma}$.

(3) If A is $\tau\sigma$-closed set then $\overline{A}^{\tau\sigma}=A$.

(4) $A\subseteq\overline{A}^{\tau\sigma}$.

(5) If $A\subseteq B$ then $\overline{A}^{\tau\sigma}\subseteq\overline{B}^{\tau\sigma}$.

Proof.

(1) Direct from definition (2.3).

(2) Direct from (1).

(3) If A is $\tau\sigma$-closed set, then A is closed in τ or A is closed in σ. If A is closed in τ, i.e $\overline{A}^{\tau}=A$, then $\overline{A}^{\tau\sigma}=\overline{A}^{\tau}\cap\overline{A}^{\sigma}=A\cap\overline{A}^{\sigma}=A$, and if A is closed in σ i.e $\overline{A}^{\sigma}=A$, then $\overline{A}^{\tau\sigma}=\overline{A}^{\tau}\cap A=A$. So $\overline{A}^{\tau\sigma}=A$.

(4) $A\subseteq\overline{A}^{\tau}$ and $A\subseteq\overline{A}^{\sigma}$, then $A\subseteq\overline{A}^{\tau}\cap\overline{A}^{\sigma}=\overline{A}^{\tau\sigma}$, so $A\subseteq\overline{A}^{\tau\sigma}$.

(5) Since $A\subseteq B$, $\overline{A}^{\tau\sigma}\subseteq\overline{A}^{\tau}\subseteq\overline{B}^{\tau}$ and $\overline{A}^{\tau\sigma}\subseteq\overline{A}^{\sigma}\subseteq\overline{B}^{\sigma}$, i.e $\overline{A}^{\tau\sigma}\subseteq\overline{B}^{\tau}\cap\overline{B}^{\sigma}=\overline{B}^{\tau\sigma}$, then $\overline{A}^{\tau\sigma}\subseteq\overline{B}^{\tau\sigma}$.

Example 2.2. Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a,c\},\{c\}\}$, $\sigma=\{X,\varphi,\{b\}\}$, $A=\{a\}$, then $\overline{A}^{\tau\sigma}=A$ but A is not $\tau\sigma$-closed set.

Definition 2.4. [20] Let (X,τ,σ) be a bitopological space and let $B\subseteq X$. A point $x\in X$ is called a $\tau\sigma$-limit point for B if $B\cap(U_x\setminus\{x\})\neq\varphi$ for any $\tau\sigma$-open set U_x containing x. The set of all $\tau\sigma$-limit points of B denoted by $(B')^{\tau\sigma}$ and called the $\tau\sigma$-derived set of B.

Example 2.3. Let $X=\{a,b,c,d\}$, $\tau=\{X,\varphi,\{a\},\{b\},\{a,b\}\}$, $\sigma=\{X,\varphi,\{c\},\{a,c\}\}$, and $A=\{a,d\}$, then $(A')^{\tau\sigma}=\{d\}$.

Theorem 2.2. In bitopological space (X,τ,σ) if $A\subseteq X$, then $(A')^{\tau\sigma}=(A')^{\tau}\cap(A')^{\sigma}$.

Proof. Suppose $(A')^{\tau\sigma}\nsubseteq(A')^{\tau}\cap(A')^{\sigma}$, i.e there is $x\in(A')^{\tau\sigma}$ but $x\notin(A')^{\tau}\cap(A')^{\sigma}$, i.e $x\notin(A')^{\tau}$ or $x\notin(A')^{\sigma}$. If $x\notin(A')^{\tau}$, then there is open set $U_x\in\tau$ containing x with $A\cap(U_x\setminus\{x\})\neq\varphi$, since $U_x\in\tau$, i.e $U_x\in\tau\cup\sigma$ then $x\notin(A')^{\tau\sigma}$, which is impossible. If $x\notin(A')^{\sigma}$, then there is $V_x\in\sigma$ containing x with $A\cap(V_x\setminus\{x\})\neq\varphi$, since $V_x\in\sigma$, i.e $V_x\in\tau\cup\sigma$ then $x\notin(A')^{\tau\sigma}$ which is impossible.
Then $(A')^{\tau\sigma}\subseteq(A')^{\tau}\cap(A')^{\sigma}$.
Now let $x\in(A')^{\tau}\cap(A')^{\sigma}$, i.e $x\in(A')^{\tau}$ and $x\in(A')^{\sigma}$, then $A\cap(U_x\setminus\{x\})\neq\varphi$ for any open set $U_x\in\tau$ containing x, and $A\cap(V_x\setminus\{x\})\neq\varphi$ for any $V_x\in\tau$ containing x, i.e $A\cap(W_x\setminus\{x\})\neq\varphi$ for any $\tau\sigma$-open set W_x containing x, then $x\in(A')^{\tau\sigma}$, i.e $(A')^{\tau}\cap(A')^{\sigma}\subseteq(A')^{\tau\sigma}$. Then $(A')^{\tau\sigma}=(A')^{\tau}\cap(A')^{\sigma}$.

Definition 2.5. Let (X,τ_1,σ_1) and (X,τ_2,σ_2) be two bitopological spaces, then we say that (X,τ_1,σ_1) weaker than (X,τ_2,σ_2) (or (X,τ_2,σ_2) stronger than (X,τ_1,σ_1)) and written $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$, if $\tau_1\leq\tau_2$ and $\sigma_1\leq\sigma_2$.
Note that $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$ iff any open set in τ_1 is open in τ_2, and any open set in σ_1 is open in σ_2.

Theorem 2.3. If (X,τ_1,σ_1) and (X,τ_2,σ_2) are bitopological spaces, then $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$ if any closed set in τ_1 is closed in τ_2, and any closed set in σ_1 is closed in σ_2.

Definition 2.6. Let (X,τ,σ) be a bitopological space and let $A\subseteq X$, then (A,τ_A,σ_A) is said to be subspace of (X,τ,σ) where $\tau_A=\{U\cap A:U\in\tau\}$, $\sigma_A=\{V\cap A:V\in\sigma\}$.

Theorem 2.4. If (X,τ,σ) is a bitopological space, $A\subseteq X$ and $B\subseteq A$, then $\overline{B}^{\tau_A\sigma_A}=\overline{B}^{\tau\sigma}\cap A$.

