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Sobolev Spaces in Metric Spaces

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Abstract

We study Sobolev type spaces (called Newtonian spaces) in metric measure spaces equipped with a doubling measure and supporting a p −Poincaré inequality. The Sobolev spaces are defined using the minimal upper gradient which is a substitute of the modulus of the usual gradient. We show that they are the right extension of the usual Sobolev spaces in R^n. In particular Newtonian functions are quasicontinuous and that they are absolutely continues on almost every curve. Moreover, Newtonian functions are continuous on the complement of small sets.

Keywords: Newtonian functions; doubling measure; metric space; nonlinear; Sobelev spaces; Poincaré inequality.

المستخلص

في هذا البحث نقوم بدراسة فضاءات نيوتن وهي تعميم لفضاءات سوبوليف من الفضاءات الاقليدية الى الفضاءات المترية التي تحقق بعض الشروط الاساسية لهذا التعميم. هذه الفضاءات تبنى على استبدال القيمة المطلقة لتدرج الدالة بدالة جديدة تسمى upper gradient والتي تكون الامتداد الصحيح حيث تجعل الفضاءات تتطابق عند الرجوع الى الفضاءات الاقليدية. كذلك نستنتج ان فضاءات نيوتن تحتفظ بمعظم الخواص الاساسية في هذا التعميم.

1. Introduction

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric spaces endowed with a metric d and a positive complete Borel measure μ which is doubling, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x, r) := \{y \in X: d(x, y) < r\}$ in X we have

$$0 < \mu(2B) \leq C\, \mu(B) < \infty,$$

where $2B = B(x, 2r)$.

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In a metric space the gradient has no obvious meaning as in domains in $\boldsymbol{R^n}$. Therefore the concept of an upper gradient was introduced in Heinonen– Koskela [1] as a substitute of the usual gradient, based on the following observation: It is well known from the fundamental theorem of calculus that, for every $x, y \in \mathbf{R}^n$ and smooth function u on $\boldsymbol{R^n}$, on the line segment $[x, y]$ we have

$$|u(y) - u(x)| \leq \int_{[x,y]} |\nabla u| \, ds$$

In fact, for every rectifiable curve γ with end points x and y we have

$$|u(y) - u(x)| \leq \int_{\gamma} |\nabla u| ds \tag{1}$$

Similarly, a nonnegative Borel function g is an upper gradient of u if (1) holds, for all rectifiable curves γ, when ∇u is replaced by g. It has many useful properties similar to those of the usual gradient. This makes the variational approach of the Dirichlet problem available in metric spaces and Sobolev spaces can then be extended to metric spaces.

There are several notions of Sobolev spaces in metric spaces; see for example Cheeger [2], Hajłasz [3] and Shanmugalingam [4-5]. The definitions in these references are different but by [4] they give the same Sobolev spaces, under mild assumption. We shall follow the definition of Shnmugalingam [4], where the Sobolev space $N^{1,p}(X)$ (called Newtonian space) was defined as the collection of p–integrable functions with p–integrable upper gradients.

This paper is organized as follows. In Section 2, we present the upper gradient as introduced in Heinonen–Koskela [1], and use an equivalent definition, of Newtonian spaces, to the one used in Shanmugalingam [4]. Moreover, we give some of the most useful property of Newtonian functions. In Section 3, we consider the Newtonian spaces in a subsets of X (with the restrictions of d and μ) as a metric spaces in their own right. We also define the Newtonian space with zero boundary values $N_0^{1,p}(X)$ which makes it possible to compare boundary values of Newtonian functions. Under rather mild assumptions on X it has been shown that Lipschitz functions with compact support are dense in $N_0^{1,p}(X)$.

