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Information Geometry of Frechet Distributions

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ABSTRACT

Using the Fisher information matrix (FIM) as a Riemannian metric, the family of Frechet distributions determines a two dimensional Riemannian manifold. In this paper we illustrates the information geometry of the Frechet space, and derive the α-geometry as; α-connections, α-curvature tensor, α-Ricci curvature with its eigenvalues and eigenvectors, α-sectional curvature, α-mean curvature, and α-scalar curvature, where we show that Frechet space has a constant α-scalar curvature. The special case where $\alpha = 0$ corresponds to the geometry induced by the Levi-Civita connection. In addition, we consider three special cases of Frechet distributions as submanifolds with dimension one, and discuss their geometrical structures, then we prove that one of these submanifolds is an isometric isomorph of the exponential manifold, which is important in stochastic process since exponential distributions represent intervals between events for Poisson processes. After that, we introduce log-Frechet distributions, and show that this family of distributions determines a Riemannian 2-manifold which is isometric with the origin manifold. Finally, an explicit expressions for some distances in Frechet space are obtained as, Kullback-Leibler distance, and J-divergence.

***Keywords*:** information geometry, statistical manifold, Frechet distribution, extreme value distributions

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1. INTRODUCTION

Statistical manifolds are representations of families of probability distributions that allow differential geometric methods to be applied to problems in stochastic processes, mathematical statistics and information theory. The origin work was due to Rao [1], who considered a family of probability distributions as a Riemannian manifold using the Fisher information matrix (FIM) as a Riemannian metric. In 1975, Efron [2] defined the curvature in statistical manifolds, and gave a statistical interpretation for the curvature with application to second order efficiency. Then Amari [3] introduced a one-parameter family of affine connections (α-connection), where the α-connection and the $(-\alpha)$-connection are dual with respect to the Fisher metric g, and in particular, the 1-connection is said to be an exponential connection, and the (-1)-connection is said to be a mixture connection. These family of connections turn out to have importance and are part of larger systems of connections for which stability results follow [4]. He further proposed a differential-geometrical framework for constructing a higher-order asymptotic theory of statistical inference. Amari defining the α-curvature of a submanifold, pointed out important roles of the exponential and mixture curvatures and their duality in statistical inference.

Several researchers studied the information geometry and its applications for some families of distributions. Amari [3] showed that the family of univariate Gaussian distributions has a constant negative curvature. Gamma manifold studied by many researcher eg [3], also Arwini and Dodson [5] proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, using an information theoretic metric topology, for more details see [6-9]. Abdel-All, Mahmoud and Add-Ellah [10] showed that the family of Pareto distributions is a space with constant positive curvature and they obtained the geodesics, and they showed the relation between the geodesic distance and the J-divergence. Weibull distribution manifold and the generalized exponential distribution manifold have been studied by Limei, Huafei and Xiaojie [11], who showed that these families has a similar but the Fisher metric and the geometrical structures are quite different.

Oller [12] studied the geometry of the two parameters extreme value distributions as, Gumbel, Weibull and Frethet distributions, and he showed that all these spaces are with constant negative curvatures, and he obtained the geodesic distances in each case. The family of Frechet distributions does not form an exponential family, hence in the present paper we extend Oller's work to derive the α-geometrical quantities, as the the α-connection and α-curvatures objects on the Frechet space without using the concept of potential function. Where here the 0-geometry corresponds to the geometry induced by the Levi-Civita connection.

2. FRECHET DISTRIBUTIONS

The Frechet distribution known as the extreme value type II distribution, and was due to Maurice Frechet who had identified one possible limit distribution for the largest order statistic in 1927 [13]. The distribution has event space $\Omega = \mathbb{R}^+$, and probability density function (pdf) given by

$$F(x,\beta,\lambda) = \frac{\lambda}{\beta}\left(\frac{\beta}{x}\right)^{1+\lambda} e^{-\left(\frac{\beta}{x}\right)^{\lambda}} \quad \text{for } x > 0 \tag{2.1}$$

where: $\beta > 0$ is the scale parameter, and $\lambda > 0$ is the shape parameter. In the case where $\beta = 1$ and $\lambda = 1$ the Frechet distribution has the standard form $(x) = \frac{1}{x^2} e^{-\frac{1}{x}}$. The mean, variance and standard deviation for the Frechet distributions are $e(x) = \gamma\,\beta + \mu$ (where $\gamma = 0.577$ is the Euler gamma constant), $var(x) = \frac{1}{6}\pi^2\beta^2$, and $std.dev(x) = 1.28255\beta$, respectively. More details on Frechet distribution can be found in [14, 15], and some of its applications in [16-20].

Figure 1 shows Frechet distributions, in the cases where $\beta = 1$ with different shape parameters $\lambda = 1,2,3$, and where $\lambda = 1$ with different scala parameters $\beta = 1,2,3$.