Proof. $\overline{B}^{\tau_A\sigma_A}=\overline{B}^{\tau_A}\cap\overline{B}^{\sigma_A}=(\overline{B}^{\tau}\cap A)\cap(\overline{B}^{\sigma}\cap A)=(\overline{B}^{\tau}\cap\overline{B}^{\sigma})\cap A=\overline{B}^{\tau\sigma}\cap A$.

Definition 2.7. [3] Let (X,τ_1,σ_1) and (Y,τ_2,σ_2) be two bitopological spaces, and let $f:(X,\tau_1,\sigma_1)\to(Y,\tau_2,\sigma_2)$ be a map, then f is called continuous (open, closed, homeomorphism) if the maps $f:(X,\tau_1)\to(Y,\tau_2)$ and $f:(X,\sigma_1)\to(Y,\sigma_2)$ are continuous (open, closed, homeomorphism).

Example 2.4. Let $X=\{a,b,c\}$, $Y=\{1,2\}$, $f=\{(a,1),(b,1),(c,2)\}$, $\tau_1=\{X,\varphi,\{a,b\},\{c\}\}$, $\tau_2=\{X,\varphi,\{a\},\{b\},\{a,b\}\}$, $\sigma_1=\{Y,\varphi,\{1\},\{2\}\}$, $\sigma_2=\{Y,\varphi,\{1\}\}$.
Then $f:(X,\tau_1,\sigma_1)\to(Y,\tau_2,\sigma_2)$ is continuous, open and closed.

Definition 2.8. Let (X,τ,σ) be a bitopological space and let $(x_n)_{n=1}^{\infty}=\{x_1, x_2, \ldots\ldots, x_n, \ldots.\}$ be a sequence in X. We say $(x_n)_{n=1}^{\infty}$ converge to a point $x\in X$ if $(x_n)_{n=1}^{\infty}$ converge to x in (X,τ) and $(x_n)_{n=1}^{\infty}$ converge to x in (X,σ).

Example 2.5. Let X=IN, and let (X,τ,σ) be a bitopological space where $\tau=\{X,\varphi\}$ and $\sigma=\{X,\varphi,\{2\},\{2,3\},\{2,3,4\},\ldots\ldots\}$, $(n)_{n=1}^{\infty}=\{1,2,3,\ldots\ldots..\}$, then $(n)_{n=1}^{\infty}\to 1$, but $(n)_{n=1}^{\infty}\nrightarrow 2$.

3. SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

Here we introduce the separation axioms in bitopological spaces as; T_i-spaces (i=0, 1, 2, 3, 4) and R_i-spaces (i=0, 1), and then we discuss their properties. Definitions and results in this section are taken form [1, 2, 21].

3. 1. T_i-Spaces (i= 0, 1, 2, 3, 4)

Definition 3.1.1. A bitopological space (X,τ,σ) is called T_i-space where i=0,1,2,3,4 (regular, normal) if (X,τ) and (X,σ) are T_i-spaces (regular, normal).

Remarks:

(1) Every T_{i+1} bitopological space is T_i (i=0, 1, 2, 3), but not conversely.

(2) If (X,τ_1,σ_1) is T_i where i=0, 1, 2 and $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$, then (X,τ_2,σ_2) is T_i .

Theorem 3.1.1. Every subspace of T_i (or regular) bitopological space is T_i where i=0, 1, 2, 3 (regular).

Example 3.1.1. If X={a,b,c,d}, τ={X, φ,{a},{a,b},{a,c},{a,b,c}}, σ={X,φ,{b,c,d},{c,d},{b,d},{d}}, A={a,b,c} is τσ-closed set, τ_A={A, φ,{a},{a,b}, {a,c}}, σ_A={X, φ,{b,c},{c},{b}}. Then (X,τ,σ) is normal but (A,τ_A,σ_A) is not normal.

Theorem 3.1.2. If (X,τ,σ) is normal space, and F is closed set in τ and closed in σ, then (A,τ_F,σ_F) is also normal space.

Theorem 3.1.3. A bitopological space (X,τ,σ) is T_0-space iff $\overline{\{x\}}^{\tau}\neq\overline{\{y\}}^{\tau}$ and $\overline{\{x\}}^{\sigma}\neq\overline{\{y\}}^{\sigma}$ for every distinct points $x,y\in X$.

Theorem 3.1.4. If (X,τ,σ) is T_1-space, then any finite set is τσ-closed.

Example 3.1.2. Let X={a,b,c}, τ=p(X), σ={X,φ}, then {a}, {b}, {c} are τσ-closed sets but (X,τ,σ) is not T_1-space.

Theorem 3.1.5. If (X,τ,σ) is T_1-space, and A is a finite subset of X, then $(A')^{\tau\sigma}=\varphi$.

Proof. Since (X,τ,σ) is T_1-space i.e (X,τ) and (X,σ) are T_1-space, then we have $(A')^{\tau}=\varphi$ and $(A')^{\sigma}=\varphi$, $(A')^{\tau\sigma}=(A')^{\tau}\cap(A')^{\sigma}$, so $(A')^{\tau\sigma}=\varphi$.

Theorem 3.1.6. The closed continuous image of normal bitopological space is normal.

3. 2. R_i-Spaces (i= 0, 1)

Definition 3.2.1. A bitopological space (X,τ,σ) is called R_i-space if (X,τ) and (X,σ) are R_i (i=0, 1).

Example 3.2.1. Let X={a,b,c}, τ =p(X), σ ={X,φ}, then (X,τ,σ) is R_1 space but not T_0.

Theorem 3.2.1.

(1) Every T_1 bitopological space is R_0.

(2) Every R_1 bitopological space is R_0.

(3) Every T_2 bitopological space is R_1.

Theorem 3.2.2. A bitopological space (X,τ,σ) is T_1-space iff (X,τ,σ) is T_0 and R_0-space.

Theorem 3.2.3. A bitopological space (X,τ,σ) is T_2-space iff (X,τ,σ) is T_0 and R_1-space.

4. PAIRWISE IN BITOPOLOGICAL SPACES

In the present section we introduce some pairwise concepts in bitopological spaces as pairwise continuous (open, closed, homeomorphism) functions. Moreover, we define the notion of pairwise comparison between bitopological spaces.

