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Smarandache zero and weak zero divisors

Shaban A. Traina [a],*, Abir K. Salibi [b]

[a] *Department of mathematics, University of Benghazi*

[b] *Department of mathematics, University of Omar-El Muktar*

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* Corresponding author:
E-mail address: Shaban.traina@uob.edu.ly
Sh. A. Traina

ABSTRACT

In this paper, conditions on n for the ring of integers modulo n have been obtained to have S-zero divisors and S-weak zero divisors. If n is a composite number of the form $n = p_1p_2p_3$, or $(n = p^m)$, where $p_1p_2p_3$ are distinct prime numbers, or (p a prime with $m \geq 3$), then it has been proved that Z_n has S-zero divisors. Further, conditions on Z_n have been obtained to have S-weak zero divisors and we have established the existence of S-zero divisor if $n = 2^m p$ (where p an odd prime, $m \geq 3$) or $n = 3^m p$ (p a prime different from 3) or in general, when $n = p^m q$ (p, q distinct primes). We also have shown that the group ring $Z_{2n}G$, where n >1, $G = \{g/g^2\} = 1$ has S-zero divisor and S-weak zero divisor. The group ring $Z_{2n+1}G$, $G = \{g/g^2\} = 1$ has only S-weak zero divisor.

1. Introduction

The concepts of Smarandache zero divisors (S-zero divisors) and Smarandache weak zero divisors (S-weak zero divisors) in a ring R *are* illustrated with examples. Both S-zero divisors and S-weak zero divisors are zero divisors but all zero divisors may not be S-zero divisors or S-weak zero divisors.

2. Basic definitions

The notions of S-zero divisors and S-weak zero divisors are introduced and several examples are provided.

Definition 1.1: (Vasantha Kandasamy, W.B. & Chetry M.K., 2005) Let R be a ring, we say that a non-zero element $x \in R$ is a *Smarandache zero divisor* (*S-zero divisor*) if there exists a non-zero element y in R such that $x \cdot y = 0$ and there exist $a, b \in R\{0, x, y\}$, with

1. $xa = 0$ or $ax = 0$
2. $yb = 0$ or $by = 0$ and
3. $ab \neq 0$ or $ba \neq 0$.

Note that in case of commutative rings we just need

$$i)\ xa = 0, \qquad ii)\ yb = 0, \qquad iii)\ ab \neq 0$$

Example 1.2: Let Z_{12} be the ring of integers modulo 12. Clearly 6, and 4 are zero divisors, Now take $a = 2$ and $b = 3$ in $\boldsymbol{Z}_{12}$, we then have

2.6 ≡ 0 (mod 12) & 3.4 ≡ 0 (mod 12), but 2.3 ≢ 0 (mod 12).
So 6 and 4 are S-zero divisors in Z_{12}.

Example 1.3: Let $M_{2\times2} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in Z_2\right\}$ be the set of all 2×2 matrices with entries from the ring of integers $\boldsymbol{Z}_2$. Consider

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $x, y \in M_{2\times2}$ are zero divisors of $M_{2\times2}$as

$$xy = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } xy = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Now take

$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we then have

$$ax = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but}$$

$$xa = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$by = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but}$$

$$yb = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Finally

$$ab = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$ab = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

Hence $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are S-zero divisors of the ring $M_{2\times2}$.

Theorem 1.4: (Vasantha Kandasamy.W.B, 2002) Let R be a ring. Every S-zero divisor is a zero divisor but a zero divisor in general is not a S-zero divisor.

Example 1.5: Let $\boldsymbol{Z}_6$ be the ring of integers modulo 6. Clearly 2 and 3 are zero divisors but are not S-zero divisors.

Theorem 1.6: (Vasantha Kandasamy.W.B. 2004) Let R be a non-commutative ring. $x, y \in R\{0\}$ are S-zero divisors with $a, b \in R\backslash\{0, x, y\}$ satisfying the following conditions:

1. $ax = 0$ and $xa \neq 0$
2. $yb = 0$ and $by \neq 0$
3. $ab = 0$ and $ba = 0$

Then $(xa + by)^2 = 0$, i.e. $xa + by$ is a nilpotent element of R.

Example 1.7: In example 1.3.

Take $x = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, then we have $xy = yx = 0$

Now take $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. So $ax = 0, xa \neq 0$ and $yb = 0, by \neq 0$ and $ab = 0, ba \neq 0$ Then $(xa + by)^2 = 0$ Therefore $xa + by$ is a nilpotent element of R.