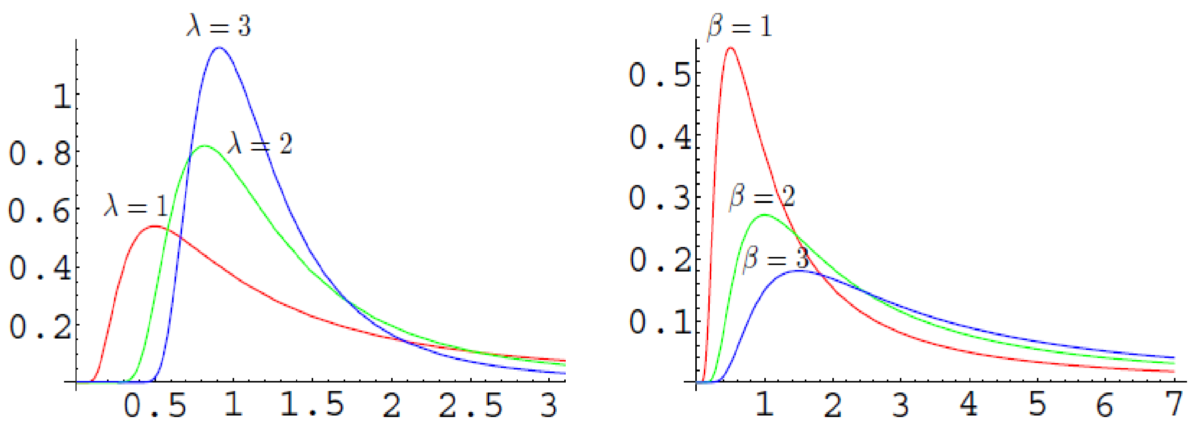

Figure 1. In the left: Frechet distributions with unit scalar parameter β, where $\lambda = 1,2,3$ for the range $x \in (0,3)$. In the right: Frechet distributions with unit shape parameter λ, where $\beta = 1,2,3$ for the range $x \in (0,7)$.

2. 1. Log-likelihood function and Shannon's entropy

The log-likelihood function for the Frechet distribution (2.1) is

$$l(x;\beta,\lambda) = \log\big(F(x;\beta,\lambda)\big)$$

$$= -\left(\frac{\beta}{x}\right)^{\lambda} - (\lambda - 1)\log(x) + \log(\lambda) + \lambda\log(\beta). \qquad (2.2)$$

By direct calculation Shannon's information theoretic entropy for the Frechet distribution, which is the negative of the expectation of the log-likelihood function, is given by

$$S_F(\beta,\lambda) = -\int_0^{\infty} l(x;\beta,\lambda) F(x;\beta,\lambda)\, dx$$

$$= 1 + 2\gamma + \log(\beta) \qquad (2.3)$$

where γ is the Euler gamma constant. Note that the Shannon's entropy is an increasing function, and independent of the parameter λ, moreover $S_F(\beta,\lambda)$ tends to zero when $\beta = e^{-1-2\gamma}$.

Figure 2 shows a plot of S_F as a function of β in the domain $\beta \in (0,3)$.

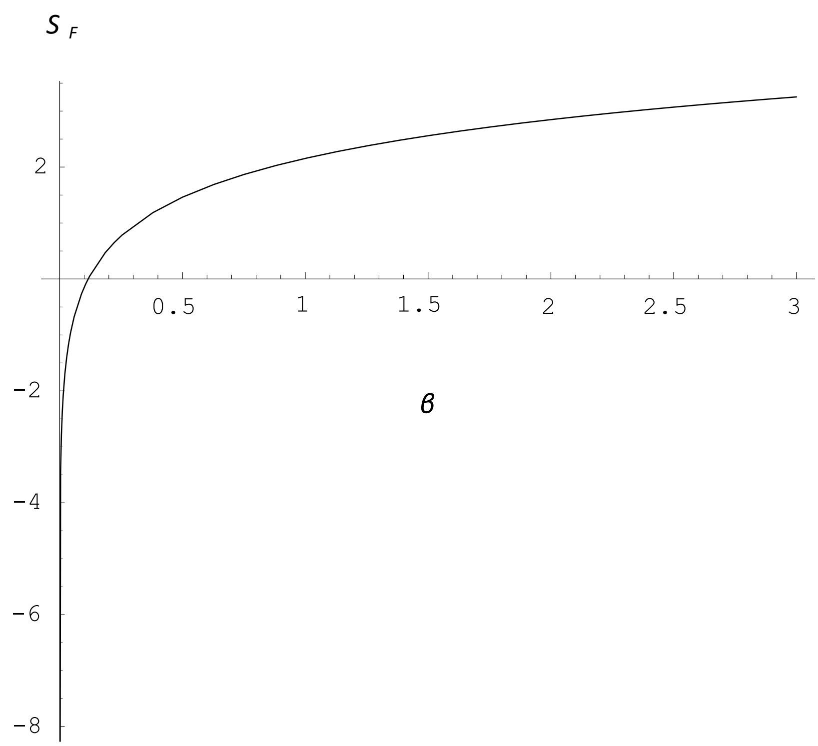

Figure 2. The Shannon's information entropy S_F, for Frechet distributions in the domain $\beta \in (0,3)$.