Definition 4.1. [3] Let (X,τ_1,σ_1) and (Y,τ_2,σ_2) be two bitopological spaces and let f: $(X,\tau_1,\sigma_1)\rightarrow(Y,\tau_2,\sigma_2)$ be a map, then f is called:

(1) Pairwise continuous (p-continuous for short) if $f^{-1}(V) \in\tau_1\cup\sigma_1$ for any $V\in\tau_2\cup\sigma_2$.

(2) Pairwise open (p-open for short) if $f(V)\in\tau_2\cup\sigma_2$ for any $V\in\tau_1\cup\sigma_1$.

(3) Pairwise closed (p-closed for short) if f(F) is $\tau_2\sigma_2$-closed set in (Y,τ_2,σ_2) for any $\tau_1\sigma_1$-closed set F in (X,τ_1,σ_1).

(4) Pairwise homeomorphism (p-homeomorphism for short) if f is bijective function, and f, f^{-1} are p-continuous.

Examples 4.1.

(1) Let X={a,b,c,d}, τ_1={X,φ,{a,b},{c,d}}, σ_1={X,φ,{a,c}}, Y={1,2,3}, τ_2={Y,φ,{1},{2,3}}, σ_2={Y,φ,{2},{1,2}}, f={(a,1),(b,1),(c,2),(d,2)}, then f : $(X,\tau_1,\sigma_1)\rightarrow(Y,\tau_2,\sigma_2)$ is p-continuous but is not continuous because f : $(X,\sigma_1)\rightarrow(Y,\sigma_2)$ is not continuous, since {2} is open in (Y,σ_2) but f^{-1}({2})={c,d} is not open in (X,σ_1). f is p-open but is not open because f : $(X,\tau_1)\rightarrow(Y,\tau_2)$ is not open, since {c,d} $\in\tau_1$ but f({c,d})={2}$\notin\tau_2$. f is p-closed but is not closed because f: $(X,\tau_1)\rightarrow(Y,\tau_2)$ is not closed, since {c,d} is closed set in (X,τ_1) but f({c,d)}={2} is not closed set (Y,τ_2).

(2) Let X={a,b,c,d}, τ_1={X,φ,{a,b}}, σ_1={X,φ,{c,d}}, Y={1,2}, τ_2={Y,φ,{2}}, σ_2={Y,φ,{1}}, f={(a,1),(b,1),(c,2),(d,2)}. Note that f: $(X,\tau_1)\rightarrow(Y,\tau_2)$ and f : $(X,\sigma_1)\rightarrow(Y,\sigma_2)$ are not continuous but f: $(X,\tau_1,\sigma_1)\rightarrow(Y,\tau_2,\sigma_2)$ is p-continuous.

Theorem 4.1. Let f: $(X,\tau_1,\sigma_1)\rightarrow(Y,\tau_2,\sigma_2)$ be a continuous (open, closed, homeomorphism), then f is p-continuous (p-open, p-closed, p-homeomorphism).

Definition 4.2. [20] Let (X,τ,σ) be a bitopological space and let $(x_n)_{n=1}^{\infty} = \{x_1, x_2, \dots\dots, x_n, \dots.\}$ be a sequence in X, we say $(x_n)_{n=1}^{\infty}$ p-converge to a point $x\in X$ if for any $V\in\tau\cup\sigma$ containing x there is $n_0\in IN$ such that $x_n\in V$ for any $n_0\geq n$.

Theorem 4.2. Let (X,τ,σ) be a bitopological space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X, then $(x_n)_{n=1}^{\infty}$ converge to x iff $(x_n)_{n=1}^{\infty}$ p-converge to $x\in X$.

Proof. "⇒" Suppose $(x_n)_{n=1}^{\infty}$ is not p-converge to $x\in X$, i.e there is $\tau\sigma$-open set V such that $x\in V$ and infinite members of $(x_n)_{n=1}^{\infty}$ do not belong to V. $V\in\tau\cup\sigma$ then $V\in\tau$ or $V\in\sigma$. If $V\in\tau$

i.e $(x_n)_{n=1}^{\infty}$ is not converge to x in (X,τ), and if V∈σ i.e $(x_n)_{n=1}^{\infty}$ is not converge to x in (X, σ). So $(x_n)_{n=1}^{\infty}$ is not converge to x in (X,τ,σ), which is impossible.
"⇐" Suppose $(x_n)_{n=1}^{\infty}$ is not converge to x∈X, i.e $(x_n)_{n=1}^{\infty}$ is not converge to x in (X,τ) or in (X,σ), then there is V∈τ∪σ such that infinite members of $(x_n)_{n=1}^{\infty}$ do not belong to V, i.e $(x_n)_{n=1}^{\infty}$ is not p-converge to x, which is impossible.

Theorem 4.3. Let f: $(X,\tau_1,\sigma_1)\to(Y,\tau_2,\sigma_2)$ be a p-continuous function from a bitopological space (X,τ_1,σ_1) to a bitopological space (Y,τ_2,σ_2) and let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x\in X$, then $f(x_n)\to f(x)$.

Proof. Let $V\in\tau_2\cup\sigma_2$ such that f(x)∈V, then $x\in f^{-1}(V)\in\tau_1\cup\sigma_1$ (f is p-continuous), since $x_n \to x$, then there is $n_0\in$IN such that $x_n\in f^{-1}(V)$ for any $n\geq n_0$, then $f(x_n)\in V$ for any $n\geq n_0$, i.e $f(x_n)\to f(x)$.

Definition 4.3. [1] Let (X,τ_1,σ_1) and (X,τ_2,σ_2) be two bitopological spaces, then we say that (X,τ_1,σ_1) p-weaker than (X,τ_2,σ_2) (or (X,τ_2,σ_2) p-stronger than (X,τ_1,σ_1)) and written $(X,\tau_1,\sigma_1) \leqq (X,\tau_2,\sigma_2)$ if $\tau_1\cup\sigma_1\subseteq\tau_2\cup\sigma_2$.