Definition 1.8: (Vasantha Kandasamy and Chetry, 2005) An element $x \in R\{0\}$ is a *Smarandache weak zero divisor* (*S-weak zero divisor*) if x is a zero divisor, i.e. $xy = 0$ for $y \in R\{0, x\}$, and there exists $a, b \in R\{0, x, y\}$ such that

1. $xa = 0$ or $ax = 0$.
2. $yb = 0$ or $by = 0$.
3. $ab = 0$ or $ba = 0$

Example 1.9: In $\boldsymbol{Z}_{20}$; we have
$4.5 \equiv 0 \pmod{20}$, $10.4 \equiv 0 \pmod{20}$ and $8.5 \equiv 0 \pmod{20}$, also $10.8 \equiv 0 \pmod{20}$. So 4 and 5 are S-weak zero divisors in $\boldsymbol{Z}_{20}$. We can also check whether $\boldsymbol{Z}_{20}$ has S-zero divisors. For $4{,}10 \in \boldsymbol{Z}_{20}$, we have $4.10 \equiv 0 \pmod{20}$. Now take $2{,}5 \in \boldsymbol{Z}_{20}$ such that $5.4 \equiv 0 \pmod{20}$, and $2.10 \equiv 0 \pmod{20}$, but $2.5 \not\equiv 0 \pmod{20}$ Thus $\boldsymbol{Z}_{20}$ has both S-zero divisor and S-weak zero divisor.

Theorem 1.10: Let R be a non-commutative ring. $x, y \in R\{0\}$ are S-weak zero divisors with $a, b \in R\{0, x, y\}$ satisfying the following conditions:

1. $ax = 0$ and $xa \neq 0$
2. $yb = 0$ and $by \neq 0$
3. $ab = 0$ and $ba = 0$

Then $(xa + by)^2 = 0$ i.e. $xa + by$ is a nilpotent element of R.

Proof. Consider $(xa + by)^2 = xaxa + xaby + byxa + byby$; $ax = 0$, using $ax = 0$, $ax = 0$; $ab = 0$, $xy = yx = 0$, and $yb = 0$ we get $xa + by$ to be a nilpotent element of order 2.

Example 1.11: In example 1.3.

Take $x = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, then we have $xy = yx = 0$

Now take $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. So $ax = 0$, $xa \neq 0$ and $yb = 0$, $by \neq 0$ and $ab = 0$, $ba = 0$

Then $(xa + by)^2 = 0$. Thus, $xa + by$ is a nilpotent element of R.

3. S-zero divisors and S-weak zero divisors in $\boldsymbol{Z}_n$

In this section we find conditions for $\boldsymbol{Z}_n$ to have S-zero divisors and S-weak zero divisors. We show that if n is of the form pq, where pq distinct odd primes, then $\boldsymbol{Z}_n$ has no S-zero divisors but it has S-weak zero divisors. More results are proved.

Theorem 2.12 (Vasantha Kandasamy and Chetry, 2005)**:** Let $\boldsymbol{Z}_p$ be the ring of integers modulo p, where p is a prime, $\boldsymbol{Z}_p$ has no S-zero divisor and S-weak zero divisor.

Theorem 2.13: Let $\boldsymbol{Z}_{2p}$ be the ring of integers modulo $2p$, p a prime, $\boldsymbol{Z}_{2p}$ has no S-zero divisor or S-weak zero divisor.
Proof. Let *x, y* be two zero divisors in $\boldsymbol{Z}_{2p}$ such that $xy \equiv 0 \pmod{2p}$. Then x, y must be of the following form: Take $x = p$, $x = p$, $y = 2m$, $m \in \boldsymbol{Z}$.; x and y cannot be S-zero divisors since there is no $a, b \in \boldsymbol{Z}_{2p} \backslash \{p\}$ such that $xa \equiv 0 \pmod{2p}$, $by \in 0 \pmod{2p}$, $ab \equiv 0 \pmod{2p}$. Hence the claim.

Example 2.14: In $\boldsymbol{Z}_6$; The only zero divisors are 2,3 and 4. However, they cannot be S-zero divisors or S-weak zero divisors.

Theorem 2.15: Let $\boldsymbol{Z}_{pq}$ be the ring of integers modulo *pq, where p, q* are distinct odd primes, then

1. $\boldsymbol{Z}_{pq}$ has S-weak zero divisors.
2. $\boldsymbol{Z}_{pq}$ has no S-zero divisors.

Proof.

(i). Let x, y be non-zero element in $\boldsymbol{Z}_{pq}$, take $x = p$ and $y = q$, then $x.y \equiv 0 \pmod{pq}$.