2. 2. Fisher Information Matrix (FIM)

The Fisher Information Matrix (FIM) is given by the expectation of the covariance of partial derivatives of the log-likelihood function. Here we give the Fisher information components of the family of two parameter Frechet distributions with respect to the coordinate system $(\theta) = (\theta_1, \theta_2) = (\beta, \lambda)$

$$g_{ij} = \int_0^\infty \frac{\partial^2 l(x,\theta)}{\partial\theta_i \partial\theta_j} F(x,\theta) dy dx. \tag{2.4}$$

hence:

$$g = [g_{ij}] = \begin{bmatrix} \frac{\lambda^2}{\beta^2} & \frac{1-\gamma}{\beta} \\ \frac{1-\gamma}{\beta} & \frac{6(\gamma-1)^2+\pi^2}{6\lambda^2} \end{bmatrix}. \tag{2.5}$$

and the variance covariance matrix is

$$g^{-1} = \left[g^{ij}\right] = \begin{bmatrix} \frac{(6(\gamma-1)^2+\pi^2)\beta^2}{\pi^2\lambda^2} & \frac{6(\gamma-1)\beta}{\pi^2} \\ \frac{6(\gamma-1)\beta}{\pi^2} & \frac{6\lambda^2}{\pi^2} \end{bmatrix}. \tag{2.6}$$

3. GEOMETRY OF FRECHET MANIFOLD

In this section we consider the family of Frechet distributions as a Riemannian 2-manifold, equipped with the Fisher information metric, and obtain the α-connection, α-curvature tensor, α-Ricci curvature with its eigenvalues and eigenvectors, α-sectional curvature, α-mean curvature and α-scalar curvature. Where here the 0-geometry corresponds to the geometry induced by the Levi-Civita connection, and we show that this space has a negative constant curvature.

3. 1. Frechet 2-manifold

Let M be the family of all Frechet probability density functions

$$M = \left\{ F(x,\beta,\lambda) = \frac{\lambda}{\beta}\left(\frac{\beta}{x}\right)^{1+\lambda} e^{-\left(\frac{\beta}{x}\right)^{\lambda}} \middle| \beta,\lambda \in \mathbb{R}^+ \right\}, x \in \mathbb{R}^+ \tag{3.7}$$

so the parameter space is $\mathbb{R}^+ \times \mathbb{R}^+$ and the random variable is $x \in \Omega = \mathbb{R}^{+\cdot}$

Following the Rao's idea to use the Fisher information matrix (FIM) as a Riemannian metric, we can consider the family of Frechet distributions M as a Riemannian 2-manifold with coordinate system $(\theta_1,\theta_2) = (\beta,\lambda)$ and Fisher metric g (2.5).

3. 2. α-Connections

For each $\alpha \in R$, the *α (or $\nabla^{(\alpha)}$)-connection* is the torsion-free affine connection with components

$$\Gamma_{ij,k}^{(\alpha)} = \int_0^\infty \left(\frac{\partial^2 \log F}{\partial\theta_i\,\partial\theta_j}\frac{\partial \log F}{\partial\theta_k} + \frac{1-\alpha}{2}\frac{\partial\, log\, F}{\partial\theta_i}\frac{\partial \log F}{\partial\theta_j}\frac{\partial \log F}{\partial\theta_k} \right) F\, dx.$$

We have an affine connection $\nabla^{(\alpha)}$ defined by:

$$\langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)} \qquad \text{where} \qquad \partial_i = \frac{\partial}{\partial\theta_i}$$

So by solving the equations

$$\Gamma_{ij,k}^{(\alpha)} = \sum_{h=1}^{2} g_{kh}\, \Gamma_{ij}^{h(\alpha)}\ , \quad (k = 1,2)$$

.

we obtain the components of $\nabla^{(\alpha)}$.

Here we give the analytic expressions for the α-connections and then the 0-connections, with respect to coordinates $(\theta_1, \theta_2) = (\beta, \lambda)$.

The nonzero independent components $\Gamma^{(\alpha)}_{ij,k}$, $(i, j, k = 1,2)$ are

$$\Gamma^{(\alpha)}_{11,1} = \frac{(\alpha\lambda - 1)\lambda^2}{\beta^3},$$

$$\Gamma^{(\alpha)}_{21,1} = \frac{(1-(\gamma-2)\alpha)\lambda}{\beta^2},$$

$$\Gamma^{(\alpha)}_{22,1} = \Gamma^{(\alpha)}_{12,2} = \frac{(12 + 6(\gamma - 4)\gamma + \pi^2)\alpha}{6\beta\lambda},$$

$$\Gamma^{(\alpha)}_{11,2} = \frac{\gamma - 1 - (1 + (\gamma - 2)\alpha)\lambda}{\beta^2},$$

$$\Gamma^{(\alpha)}_{22,2} = \frac{-6\gamma^3\alpha+(6\gamma^2+\pi^2)(6\alpha-1)-3\gamma\big((12+\pi^2)\alpha-4\big)+6\big(\alpha-1-2\alpha\zeta(3)\big)}{6\lambda^3}. \tag{3.8}$$

where $\zeta(s)$ is the Riemann zeta function, which is defined as $\zeta(s) = \sum_{x=1}^{\infty} x^{-s}$.

The components $\Gamma^{i(\alpha)}_{jk}$, $(i, j, k = 1,2)$ of the $\nabla^{(\alpha)}$-connections are given by

$$\Gamma^{(\alpha)1} = \left[\Gamma^{(\alpha)1}_{ij}\right] = \begin{bmatrix} \frac{6(\gamma-1)(\alpha-1)\lambda+\pi^2(\alpha\lambda-1)}{\pi^2\beta} & \frac{6+6\gamma(\alpha-2)-6\gamma^2(\alpha-1)+\pi^2(\alpha+1)}{\pi^2\lambda} \\ \frac{6+6\gamma(\alpha-2)-6\gamma^2(\alpha-1)+\pi^2(\alpha+1)}{\pi^2\lambda} & \Gamma^{(\alpha)1}_{22} \end{bmatrix}, \tag{3.9}$$