Example 4.2. Let X={a,b,c}, τ_1={X,φ,{a},{b,c}}, σ_1={X,φ,{a,b}}, τ_2={X, φ,{a}}, σ_2={X, φ,{b,c}}, then $(X,\tau_1,\sigma_1) \leqq (X,\tau_2,\sigma_2)$ but $(X,\tau_1,\sigma_1)\nleq(X,\tau_2,\sigma_2)$ because $\sigma_1\nleq\sigma_2$.

Theorem 4.4. If (X,τ_1,σ_1) and (X,τ_2,σ_2) are bitopological spaces, then $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$ iff any $\tau_1\sigma_1$-closed set is $\tau_2\sigma_2$-closed set.

Proof. "⟹" Let F be $\tau_1\sigma_1$-closed set, i.e F^c is $\tau_1\sigma_1$-open set, then $F^c\in\tau_1\cup\sigma_1$ since $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$, i.e $\tau_1\cup\sigma_1\subseteq\tau_2\cup\sigma_2$, then $F^c\in\tau_2\cup\sigma_2$. So F is $\tau_2\sigma_2$-closed set.
"⟸" Let U be $\tau_1\sigma_1$-open set, i.e U^c is $\tau_1\sigma_1$-closed set, so U^c is $\tau_2\sigma_2$-closed set, then U^c is $\tau_2\sigma_2$-closed set, then U is $\tau_2\sigma_2$-open set i.e $\tau_1\cup\sigma_1\subseteq\tau_2\cup\sigma_2$. So $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$.

Theorem 4.5. If $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$, then $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$.

Proof. $(X,\tau_1,\sigma_1)\leq(X,\tau_2,\sigma_2)$ i.e $\tau_1\leq\tau_2$ and $\sigma_1\leq\sigma_2$, then $\tau_1\subseteq\tau_2$ and $\sigma_1\subseteq\sigma_2$, i.e $\tau_1\cup\sigma_1\subseteq\tau_2\cup\sigma_2$.

5. NEW PAIRSIWE SEPARATION AXIOMMS BITOPOLOGICAL SPACES

In this section we define new pairwise separation axioms in bitopological spaces as; p'-T_i spaces (i=0, 1, 2, 3, 4), p'-R_i spaces (i=0, 1), then we investigate the properties for these pairwise separation axioms in BTS. In addition, we study the relation between the separation axioms and the new pairwise separation axioms in BTS.

Our definitions for these pairwise bitopological spaces are different from Kelly and Murdeshwar's definitions [1, 10], except the axiom of p'-T_0 space which is due to Murdeshwar and Naimpally [10].

5. 1. p'-T_i Bitopological Spaces (i= 0, 1, 2, 3, 4)

Definition 5.1.1. [10] A bitopological space (X,τ,σ) is called pairwise T_0 space (p'-T_0 space for short) if whenever x and y are distinct points in X there is $\tau\sigma$-open set U ($U\in\tau\cup\sigma$) containing one point and not the other.

Remarks:

(1) Any T_0 bitopological space is p'-T_0, but converse is not true.

(2) If (X,τ) or (X,σ) is T_0-space, then (X,τ,σ) is p'-T_0.

(3) If (X,τ_1,σ_1) is p'-T_0 bitopological space, $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$, then (X,τ_2,σ_2) is p'-T_0.

Example 5.1.1. Let X={a,b,c }, $\tau=\{X,\varphi,\{a,b\}\}$, $\sigma=\{X,\varphi,\{b\}\}$, then (X,τ,σ) is p'-T_0 space but is not T_0.

Theorem 5.1.1. The following statements are equivalent, for a bitopological space (X,τ,σ).

(1) (X,τ,σ) is p'-T_0 space.

(2) $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$ whenever $x\neq y$, $x,y\in X$.

(3) $\overline{\{x\}}^{\tau}\neq\overline{\{y\}}^{\tau}$ or $\overline{\{x\}}^{\sigma}\neq\overline{\{y\}}^{\sigma}$ whenever $x\neq y$, $x,y\in X$.

Proof. "1→2" Let (X,τ,σ) be a p'-T_0 space and let $x,y\in X$, $x\neq y$, then there is $\tau\sigma$-open set U containing one and not the other. Suppose $x\in U\not\ni y$, since $U\in\tau\cup\sigma$, then $U\in\tau$ or $U\in\sigma$. If $U\in\tau$ i.e $x\notin\{y\}'^{\tau}$, $x\notin\{y\}$, so $x\notin\overline{\{y\}}^{\tau}$, but $\overline{\{y\}}^{\tau\sigma}\subseteq\overline{\{y\}}^{\tau}$, i.e $x\notin\overline{\{y\}}^{\tau\sigma}$, $x\in\overline{\{x\}}^{\tau\sigma}$ then $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$, and similarity if $U\in\sigma$, then $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$. So $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$.

"2→3" Let $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$, i.e there is $z\in\overline{\{x\}}^{\tau\sigma}$ and $z\notin\overline{\{y\}}^{\tau\sigma}$ or there is $z\in\overline{\{y\}}^{\tau\sigma}$ and $z\notin\overline{\{x\}}^{\tau\sigma}$. In the first case: $z\in\overline{\{x\}}^{\tau\sigma}$ and $z\notin\overline{\{y\}}^{\tau\sigma}$, i.e there is $\tau\sigma$-closed set F such that $y\in F\not\ni z$, then $x\notin F$ so $x\notin\overline{\{y\}}^{\tau\sigma}=\overline{\{y\}}^{\tau}\cap\overline{\{y\}}^{\sigma}$, i.e $x\notin\overline{\{y\}}^{\tau}$ or $x\notin\overline{\{y\}}^{\sigma}$, then $\overline{\{x\}}^{\tau}\neq\overline{\{y\}}^{\tau}$ or $\overline{\{x\}}^{\sigma}\neq\overline{\{y\}}^{\sigma}$. Similarity in the second case: $z\in\overline{\{y\}}^{\tau\sigma}$ and $z\notin\overline{\{x\}}^{\tau\sigma}$.