Now take $a \equiv k_1 q \pmod{pq}$, $k_1 \in Z$, and $b \equiv k_2 p \pmod{pq}$, $k_2 \in Z$, then $xa \equiv 0 \pmod{pq}$, $by \equiv 0 \pmod{pq}$ and $ab \not\equiv 0 \pmod{pq}$ or $ab \equiv 0 \pmod{pq}$.
So $\boldsymbol{Z}_{pq}$ has S-weak zero divisors.

(ii) Let x, y be a non-zero element in $\boldsymbol{Z}_{pq}$, $x.y \equiv 0 \pmod{pq}$. So, x and y must be as in the following form $x \equiv k_1 p \pmod{pq}$, $k_1 \in \boldsymbol{Z}$, and $y \equiv k_2 q \pmod{pq}$.
Now to find $a, b \in \boldsymbol{Z}_{pq}\{0, x, y\}$ such that $a.x \equiv 0 \pmod{pq}$, and $b.y \equiv 0 \pmod{pq}$, a and b must be of the following form $a = k_3 q \pmod{pq}$, $k_3 \in \boldsymbol{Z}$ and $b = k_4 q \pmod{pq}$, $k_4 \in \boldsymbol{Z}$. From this, we get $a.b \equiv 0 \pmod{pq}$. So, $\boldsymbol{Z}_{pq}$ has no S-zero divisors.

Example 2.16: Let $\boldsymbol{Z}_{15}$ be the ring of integers modulo 15. Clearly 3, 5, 6, 9, 10 and 12 are S-weak zero divisors in $\boldsymbol{Z}_{15}$.

Corollary 2.17 (Vasantha Kandasamy and Chetry, 2005)**:** $\boldsymbol{Z}_{p^2}$, p be an odd prime greater than 3, has no S-zero divisors and has S-weak zero divisors.

Example 2.18: In $\boldsymbol{Z}_{25}$; take $x = 5, y = 10$, and $a = 15, b = 20$. Then $5.10 \equiv 0 \pmod{25}$, $5.15 \equiv 0 \pmod{25}$, $10.20 \equiv 0 \pmod{25}$ and $15.20 \equiv 0 \pmod{25}$.

Theorem 2.19 [Vasantha Kandasamy and Chetry, 2005]**:** $\boldsymbol{Z}_{p^n}$ has S-zero divisors, p be a prime and $n \geq 3$.

Example 2.20: In $\boldsymbol{Z}_8$; we have $x = 4, y = 4$ and $x.y \equiv 0 \pmod 8$. Take $a = 2$, $b = 6$, then $a.x \equiv 0 \pmod 8$, $b.y \equiv 0 \pmod 8$ but $a.b \not\equiv 0 \pmod 8$.

Corollary 2.21: $\boldsymbol{Z}_{p^n}$ has S-weak zero divisor, where p odd prime, $n \geq 3$
Proof. Take $x = p, y = p^{n-1}$ and take $a = kp^{n-1}, b = kp$
So we have $ax \equiv 0 \pmod{p^n}$, $by \equiv 0 \pmod{p^n}$, and $ab \equiv 0 \pmod{p^n}$. Hence the claim.

Theorem 2.22: $\boldsymbol{Z}_n$ has S-zero divisor when $n = p_1 p_2 p_3$, where p_1, p_2, p_3 are distinct primes.
Proof. Take $x = p_1 p_2$ and $y = p_1 p_3$, then $xy \equiv 0 \pmod n$, and take $a = p_3$ and $b = p_2$,then $ax \equiv 0 \pmod n$,and $by \equiv 0 \pmod n$, but $ab \not\equiv 0 \pmod n$. Hence the claim.
On the other hand, Take $x = p_1 p_2, y = p_3$ and $a = kp_3, b = kp_1 p_2$, $k \in \boldsymbol{Z}$ Therefore, $ax \equiv 0 \pmod n$, $by \equiv 0 \pmod n$, and $ab \equiv 0 \pmod n$.

Example 2.23: In $\boldsymbol{Z}_{30}$, $(30 = 2.3.5)$, we have
$x = 6, y = 10$, and $6.10 \equiv 0 \pmod{30}$
$a = 5, b = 3, 6.5 \equiv 0 \pmod{30}, 10.3 \equiv 0 \pmod{30}$,
But $5.3 \not\equiv 0 \pmod{30}$. So, 6 and 10 are S-zero divisors.
Also, $x = 5, y = 6$ are S-weak zero divisors with $a = 12, b = 10$
We can generalize theorem 2.22 as follows:

Theorem 2.24: $\boldsymbol{Z}_n$ has S-zero divisor when $n = p_1 p_2 \cdots p_t$, where $p_1 p_2 \cdots p_t$ are distinct primes.
Proof. For S-zero divisors, the proof is similar to the previous theorem.
Now take $x = p_1 p_2 ... p_{t-1}$, $n = p_1 p_2 \cdots p_t$ and $n = p_3 p_4 \cdots p_t$, $b = p_1 p_2$. Hence the claim.