where

$$\Gamma^{(\alpha)1}_{22} = \frac{(\gamma - 1)\beta\left(6\gamma^2(\alpha - 1) + \pi^2(5\alpha - 1) - 2\gamma(-6 + \pi^2\alpha) - 6\big(1 + \alpha + 2\alpha\zeta(3)\big)\right)}{\pi^2\lambda^3} + \frac{\big(12+6(\gamma-4)\gamma+\pi^2\big)\alpha\beta}{6\lambda^3}.$$

$$\Gamma^{(\alpha)2} = \left[\Gamma^{(\alpha)2}_{ij}\right] = \begin{bmatrix} \frac{6(\alpha-1)\lambda^3}{\pi^2\beta^2} & \frac{(-6-6\gamma(\alpha-1)+\pi^2\alpha)\lambda}{\pi^2\beta} \\ \frac{(-6-6\gamma(\alpha-1)+\pi^2\alpha)\lambda}{\pi^2\beta} & \frac{6\gamma^2(\alpha-1)+\pi^2(5\alpha-1)-2\gamma(\pi^2\alpha-6)-6\big(1+\alpha+2\alpha\zeta(3)\big)}{\pi^2\lambda} \end{bmatrix}, \tag{3.10}$$

The components of the Christoffel symbols Γ^{i}_{jk} are given in the case where $\alpha = 0$; as

$$\Gamma^1 = \begin{bmatrix} -\dfrac{6\lambda(\gamma-1)+\pi^2}{\pi^2\beta} & \dfrac{6(\gamma-1)^2+\pi^2}{\pi^2\lambda} \\ \dfrac{6(\gamma-1)^2+\pi^2}{\pi^2\lambda} & -\dfrac{(\gamma-1)\beta(6(\gamma-1)^2+\pi^2)}{\pi^2\lambda^3} \end{bmatrix},$$

$$\Gamma^2 = \begin{bmatrix} -\dfrac{6\lambda^3}{\pi^2\beta^2} & \dfrac{6(\gamma-1)\lambda}{\pi^2\beta} \\ \dfrac{6(\gamma-1)\lambda}{\pi^2\beta} & -\dfrac{6(\gamma-1)^2+\pi^2}{\pi^2\lambda} \end{bmatrix}.$$

3. 3. α-Curvatures

By direct calculation we provide various curvature objects of the Frechet manifold M, as: the α-curvature tensor, the α-Ricci curvature, the α-scalar curvature, the α-sectional curvature, and the α-mean curvature. After that, we give the expressions for the α-curvatures in the special case $\alpha = 0$.

The (α)-curvature tensor components, which are defined as:

$$R_{ijkl}^{(\alpha)} = \sum_{h=1}^{2} g_{hl}\left(\partial_i \Gamma_{jk}^{(\alpha)h} - \partial_j \Gamma_{ik}^{(\alpha)h} + \sum_{m=1}^{2} \Gamma_{im}^{(\alpha)h}\Gamma_{jk}^{(\alpha)m} - \Gamma_{jm}^{(\alpha)h}\Gamma_{ik}^{(\alpha)m}\right) \quad ,(i,j,k,l = 1,2)$$

are given by

$$R_{1211}^{(\alpha)} = \frac{(\pi^2-12)\alpha\lambda^2}{\pi^2\beta^3},$$

$$R_{1212}^{(\alpha)} = \frac{\pi^2\big(1-\alpha(\gamma+3\alpha-2)\big)+6\alpha\big(2\gamma-2+\alpha+2(\alpha-1)\zeta(3)\big)}{\pi^2\beta^2},$$

$$R_{1221}^{(\alpha)} = \frac{\pi^2\big(-1+\alpha(2-\gamma+3\alpha)\big)-6\alpha\big(2-2\gamma+\alpha+2(1+\alpha)\zeta(3)\big)}{\pi^2\beta^2},$$

$$R_{1222}^{(\alpha)} = \frac{-\alpha\Big(24-12\pi^2+\pi^4-2\gamma^2(\pi^2-12)+48\zeta(3)+8\gamma\big(\pi^2-6(1+\zeta(3))\big)\Big)}{2\pi^2\beta\lambda^2}. \quad (3.11)$$

while the other independent components are zero.

The nonzero independent 0-curvature tensor components are given by:

$$R_{1212} = -R_{1221} = \frac{1}{\beta^2}.$$

Contracting $R_{ijkl}^{(\alpha)}$ with g^{il}we obtain the components $R_{jk}^{(\alpha)}$ of the Ricci tensor

$$R_{11}^{(\alpha)} = \frac{-6\lambda^2\left(\pi^2(1+\alpha-3\alpha^2)+6\alpha\big(\alpha+2(\alpha-1)\zeta(3)\big)\right)}{\pi^4\beta^2},$$

$$R_{12}^{(\alpha)} = \frac{3\left(\pi^4\alpha+2\pi^2(\gamma-1+(\gamma-4)\alpha-3(\gamma-1)\alpha^2)+12(\gamma-1)\alpha\big(\alpha+2(\alpha-1)\zeta(3)\big)\right)}{\pi^4\beta},$$

$$R_{22}^{(\alpha)} = \frac{\pi^4\big(-1+\alpha(5-4\gamma+3\alpha)\big)-36(\gamma-1)^2\alpha\big(\alpha+2(\alpha-1)\zeta(3)\big)}{\pi^4\lambda^2},$$

$$+\frac{-6\Big(1+\gamma^2(1+\alpha-3\alpha^2)+\gamma\big(-2+\alpha(6\alpha-7)\big)+2\alpha\big(3+\alpha(\zeta(3)-1)+\zeta(3)\big)\Big)}{\pi^2\lambda^2}. \quad (3.12)$$