"3→1" Suppose (X,τ,σ) is not p'-T_0 space, then there is $x\neq y$ such that any $\tau\sigma$-open set containing x containing y and any $\tau\sigma$-open set containing y containing x and let $U\in\tau$ (or σ) that contains x. Then U is $\tau\sigma$-open set, since $x\in U$ and (X,τ,σ) is not p-T_0 space, $y\in U$, i.e $x\in\{y\}'^{\tau}\subseteq\overline{\{y\}}^{\tau}$ (or $x\in\{y\}'^{\sigma}\subseteq\overline{\{y\}}^{\sigma}$),then $\overline{\{x\}}^{\tau}\subseteq\overline{\{y\}}^{\tau}$ (or $\overline{\{x\}}^{\sigma}\subseteq\overline{\{y\}}^{\sigma}$). Similarity, if $V\in\tau$ (or $V\in\sigma$) that contains y, then $\overline{\{y\}}^{\tau}\subseteq\overline{\{x\}}^{\tau}$ (or $\overline{\{y\}}^{\sigma}\subseteq\overline{\{x\}}^{\sigma}$). Then $\overline{\{x\}}^{\tau}=\overline{\{y\}}^{\tau}$ (or $\overline{\{x\}}^{\sigma}=\overline{\{y\}}^{\sigma}$). Contradiction

Definition 5.1.2. A bitopological space (X,τ,σ) is called pairwise T_1 space (p'-T_1 space for short) if whenever x and y are distinct points in X there are two $\tau\sigma$-open sets one containing x but not y, and the other containing y but not x.

Remarks:

(1) Any T_1 bitopological space is p'-T_1, but the converse is not true.

(2) If (X,τ) or (X,σ) is T_1-space, then (X,τ,σ) is p'-T_1.

(3) If (X,τ_1,σ_1) is p'-T_1 space, $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$, then (X,τ_2,σ_2) is p'-T_1.

Example 5.1.2. Let $X=\{1,2\}$, $\tau=\{X,\varphi,\{1\}\}$, $\sigma=\{X,\varphi,\{2\}\}$, $\tau\cup\sigma=\{X,\varphi,\{1\},\{2\}\}$, then a bitopological space (X,τ,σ) is p'-T_1 space but not T_1.

Theorem 5.1.2. In bitopological space (X,τ,σ) if $\{x\}$ is $\tau\sigma$-closed set for any $x\in X$, then (X,τ,σ) is p'-T_1 space.

Proof. Let x, $y\in X$, $x\neq y$, then $\{x\}^c$and $\{y\}^c$are $\tau\sigma$-open sets and $x\notin\{x\}^c \ni y$, $y\notin\{y\}^c \ni x$, so (X,τ,σ) is p'-T_1.

Examples 5.1.3.

(1) Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a\},\{b\},\{a,b\}\}$, $\sigma=\{X,\varphi,\{c\}\}$, then (X,τ,σ) is p'-T_1 space but $\{a\}$ is not $\tau\sigma$-closed set.

(2) Let $X=\{a,b,c\}$, $\tau=\{X, \varphi,\{a,b\},\{c\}\}$, $\sigma=\{X, \varphi,\{a,c\},\{b\}\}$, $A=\{a,b,c\}$, note that (X,τ,σ) is p'-T_1, but $(A')^{\tau\sigma} =\{a\}\neq \varphi$. Moreover $a\in(A')^{\tau\sigma}$ but any $\tau\sigma$-open set contains a is finite.

Theorem 5.1.3. If (X,τ) or (X,σ) is T_1, then $(A')^{\tau\sigma} = \varphi$ where A is a finite subset of X.

Proof. Since (X,τ) (or (X,σ)) is T_1-space, then $(A')^{\tau}= \varphi$ (or $(A')^{\sigma}= \varphi$), i.e $(A')^{\tau\sigma}= \varphi \cap(A')^{\sigma}= \varphi$ (or $(A')^{\tau\sigma}=(A')^{\tau}\cap \varphi= \varphi$), so $(A')^{\tau\sigma}= \varphi$.

Theorem 5.1.4. Let (X,τ,σ) be a bitopological space which satisfy condition that any convergence sequence has a unique limit point, then (X,τ,σ) is p'-T_1.

Proof. Let $x,y\in X$, $x\neq y$ note that $\{x, x, \ldots, x, \ldots\}=(x)_1^{\infty}\to x$, $\{y,y,\ldots,y,\ldots\}= (y)_1^{\infty}\to y$, then $(x)_1^{\infty} \nrightarrow y$, i.e there is $U_y\in\tau\cup\sigma$ such that $y\in U_y \not\ni x$, $(y)_1^{\infty} \nrightarrow x$ i.e there is $U_x\in\tau\cup\sigma$ such that $x\in U_x \not\ni y$. So (X,τ,σ) is p'-T_1.

Example 5.1.4. Let X=IN, (X,τ_c) be the cofinite topological space, (X,σ) be the trivial space, then (X,τ_c,σ) is p'-T_1 space, because (X,τ_c) is T_1 space, but $(n)_1^{\infty}$ is a sequence in (X,τ_c,σ), and $(n)_1^{\infty}$ converge to n for any $n\in$IN.

Definition 5.1.3. A bitopological space (X,τ,σ) is called pairwise T_2 space (p'-T_2 space for short) if whenever x and y are distinct points in X there are disjoint $\tau\sigma$-open sets U and V with $x\in U$, $y\in V$.

Remarks:

(1) Any T_2 bitopological space is p'-T_2 but converse is not true.

(2) If (X,τ) or (X,σ) is T_2-space, then (X,τ,σ) is p'-T_2.

(3) If (X,τ_1,σ_1) is p'-T_2 space, $(X,\tau_1,\sigma_1)\leqq(X,\tau_2,\sigma_2)$, then (X,τ_2,σ_2) is p'-T_2.

Example 5.1.5. Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a\},\{b\},\{a,b\}\}$, $\sigma=\{X,\varphi,\{c\}\}$, then a bitopological space (X,τ,σ) is p'-T_2-space but is not T_2.

Theorem 5.1.5. If (X,τ,σ) is p'-T_2 space, then $\{x\}=\cap\{\overline{U}^{\tau\sigma} : U$ is $\tau\sigma$-open set, $x\in U\}$.