Theorem 2.25: Let $\boldsymbol{Z}_{2^m p}$ be the ring of integers modulo $2^m p$, where p be an odd prime and $m \geq 2$, then $2p$, 2^m and $2kp$, $2^m k$ ($k \in \mathbf{Z}$) are S-zero divisors in $Z_{2^m p}$.

Proof. $2p^m \equiv 0 \pmod{2^m p}$.

Take $a = 2^{m-1}$ and $b = p$.

Then $2p2^{m-1} \equiv 0 \pmod{2^m p}$ and $2^m p \equiv 0 \pmod{2^m p}$ and we have:

$2^{m-1} p \not\equiv 0 \pmod{2^m p}$.

Therefore $2p$ and 2^m are S-zero divisors in $\boldsymbol{Z}_{2^m p}$.

Note: The number of S-zero divisor in $Z_{2^m p}$ is $(2^{n-1} - 2 + p)$.

Example 2.26: In $\boldsymbol{Z}_{24}$, $(24 = 2^3.3)$, 6, 8, 12, 16 and 18 are all the S-zero divisors in $\boldsymbol{Z}_{24}$, i.e. the number of S-zero divisors in $\boldsymbol{Z}_{24}$ is 5 which can be calculated using the formula in the last note as $(2^{3-1} - 2 + 3) = 5$.

Theorem 2.27: Let $\boldsymbol{Z}_{3^m p}$ be the ring of integers modulo $3^m p$, p be a prime such that $p \neq 3$ and $m \geq 2$,then $3p, 3^m$ are S-zero divisors in $Z_{3^m p}$ also $3kp$ and, $k \in \boldsymbol{Z}$, are S-zero divisors in $\boldsymbol{Z}_{3^m p}$.
Proof. One can show that $3p, 3^m$ are S-zero divisors in $\boldsymbol{Z}_{3^m p}$, and the number of S-zero divisors in $\boldsymbol{Z}_{3^m p}$ is $(3^{m-1} - 2 + p)$.

Example 2.28: In $\boldsymbol{Z}_{45}$, ($45 = 3^2.5$), 9, 15, 18, 27, 30 and 36 are all the S-zero divisors in $\boldsymbol{Z}_{45}$ and the number of S-zero divisors in Z_{45} is $(2^{2-1} - 2 + 5) = 6$. We can generalize theorem 2.25, 2.27 as follows:

Theorem 2.29: Let $\boldsymbol{Z}_{p^m q}$ be the ring of integers modulo $p^m q$, where p, q are distinct primes and $m \geq 2$, then kpq, $p^m k$, $k \in \boldsymbol{Z}$, are S-zero divisors.

Theorem 2.30: Let $\boldsymbol{Z}_{p^m}$ be the ring of integers modulo p^m, where p be a prime and $m \geq 2$, then p^2, p^{m-1} are S-zero divisors.
Proof. Take $a = p^{m-2}$ and $b = p$, so we have $p^2.p^{m-2} \equiv 0$ (mod p^m), and $p.p^{m-1} \equiv 0$ (mod p^m), but $p^{n-2}.p \equiv 0$ (mod p^m). Hence, p^2 and p^{m-1} are S-zero divisors.

Remark 2.31: One can also see that kp^2, $k \in \boldsymbol{Z}$ is S-zero divisors in $\boldsymbol{Z}_{p^m}$.
Note: The number of S-zero divisors in $\boldsymbol{Z}_{p^m}$ is $(p^{m-2} - 1)$

Example 2.32: In $\boldsymbol{Z}_{32}(32 = 2^5)$, 4, 8, 12, 16, 20, 24 and 28 are all the S-zero divisors in Z_{32}, and the number of S-zero divisors is $(2^3 - 1 = 7)$.

3. S-zero divisors in the group ring $\boldsymbol{Z_2 G}$

Here we will show that the group ring $Z_2 G$ where G is a finite cyclic group of non-prime order has S-zero divisor. We illustrate by certain examples the non-existence of S-zero divisors before this, and we prove the group ring $\boldsymbol{Z}_2 G$ where n >1, $G = \{g/g^2 = 1\}$ has S-zero divisor and S-weak zero divisor. Further the group ring $\boldsymbol{Z}_{2n+1}G, = \{g/g^2 = 1\}$ has S-weak zero divisor.