In the case where $\alpha = 0$, we obtain the 0-Ricci curvature R :

$$R = \begin{bmatrix} \frac{-6\lambda^2}{\pi^2\beta^2} & \frac{6(\gamma-1)}{\pi^2\beta} \\ \frac{6(\gamma-1)}{\pi^2\beta} & -\frac{6(\gamma-1)^2+\pi^2}{\pi^2\lambda^2} \end{bmatrix}.$$

with eigenvalues and eigenvectors:

$$\begin{pmatrix} \frac{-\beta^2(6(\gamma-1)^2+\pi^2)-6\lambda^4-\sqrt{\left((6(\gamma-1)^2+\pi^2)\beta^2+6\lambda^4\right)^2-24\pi^2\beta^2\lambda^4}}{2\pi^2\beta^2\lambda^2} \\ \frac{-\beta^2(6(\gamma-1)^2+\pi^2)-6\lambda^4+\sqrt{\left((6(\gamma-1)^2+\pi^2)\beta^2+6\lambda^4\right)^2-24\pi^2\beta^2\lambda^4}}{2\pi^2\beta^2\lambda^2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\beta^2(6(\gamma-1)^2+\pi^2)-6\lambda^4-\sqrt{\left((6(\gamma-1)^2+\pi^2)\beta^2+6\lambda^4\right)^2-24\pi^2\beta^2\lambda^4}}{12(\gamma-1)\beta\lambda^2} & 1 \\ \frac{\beta^2(6(\gamma-1)^2+\pi^2)-6\lambda^4+\sqrt{\left((6(\gamma-1)^2+\pi^2)\beta^2+6\lambda^4\right)^2-24\pi^2\beta^2\lambda^4}}{12(\gamma-1)\beta\lambda^2} & 1 \end{pmatrix}$$

By contracting the Ricci curvature components $R_{ij}^{(\alpha)}$with the inverse metric components g^{il} we obtain the scalar curvature $R^{(\alpha)}$:

$$R^{(\alpha)} = \frac{12\left(\pi^2(3\alpha^2-1)-6\alpha^2\left(1+2\zeta(3)\right)\right)}{\pi^4}. \tag{3.13}$$

Note that the Frechet Manifold M is a space with constant α-scalar curvature, where $R^{(\alpha)}$ tends to zero where $\alpha = \pm 1.03665$. Figure 3 shows a plot of $R^{(\alpha)}$ where the 0-scalar curvature is negative constant, and given by:

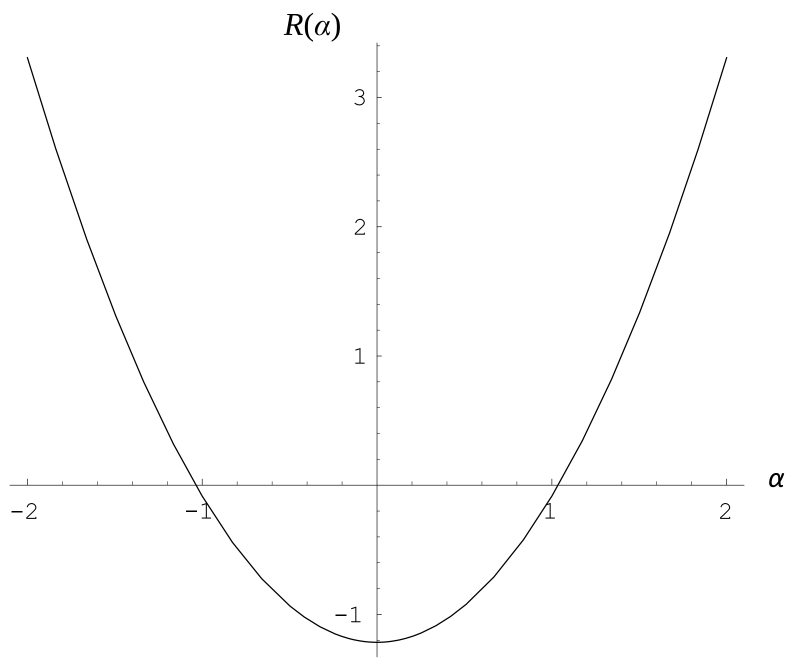

Figure 3. The scalar curvature $R^{(\alpha)}$ for Frechet manifold, in the range $\alpha \in [-2,2]$. $R^{(\alpha)}$ tends to zero where $= \pm 1.03665$. In the case when $\alpha = 0$ the scalar curvature $R^{(0)} = -\frac{12}{\pi^2}$.

$$R = -\frac{12}{\pi^2}.$$

The sectional curvatures $\varrho^{(\alpha)}(i,j)$, where $\varrho^{(\alpha)}(i,j) = \frac{R^{(\alpha)}_{ijij}}{g_{ii}g_{jj}-g_{ij}^2}, (i,j = 1,2)$, are

$$\varrho^{(\alpha)}(1,2) = \frac{6\pi^2\left(-1+\alpha(\gamma-2+3\alpha)\right)-36\alpha\left(2\gamma-2+\alpha+2(\alpha-1)\zeta(3)\right)}{\pi^4}. \tag{3.14}$$