Proof. Let $x\in X$, then for any $y\in X$ such that $x\neq y$, there are $\tau\sigma$-open sets U_x, V_y such that $x\in U_x$, $y\in V_y$ and $U_x\cap V_y=\varphi$, then $U_x\subseteq V_y^c$, i.e $\overline{U}_x^{\tau\sigma}\subseteq V_y^c$, $y\notin V_y^c$, i.e $y\notin\overline{U}_x^{\tau\sigma}$, so $y\notin\cap\{\overline{U}_x^{\tau\sigma} : U_x$ is $\tau\sigma$-open set, $x\in U_x\}$. So $\{x\}=\cap\{\overline{U}_x^{\tau\sigma} : U$ is $\tau\sigma$-open set, $x\in U_x\}$.

Theorem 5.1.6. In p'-T_2 space (X,τ,σ) any convergence sequence has a unique limit point.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X, and suppose $x_n\to x$ and $x_n\to y$, $x\neq y$. Since (X,τ,σ) is p'-T_2, then there are $U_x,U_y\in\tau\cup\sigma$ with $x\in U_x$, $y\in U_y$ and $U_x\cap U_y=\varphi$, $x\in U_x$, i.e there is $n_0\in IN$ such that $x_n\in U_x$ for any $n\geq n_0$, and since $U_x\cap U_y=\varphi$, then infinite members of $(x_n)_{n=1}^{\infty}$ do not belong to U_y, then $(x_n)_{n=1}^{\infty}\nrightarrow y$ which is impossible.

Definition 5.1.4. A bitopological space (X,τ,σ) is called pairwise regular space (p'-regular space for short) if whenever A is $\tau\sigma$-closed set and $x\notin A$, then there are two disjoint $\tau\sigma$-open sets U and V with $x\in U$, $A\subseteq V$.
A p'-T_3 space is a p'-regular and p'-T_1 space.

Remark. Any regular bitopological space (T_3 bitopological space) is p'-regular (p'-T_3) but converse is not true.

Examples 5.1.6.

(1) Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a\}\}$, $\sigma=\{X,\varphi,\{b,c\}\}$, then (X,τ,σ) is p'-regular space but is not regular.

(2) Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a\},\{b\},\{a,b\}\}$, $\sigma=\{X,\varphi,\{c\},\{a,c\},\{b,c\}\}$, then (X,τ,σ) is p'-T_3 space, but is not T_3.

Definition 5.1.5. A bitopological space (X,τ,σ) is called pairwise normal space (p'-normal space for short) if whenever E and F are disjoint $\tau\sigma$-closed sets there are disjoint $\tau\sigma$-open sets U and V with $E\subseteq U$ and $F\subseteq V$.
A p'-T_4 space is p'-normal and p'-T_1 space.

Remark. Any normal bitopological space (T_4 bitopological space) is p'-normal space (p'-T_4) but converse is not true.

Examples 5.1.7.

(1) Let X={a,b,c}, τ={X,φ,{b,c},{a,c},{c}}, σ={{X,φ,{a,c},{a,b},{a}}, then (X,τ,σ) is p'-T_4 space but is not T_4.

(2) Let X={a,b,c,d}, τ={X, φ,{a},{a,b},{a,c},{a,b,c}}, σ={X, φ,{d}}, A={a,b,c} is τσ-closed set, τ_A={A, φ,{a},{a,b},{a,c}}, σ_A={A, φ }, then (X,τ,σ) is a p'-normal but (A,τ_A,σ_A) is not p'-normal.

Theorem 5.1.7. Any p'-T_{i+1} bitopological space is p'-T_i (i=0, 1, 2, 3), but the converse is not true.

Theorem 5.1.8. Any subspace of p'-T_i (p-regular) space is p'-T_i where i=0, 1, 2, 3 (p'-regular).

Examples 5.1.8.

(1) Let X={a,b,c}, τ={X,φ,{a}}, σ={X,φ,{a,c},{b}}, then (X,τ,σ) is p'-T_0 space but is not p'-T_1.

(2) Let X be an infinite set and let (X,τ) be a trivial space and τ_c be a cofinite topology, then (X,τ,τ_c) is p'-T_1 space but is not p'-T_2.

(3) Let X={a,b,c}, τ={X,φ, {a},{b},{a,b}}, σ={X,φ,{c}}, then (X,τ,σ) is p'-T_2 space but is not p'-T_3.

(4) Let X={a,b,c}, and let (X,τ) be a trivial space, σ={X,φ,{a}}, then (X,τ,σ) is p'-normal space but is not p'-regular.

(5) Let X={a,b,c}, τ={X,φ,{a},{b},{a,b}}, σ={X,φ}, then (X,τ) is not regular, (X,σ) is regular space but (X,τ,σ) is not p'-regular.

(6) Let X={a,b,c}, τ={X,φ,{b,c},{a,c},{c}}, σ={X,φ}, then (X,τ) is not normal space, (X,σ) is normal space but (X,τ,σ) is not p'-normal.

5. 2. p'-R_i Bitopological Spaces (i= 0, 1)

Definition 5.2.1. A bitopological space (X,τ,σ) is called pairwise R_0 (p'-R_0 for short) if for each V∈τ∪σ and x∈V, then $\overline{\{x\}}^{\tau\sigma}\subseteq V$.

Examples 5.2.1.

(1) Let X={a,b,c}, τ={X,φ,{a},{b,c}}, σ={X,φ}, then (X,τ,σ) is p'-R_0 space but is not p'-T_0.

(2) Let X={a,b,c}, τ={X,φ,{a},{b},{a,b}}, σ={X,φ,{b,c},{a,c},{c}}, then (X,τ,σ) is p'-R_0 space but is not R_0.

(3) Let X={a,b,c}, τ={X, φ }, σ={X, φ,{a},{b},{a,b}}, then (X,τ) is R_0 but (X,τ,σ) is not p'-R_0.

Remark: Any R_0 bitopological space is p'-R_0.

Theorem 5.2.1. Any p'-T_1 bitopological space is p'-R_0.