Example 3.33: Consider the group ring $\boldsymbol{Z}_2 G$ where $G = \{g/g^2 = 1\}$ over $\boldsymbol{Z}_2$. Clearly, $(1+g)^2 = 0$ is the only zero divisor, so it cannot have S-zero divisors or S-weak zero divisors. Similarly, $\boldsymbol{Z}_2 G$ where $G = \{g/g^3 = 1\}$ has no S-zero divisors or S-weak zero divisors.

Example 3.34: Consider the group ring $\boldsymbol{Z}_2 G$ where $G = \{g/g^4 = 1\}$ is the cyclic group of order 4. Then

$$(1+g)(1+g+g^2+g^3) = 0$$
$$(1+g^2)(1+g+g^2+g^3) = 0$$
$$(1+g^3)(1+g+g^2+g^3) = 0$$
$$(g+g^2)(1+g+g^2+g^3) = 0$$
$$(g+g^3)(1+g+g^2+g^3) = 0$$
$$(g^2+g^3)(1+g+g^2+g^3) = 0$$

are some of the zero divisors in $\boldsymbol{Z}_2 G$ So it has S-zero divisors and no S-weak zero divisors.

Theorem 3.35 (Vasantha Kandasamy and Chetry, 2005)**:** Let $\boldsymbol{Z}_2 G$ be the group ring where G is a cyclic group of prime order p. Then the group ring $\boldsymbol{Z}_2 G$ has no S-zero divisors or S-weak zero divisors.

Theorem 3.36 (Vasantha Kandasamy and Chetry, 2005)**:** Let $\boldsymbol{Z}_2 S_n$ be the group ring of the symmetric group S_n over $\boldsymbol{Z}_2$. Then $\boldsymbol{Z}_2 S_n$ has S-zero divisors.

Example 3.37: The group ring $\boldsymbol{Z}_2 S_3$ where S_3 is the symmetric group of order 4, has S-zero divisors.

Let $a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$,
$d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

Put $A = 1 + a + b + c + d + e$ and $B = d + e$ (1 is the identity permutation).
Clearly $AB = 0$.
Take
$X = 1 + a$ and $Y = 1 + d + e$ then
$AX = 0$ and $BY = 0$, but $XY \neq 0$.
Hence $Z_2 S_3$ has S-zero divisors.

Theorem 3.38: the group ring $\boldsymbol{Z}_{2n} G$, where n >1, has S-zero divisor.

Proof. To prove that $\boldsymbol{Z}_{2n} G$, where n >1, $G = \{g/g^2 = 1\}$ has S-zero divisor,
Take $x = (2n-1) + g$, $y = n + ng$ and take $a = 1 + g$, $b = (2n-1) + (2n-1)g$
Now we have $xy = (n(2n-1) + n) + (n + n(2n-1))g$
$$xy = n(2n) + n(2n)g \equiv 0 \text{ (mod 2n)}$$

and

$$ax = (1+g)((2n-1)+g) = (2n) + (2n)g \equiv 0 \text{ (mod 2n)}$$
$$by = ((2n-1) + (2n-1)g)(n + ng)$$
$$= (n(2n-1) + n(2n-1)) + 2n(2n+1) \equiv 0 \text{ (mod 2n)}$$
$$ab = (1+g)((2n-1) + (2n-1)g) = (2n-1+2n-1)$$
$$+((2n-1) + (2n-1))g$$
$$= (4n-2) + (4n-2)g \not\equiv 0 \text{ (mod 2n)}$$

To show that $\boldsymbol{Z_{2n}} G$, where n >1, $G = \{g/g^2 = 1\}$ has S-weak zero divisor, take $x = 1 + g$, $y = n + ng.x$, and take $a = (2n-1) + g$, $b = (2n-1) + (2n-1)g$, i.e.
$xy \equiv 0$, (mod 2n), $ax \equiv 0$ (mod 2n), $by \equiv 0$ (mod 2n) and $ab \equiv 0$ (mod 2n). Hence the claim.

Theorem 3.39: The group ring $\boldsymbol{Z}_{2n+1} G$, $G = \{g/g^2 = 1\}$ has S-weak zero divisor.

Proof. Take $x = 2n + 2ng$, $y = 1 + 2ng$ and take $a = 2n + g$, $b = 1 + g$. So we have
$xy \equiv 0$ (mod $2n+1$), $ax \equiv 0$ (mod $2n+1$), $by \equiv 0$ (mod $2n+1$)
And $ab \equiv 0$ (mod $2n+1$).
Then the group ring $\boldsymbol{Z}_{2n+1} G$, $G = \{g/g^2 = 1\}$ has S-weak zero divisor.

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