The 0-sectional curvature is given by

$$\varrho^{(\alpha)}(1,2) = -\frac{6}{\pi^2}$$

The mean curvatures $\varrho^{(\alpha)}(i)$ where $\varrho^{(\alpha)}(i) = \sum_{j=1}^{2} \frac{1}{3}\ \varrho^{(\alpha)}(i,j), (i = 1,2)$ are

$$\varrho^{(\alpha)}(1) = \frac{2\pi^2\big(-1+\alpha(\gamma-2+3\alpha)\big) - 12\alpha\big(-2+2\gamma+\alpha+2(\alpha-1)\zeta(3)\big)}{\pi^4}.$$

$$\varrho^{(\alpha)}(2) = \frac{-2\pi^2(1+(\gamma-2-3\alpha)\alpha)-12\alpha\big(2-2\gamma+\alpha+2(\alpha+1)\zeta(3)\big)}{\pi^4}. \quad (3.15)$$

Hence, the 0-mean curvatures are constant and given by:

$$\varrho(1) = \varrho(2) = -\frac{2}{\pi^2}.$$

4. SUBMANIFOLDS

In the present section we study three special cases of the family of Frechet distributions as submanifolds; M_1 where the shape parameter = 1, M_2 where the scala parameter = 1, and M_3 where the shape and scala parameters are identical. These submanifolds have dimension 1 and so that all the curvatures are zero. Moreover, we prove that the space M_1 is an isometric isomorph of the space of exponential distributions.

4. 1. Submanifolds $M_1 : \lambda = 1$

In the case where the shape parameter $\lambda = 1$, the Frechet probability density function (2.1) reduces to the form

$$F(x,\beta) = \frac{\beta}{x^2} e^{\frac{-\beta}{x}}, \quad x > 0, \quad \beta > 0. \quad (4.16)$$

Using $(\theta_1) = (\beta)$ as a local coordinate system, we can consider the family of Frechet distributions with unit shape parameter as a Riemannian 1-manifold, with Fisher metric:

$$g = \left[\ \frac{1}{\beta^2}\ \right]. \quad (4.17)$$

Here we note that, the submanifold M_1 and the manifold of exponential distributions, which is

$$\left\{\beta e^{-\beta x} \,\middle|\, \beta \in \mathbb{R}^+\right\}, x \in \mathbb{R}^+$$

have the same Fisher metric. Hence the submanifold M_1 is an isometric of the exponential manifold. In this manifold all the curvatures are zero, while the components $\Gamma_{11,1}^{(\alpha)}$ and $\Gamma_{11}^{1(\alpha)}$ of the $\nabla^{(\alpha)}$-connections are

$$\Gamma_{11,1}^{(\alpha)} = \frac{\alpha-1}{\beta^3},$$

$$\Gamma_{11}^{(\alpha)1} = \frac{\alpha - 1}{\beta}. \tag{4.18}$$

4. 2. Submanifolds M_2 : $\boldsymbol{\beta} = \mathbf{1}$

In the case where the scalar parameter $\beta = 1$, the Frechet probability density function (2.1) reduces to the form

$$F(x, \lambda) = \lambda \left(\frac{1}{x}\right)^{1+\lambda} e^{-\left(\frac{1}{x}\right)^{\lambda}}, \quad x > 0, \quad \lambda > 0. \tag{4.19}$$

Using $(\theta_1) = (\lambda)$ as a local coordinate system, we can consider the family of Frechet distributions with unit scala parameter as a Riemannian 1-manifold, with Fisher metric:

$$g = \left[\frac{6(\gamma - 1)^2 + \pi^2}{6\lambda^2} \right]. \tag{4.20}$$

In this manifold all the curvatures are zero, while the components of the $\nabla^{(\alpha)}$-connections are

$$\Gamma_{11,1}^{(\alpha)} = \frac{-6\gamma^3\alpha + 6\gamma^2(6\alpha - 1) + \pi^2(6\alpha - 1) - 3\gamma\big((12 + \pi^2)\alpha - 4\big) + 6\big(\alpha - 1 - 2\alpha\zeta(3)\big)}{6\lambda^3},$$

$$\Gamma_{11}^{(\alpha)1} = \frac{-6\gamma^3\alpha + 6\gamma^2(6\alpha - 1) + \pi^2(6\alpha - 1) - 3\gamma\big((12 + \pi^2)\alpha - 4\big) + 6\big(\alpha - 1 - 2\alpha\zeta(3)\big)}{(6(\gamma - 1)^2 + \pi^2)\lambda},$$

$$\Gamma_{11}^{1} = -\frac{1}{\lambda} \tag{4.21}$$

where Γ_{11}^{1} is the Christoffel symbol, in the case where $\alpha = 0$.

4. 3. Submanifolds M_3: $\lambda = \boldsymbol{\beta}$

In the case where $\lambda = \beta$, the Frechet probability density function (2.1) reduces to the form

$$F(x, \beta) = \left(\frac{\beta}{x}\right)^{1+\beta} e^{-\left(\frac{\beta}{x}\right)^{\beta}}, \quad x > 0, \quad \beta > 0. \tag{4.22}$$

Using $(\theta_1) = (\beta)$ as a local coordinate system, we can consider the family of Frechet distributions with unit scala parameter as a Riemannian 1-manifold, with Fisher metric:

$$g = \left[\frac{6\gamma^2 + \pi^2 - 12\gamma(1+\beta) + 6(1+\beta)^2}{6\beta^2} \right]. \tag{4.23}$$