Proof. Suppose (X,τ,σ) is not p'-R_0 space, i.e there is $x\in V\in\tau\cup\sigma$ but $\overline{\{x\}}^{\tau\sigma}\not\subseteq V$, then there is $y\in\overline{\{x\}}^{\tau\sigma}$ but $y\notin V$ so $x\neq y$, since (X,τ,σ) is p'-T_1, then there is $\tau\sigma$-open set U, $y\in U\not\ni x$, i.e $x\in U^c$ so $\{x\}\subseteq U^c$, then $\overline{\{x\}}^{\tau\sigma}\subseteq U^c$, $y\in\overline{\{x\}}^{\tau\sigma}\subseteq U^c$ i.e $y\in U^c$ which is impossible. So (X,τ,σ) is p'-R_0.

Theorem 5.2.2. A bitopological space (X,τ,σ) is p'-T_1 space iff (X,τ,σ) is p'-T_0 and p'-R_0-space.

Proof. "$\Rightarrow$" Direct from theorems (5.1.7) and (5.2.1).
"$\Leftarrow$" Let $x,y\in X$, $x\neq y$, since (X,τ,σ) is p'-T_0 then there is $U\in\tau\cup\sigma$ such that $x\in U\not\ni y$ or there is $V\in\tau\cup\sigma$ such that $y\in V\not\ni x$.
In the first case: $x\in U\not\ni y$, since (X,τ,σ) is p'-R_0 i.e $\overline{\{x\}}^{\tau\sigma}\subseteq U\not\ni y$, so $y\notin\overline{\{x\}}^{\tau\sigma}=\overline{\{x\}}^{\tau}\cap\overline{\{x\}}^{\sigma}$, i.e $y\notin\overline{\{x\}}^{\tau}$ or $y\notin\overline{\{x\}}^{\sigma}$, if $y\notin\overline{\{x\}}^{\tau}$, i.e $y\in(\overline{\{x\}}^{\tau})^c\in\tau$ that implies $(\overline{\{x\}}^{\tau})^c\in\tau\cup\sigma$, $y\in(\overline{\{x\}}^{\tau})^c\not\ni x$, then we have two $\tau\sigma$-open sets U, $(\overline{\{x\}}^{\tau})^c$ such that $x\in U\not\ni y$ and $y\in(\overline{\{x\}}^{\tau})^c\not\ni x$ so (X,τ,σ) is p'-T_1 space, and if $y\notin\overline{\{x\}}^{\sigma}$ i.e $y\in(\overline{\{x\}}^{\sigma})^c\in\sigma$ that implies $(\overline{\{x\}}^{\sigma})^c\in\tau\cup\sigma$, $y\in(\overline{\{x\}}^{\sigma})^c\not\ni x$, then we have two $\tau\sigma$-open sets U, $(\overline{\{x\}}^{\sigma})^c$ such that $x\in U\not\ni y$ and $y\in(\overline{\{x\}}^{\sigma})^c\not\ni x$. The second case is similar.

Theorem 5.2.3. Every subspace of p'-R_0 space is p'-R_0.

Proof. Let $A\subseteq X$, (X,τ,σ) is p'-R_0, and let $x\in U_A\in\tau_A\cup\sigma_A$, i.e there is $\tau\sigma$-open set $U\in\tau\cup\sigma$, with $U_A=U\cap A$, $x\in U$, then $\overline{\{x\}}^{\tau\sigma}\subseteq U$ (since (X,τ,σ) is p'-R_0), then $\overline{\{x\}}^{\tau_A\sigma_A}=\overline{\{x\}}^{\tau\sigma}\cap A\subseteq U\cap A=U_A$. So (A,τ_A,σ_A) is p'-R_1.

Definition 5.2.2. A bitopological space (X,τ,σ) is called pairwise R_1 space (p'-R_1 space for short) if for each pair of points $x,y\in X$ with $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$ there are $U,V\in\tau\cup\sigma$ with $x\in U$, $y\in V$ and $U\cap V=\varphi$.

Example 5.2.2. Let X=IR, (X,τ_c) be the cofinite topological space, (X,σ) be the trivial space, then (X,τ,σ) is p'-R_0 space but is not p'-R_1 because for any distinct points $a,b\in$ IR, $\overline{\{a\}}^{\tau\sigma}=\{a\}\neq\{b\}=\overline{\{b\}}^{\tau\sigma}$ and for any $\tau\sigma$-open sets U,V such that $U\cap V=\varphi$, $a\in U$, $b\in V$, then $U\subseteq V^c$, which is impossible.

Theorem 5.2.4. Any p'-R_1 space is p'-R_0.

Proof. Suppose (X,τ,σ) is not p'-R_0, i.e there is $x\in V\in\tau\cup\sigma$ but $\overline{\{x\}}^{\tau\sigma}\not\subseteq V$, so there is $y\in\overline{\{x\}}^{\tau\sigma}$, $y\notin V$ i.e $y\in V^c$, then $\overline{\{y\}}^{\tau\sigma}\subseteq V^c$, $x\notin V^c$ i.e $x\notin\overline{\{y\}}^{\tau\sigma}=\overline{\{y\}}^{\tau}\cap\overline{\{y\}}^{\sigma}$, $x\in\overline{\{x\}}^{\tau}$ and $x\in\overline{\{x\}}^{\sigma}$ so $x\in\overline{\{x\}}^{\tau\sigma}$ i.e $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$, then by p'-$R_1$ space there are $\tau\sigma$-open sets U and V, $x\in U$, $y\in W$ and $U\cap W=\varphi$ this impels $U\subseteq W^c$, $x\in W^c\not\ni y$ i.e $\overline{\{x\}}^{\tau\sigma}\subseteq W^c\not\ni y$, then $y\notin\overline{\{x\}}^{\tau\sigma}$, which is contradiction.

Theorem 5.2.5. Any p'-T_2 bitopological space is p'-R_1.

Proof. Let $x,y\in X$, $\overline{\{x\}}^{\tau\sigma} \neq \overline{\{y\}}^{\tau\sigma}$, i.e $x\neq y$, since (X,τ,σ) be a p'-T_2, then there are $\tau\sigma$-open sets $U,V\in\tau\cup\sigma$ with $x\in U$, $y\in V$ and $U\cap V=\varphi$, i.e (X,τ,σ) is p'-R_1 space.

Examples 5.2.3.