In this manifold all the curvatures are zero, while the components of the $\nabla^{(\alpha)}$-connections are

$$\Gamma_{11,1}^{(\alpha)} = \frac{6\gamma^2(-1+3\alpha(2+\beta)) + \pi^2(-1+3\alpha(2+\beta)) + 6(1+\beta)\left(-1+\alpha(1+\beta(5+\beta))\right)}{6\beta^3},$$

$$+\frac{-6\gamma^3\alpha - 3\gamma\left(-2(2+\beta)+\alpha\left(\pi^2+6(2+\beta(4+\beta))\right)\right)\_12\alpha\zeta(3)}{6\beta^3},$$

$$\Gamma_{11}^{(\alpha)1} = \frac{6\gamma^2(-1+3\alpha(2+\beta)) + \pi^2(-1+3\alpha(2+\beta)) + 6(1+\beta)\left(-1+\alpha(1+\beta(5+\beta))\right)}{\beta(6\gamma^2+\pi^2-12\gamma(1+\beta)+6(1+\beta)^2)}$$

$$+\frac{-6\gamma^3\alpha - 3\gamma\left(-2(2+\beta)+\alpha\left(\pi^2+6(2+\beta(4+\beta))\right)\right)\_12\alpha\zeta(3)}{\beta(6\gamma^2+\pi^2-12\gamma(1+\beta)+6(1+\beta)^2)},$$

$$\Gamma_{11}^{1} = \frac{6-6\gamma+6\beta}{6\gamma^2+\pi^2-12\gamma(1+\beta)+6(1+\beta)^2} - \frac{1}{\beta}. \tag{4.24}$$

where Γ_{11}^{1} is the Christoffel symbol.

5. LOG-FRECHET MANIFOLD

We introduce the log-Frechet distribution, which arises from the Frechet distribution (2.1) for non-negative random variable $y = e^{-x}$. So the log-Frechet distribution, has probability density function (pdf):

$$L_F(y;\beta,\lambda) = \lambda\,\beta^{\lambda}\frac{1}{y}(-\log(y))^{-1-\lambda}e^{-\beta^{\lambda}(-\log(y))^{-\lambda}}, \; y \in (0,1). \tag{5.25}$$

where $\beta > 0, \lambda > 0$. The standard form of the log-Frechet distribution is $L_F(y;1,1)$, with the following probability density function:

$$L_F(y) = \frac{1}{y\log^2(y)}e^{\frac{1}{\log(y)}}.$$

Figure 4 shows plots of the log-Frechet family of densities in the cases where; $\beta = 0.5$ and $\lambda = 1{,}1.5{,}2.5$, and where $\lambda = 2$ and $\beta = 0.2{,}0.4{,}0.6$, for the range $y \in (0,1)$.

Here we show that the families of Frechet and log-Frechet distributions have the same Fisher metric.

From

$$L_F(y) = F(x)\frac{dx}{dy}$$

Hence

$$\log\big(L_F(y)\big) = \log\big(F(x)\big) + \log\left(\frac{dx}{dy}\right)$$

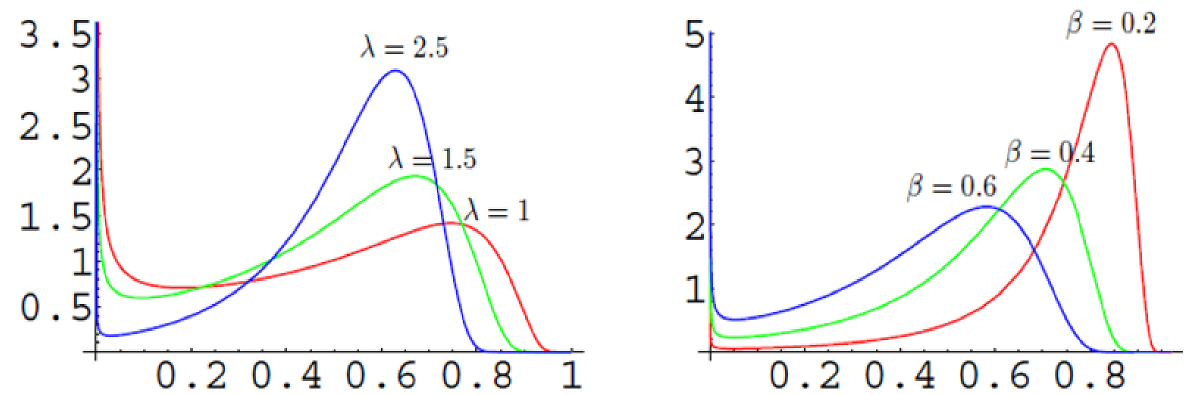

Figure 4. In the left: Log-Frechet distributions where $\beta = 0.5$ and $\lambda = 1,1.5,2.5$, for the range $y \in (0,1)$. In the right: Log-Frechet distributions where $\lambda = 2$ and $\beta = 0.2,0.4,0.6$, for the range $y \in (0,1)$.

Then double differentiation of this relation with respect to θ_i and θ_j (when $(\theta_1,\theta_2) = (\beta,\lambda)$) yields

$$\frac{\partial^2 \log\big(L_F(y)\big)}{\partial\Theta_i\,\partial\Theta_j} = \frac{\partial^2 \log\big(F(x)\big)}{\partial\Theta_i\,\partial\Theta_j}$$

from (2.4) we can see that $F(x)$ and $L_F(y)$ have the same Fisher metric. Hence Frechet and log-Frechet families of distributions have a common differential geometry through the information metric.

6. DISTANCES IN FRECHET MANIFOLD

If M is the Frechet 2-manifold, and F_1 and F_2 are two points in M where:

$$F_i(x,;\beta_i,\lambda_i) = \frac{\lambda_i}{\beta_i}\left(\frac{\beta_i}{x}\right)^{1+\lambda_i} e^{-\left(\frac{\beta_i}{x}\right)^{-\lambda}} \qquad (i = 1,2).$$

Then the Kullback-Leibler distance, and J-divergence between Frechet distributions F_1 and F_1, are given by:

- Kullback-Leibler distance:

  The KullbackLeibler distance or relative entropy is a non-symmetric measure of the difference between two probability distributions. From F_1 to F_2 the Kullbback-distance $KL(F_1,F_2)$ is given by

$$KL(F_1,F_2)=\int_0^{\infty}F_1\log\left(\frac{F_1}{F_2}\right)dx,$$

$$=-1-\gamma+\log\left(\frac{\lambda_1}{\lambda_2}\right)+\Gamma\left(1+\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\beta_2}{\beta_1}\right)^{\lambda_2}+\left(\log\left(\frac{\beta_1}{\beta_2}\right)+\frac{\gamma}{\lambda_1}\right)\lambda_2. \qquad (6.26)$$

- J-divergence:

  The J-divergence is a symmetrization of the Kullback-Leibler distance. Its given by this formula

$$J(F_1,F_2)=\int_0^{\infty}(F_1-F_2)\log\left(\frac{F_1}{F_2}\right)dx,$$

$$=KL(F_1,F_2)+KL(F_2,F_1),$$

$$=-2-2\gamma+\Gamma\left(1+\frac{\lambda_1}{\lambda_2}\right)\left(\frac{\beta_1}{\beta_2}\right)^{\lambda_1}+\Gamma\left(1+\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\beta_2}{\beta_1}\right)^{\lambda_2}$$

$$+\lambda_1\log\left(\frac{\beta_2}{\beta_1}\right)+\frac{\gamma\lambda_1}{\lambda_2}+\lambda_2\log\left(\frac{\beta_1}{\beta_2}\right)+\frac{\gamma\lambda_2}{\lambda_1}. \qquad (6.27)$$

7. FRECHET 3-MANIFOLD

Here we consider the model of Frechet distributions with location parameter as a Riemannian 3-manifold, which contains the family of two parameter Frechet distributions M as a submanifold. We provide the Fisher metric for the new manifold, but the 0-geometry objects have been calculated but are not listed because they are somewhat cumbersome.

The 3-parameter Frechet distribution has probability density function (pdf) given by:

$$F(x;\beta,\lambda,\mu)=\frac{\lambda}{\beta}\left(\frac{\beta}{x-\mu}\right)^{1+\lambda}e^{\left(\frac{\beta}{x-\mu}\right)^{-\lambda}} \quad \text{for } x>\mu \qquad (7.28)$$

where $\beta > 0$ is the scale parameter, $\lambda > 0$ is the shape parameter, and $\mu \in$ R is the location parameter. The mean for the Frechet distribution is $e(x) = \gamma\,\beta + \mu$ (where $\gamma = 0.577$ is the Euler gamma constant). In the case where $\mu = 0$, we have the 2-parameter Frechet distributions (2.1).

The family of all Frechet 3-parameter distributions can be considered as a Riemannian 3-manifold, by identifying (β,λ,μ) as a local coordinate system. Here we provide the Fisher information metric (FIM) for this 3-manifold:

$$g = [g_{ij}] = \begin{bmatrix} \frac{\lambda^2}{\beta^2} & \frac{1-\gamma}{\beta} & \frac{\lambda^2 \Gamma\left(2+\frac{1}{\lambda}\right)}{\beta^2} \\ \frac{1-\gamma}{\beta} & \frac{6(\gamma-1)^2+\pi^2}{6\lambda^2} & \frac{\Gamma\left(\frac{1}{\lambda}\right)\left(1+(\lambda+1)\psi\left(2+\frac{1}{\lambda}\right)\right)}{\beta\lambda^2} \\ \frac{\lambda^2 \Gamma\left(2+\frac{1}{\lambda}\right)}{\beta^2} & \frac{\Gamma\left(\frac{1}{\lambda}\right)\left(1+(\lambda+1)\psi\left(2+\frac{1}{\lambda}\right)\right)}{\beta\lambda^2} & \frac{(\lambda+1)^2 \Gamma\left(\frac{2+\lambda}{\lambda}\right)}{\beta^2} \end{bmatrix}, \quad (7.29)$$

where Γ is the gamma function which has the formula $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$ and $\psi(v) = \frac{\Gamma'(v)}{\Gamma(v)}$ is the digamma function.

The connections and curvature objects of the Frechet 3-manifold are known but they are omitted here.

8. CONCLUSIONS

In this paper we derived the geometrical properties for the 2-manifold of the Frechet distributions, using the Fisher information matrix (FIM) as a Riemannian metric. The α-connections and α-curvatures objects as; α-curvature tensor, α-Ricci curvature, α-sectional curvature, α-mean curvature, and α-scalar curvature are obtained, where we showed that the Frechet manifold has a constant α- scalar curvature.

Three special cases of the Frechet distribution have been studied as submanifolds with dimension 1, where we proved that one of these spaces is an isometric isomorph of the exponential manifold. Finally, the explicit expressions for Kullback-Leibler distance and J-divergence in the manifold of Frechet distribution are obtained.

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