(1) Let $X=\{a,b,c\}$, $\tau=\{X,\varphi,\{a,b\}\}$, $\sigma=\{X,\varphi,\{c\}\}$, then (X,τ,σ) is p'-R_1 space but is not p'-T_2 and is not R_1-space.

(2) Let $X=\{a,b,c\}$, $\tau=\{X, \varphi,\{a\},\{b,c\}\}$, $\sigma=\{X, \varphi,\{b\}\}$, then (X,τ) is R_1, but (X,τ,σ) is not p'-R_1.

Remark: Any R_1 bitopological space is p'-R_1.

Theorem 5.2.6. A bitopological space (X,τ,σ) is p'-T_2 space iff (X,τ,σ) is p'-T_1 and p'-R_1-space.

Proof. "$\Rightarrow$" Direct from theorems (5.1.7) and (5.2.5).
"$\Leftarrow$" Let $x,y\in X$, $x\neq y$, since (X,τ,σ) is p'-T_1 then there are $\tau\sigma$-open sets U, V such that $x\in U\not\ni y$ and $y\in V\not\ni x$, V^c is $\tau\sigma$-closed set and $x\in V^c\not\ni y$, i.e $\overline{\{x\}}^{\tau\sigma}\subseteq V^c$ so $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$, since (X,τ,σ) is p'-R_1, then there are $G, H\in\tau\cup\sigma$ such that $x\in G$, $y\in H$ and $G\cap H=\varphi$, i.e (X,τ,σ) is p'-T_2 space.

Theorem 5.2.7. A bitopological space (X,τ,σ) is p'-T_2 space iff (X,τ,σ) is p'-T_0 and p'-R_1-space.

Proof. "$\Rightarrow$" Direct from theorems (5.1.7) and (5.2.5). "$\Leftarrow$" Direct from theorems (5.2.4), (5.2.2) and (5.2.7).

Theorem 5.2.8. Every subspace of p'-R_1 space is p'-R_1.

Proof. Let $A\subseteq X$, (X,τ,σ) is p'-R_1, and let $x,y\in A$, $\overline{\{x\}}^{\tau_A\sigma_A}\neq\overline{\{y\}}^{\tau_A\sigma_A}$, i.e $\overline{\{x\}}^{\tau_A\sigma_A}=\overline{\{x\}}^{\tau_A}\cap\overline{\{x\}}^{\sigma_A}=(\overline{\{x\}}^{\tau}\cap A)\cap(\overline{\{x\}}^{\sigma}\cap A)=(\overline{\{x\}}^{\tau}\cap\overline{\{x\}}^{\sigma})\cap A=\overline{\{x\}}^{\tau\sigma}\cap A$, and $\overline{\{y\}}^{\tau_A\sigma_A}=\overline{\{y\}}^{\tau_A}\cap\overline{\{y\}}^{\sigma_A}=(\overline{\{y\}}^{\tau}\cap A)\cap(\overline{\{y\}}^{\sigma}\cap A)=(\overline{\{y\}}^{\tau}\cap\overline{\{y\}}^{\sigma})\cap A=\overline{\{y\}}^{\tau\sigma}\cap A$, then $\overline{\{x\}}^{\tau\sigma}\cap A\neq\overline{\{y\}}^{\tau\sigma}\cap A$, i.e $\overline{\{x\}}^{\tau\sigma}\neq\overline{\{y\}}^{\tau\sigma}$, but (X,τ,σ) is p'-R_1, then there are disjoint $\tau\sigma$-open sets U and V such that $x\in U$, $y\in V$, then $x\in U\cap A$, $y\in V\cap A$, let $U_A=U\cap A$, $V_A=V\cap A$, then U_A, V_A are $\tau_A\sigma_A$-open sets and $U_A\cap V_A=(U\cap A)\cap(V\cap A)=(U\cap V)\cap A=\varphi\cap A$. So (A,τ_A,σ_A) is p'-R_1.

7. CONCLUSIONS

In this paper we defined a new pairwise separation axioms in bitopological space (BTS), and they are denoted by; p'-T_i spaces (i= 0, 1, 2, 3, 4), p'-regular spaces, p'-normal spaces, and p'-R_i (i= 0, 1) spaces. We concentrate our study to derive the properties of these new axioms, and how they relate to the standard separation axioms in BTS.

We obtain some results, for instant:

1) Every p'-T_{i+1} space (i= 0, 1, 2, 3) is p'-T_i but not conversely.
2) Every T_i-space is p'-T_i space (i= 0, 1, 2, 3, 4) but not conversely.
3) Every regular-space (normal) is p'-regular space (p'-normal) but not conversely.
4) If τ or σ is T_i (i= 0, 1, 2) space, then (X,τ,σ) is p'-T_i .
5) Every p'-T_1 space is p'-R_0 but not conversely.
6) Every p'-T_1 space is p'-T_0 and p'-R_0 space and conversely.
7) Every p'-R_1 space is p'-R_0 but not conversely.
8) Every p'-T_2 space is p'-R_1 but not conversely.
9) Every p'-T_2 space is p'-T_1 and p'-R_1 space and conversely.
10) Every p'-T_2 space is p'-T_0 and p'-R_1 space and conversely
11) Every R_i (i=0,1) space is p'-R_i (i= 0, 1) but not conversely.
12) Every subspace of p'-R_i space (i= 0, 1) is p'-R_i .

In this diagram we show the relation between the separation axioms and the new pairwise separation axioms in bitopological spaces:

$$\begin{array}{ccccccccc}
T_4 & \rightarrow & T_3 & \rightarrow & T_2 & \rightarrow & T_1 & \rightarrow & T_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
p'\text{-}T_4 & \rightarrow & p'\text{-}T_3 & \rightarrow & p'\text{-}T_2 & \rightarrow & p'\text{-}T_1 & \rightarrow & p'\text{-}T_0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & p'\text{-}R_1 & \rightarrow & p'\text{-}R_0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & R_1 & \rightarrow & R_0 & &
\end{array}$$

$$\begin{array}{ccc}
 & p'\text{-}R_0 & \\
p'\text{-}T_1 & \leftrightarrow & p'\text{-}T_0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
 & p'\text{-}R_1 & \\
p'\text{-}T_2 & \leftrightarrow & p'\text{-}T_1
\end{array}$$

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