In Section 4, we study the Newtonian spaces in the Euclidean setting and show that they are the right generalization of the usual Sobolev spaces, i.e. both spaces coincide, in the sense that every $u \in N^{1,p}(\boldsymbol{R}^n)$ belongs to $W^{1,p}(\boldsymbol{R}^n)$and every $u \in W^{1,p}(\boldsymbol{R}^n)$has a representative in the Newtonian space $\boldsymbol{N}^{1,p}(\boldsymbol{R}^n)$ which is *quasicontinuous*. This is a *Luzin type phenomenon*, which means that a Sobolev function is continuous on the complement of a small set. In this setting the removed set has small capacity. We also give a simple example in the plane which shows that the function $\chi_R \in W^{1,p}(\boldsymbol{R}^2)$ and

$\chi_R \notin N^{1,p}(R^2)$. But, as the real line has two-dimensional Lebesgue measure zero, $\chi_R = 0$ a.e. in $\boldsymbol{R}^2$ and clearly $0 \in N^{1,p}(\boldsymbol{R}^2)$.

2. Upper gradients and Newtonian Spaces

The first order Sobolev spaces in $\boldsymbol{R}^n$ are defined as follows: For $1 < p < \infty$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\boldsymbol{R}^n)$ we define

$$\| f \|_{W^{1,p}}^p (\boldsymbol{R}^n) = \int_{\boldsymbol{R}^n} (|f|^p + |\nabla f|^p)\, dx,$$

where the ∇f is the weak gradient of f. $W^{1,p}(\boldsymbol{R}^n)$ is given by

$$W^{1,p}(\boldsymbol{R}^n) = \{f\colon \| f \|_{W^{1,p}}^p (\boldsymbol{R}^n) < \infty\}$$

As we see to define the $W^{1,p}(\boldsymbol{R}^n)$ one uses the gradient i.e. the directional derivative. In metric spaces we can not talk about directions. However we do not really use the vector ∇f , only the scalar $|\nabla f|$ is used. For $|\nabla f|$ there is a possible counter part in metric spaces called upper and has been introduced by Heinnonen–Koskela [1].
In this section we introduce the *upper gradient* as a substitute of the usual gradient.

Definition 2.1
A nonnegative Borel function g on X is said to be an upper gradient of an extended real–valued function f on X if for all rectifiable curve $\gamma : [0, l\gamma] \to X$ parametrized by the arc length ds, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds \tag{2}$$

whenever both $f\,(\gamma(0))$ *and* $f\,(\gamma(l\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2) holds for p −almost every curve then g is a p *−weak upper gradient* of f. The upper gradient in not unique. In particular, from (2) every Borel function greater than g will be another upper gradient of f. Moreover, the operation of taking an upper gradient is not linear. However, we have the following useful property.

Lemma 2.2
If $a, b \in \boldsymbol{R}$ and g_1, g_2 are upper gradients of u_1, u_2, respectively. Then $|a|g_1 + |b|g_2$ is an upper gradient of $au_1 + bu_2$.
We shall need the following lemma, which gives a nontrivial example of upper gradient, see Björn–Björn [6], Corollary 1.15.

Lemma 2.3
If $X = \boldsymbol{R}^n$ and $f \in C^1(\boldsymbol{R}^n)$, then $|\nabla f|$ is an upper gradient of f.

By saying that (2) holds for p–almost every curve we mean that it fails only for a curve family with zero p–modulus.

Definition 2.4
Let Γ be a family of curve on X. Then we define the p −modulus of Γ by

$$Mod_p(\Gamma) = \inf \int_X \rho \, d\mu, \tag{3}$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho \geq 1$ for all $\gamma \in \Gamma$.

If f has an upper gradient in $\boldsymbol{L}^p(X)$, then it has a minimal p–weak upper gradient $g_f \in \boldsymbol{L}^p(X)$ in the sense that for every p–weak upper gradient $g \in \boldsymbol{L}^p(X)$ of f, we have, $g_f \leq g$ a.e. see Corollary 3.7 in Shanmugalingam [5].

Proposition 2.5 (Proposition 1.37 in [6])
$Mod(\Gamma) = 0$ if and only if there is a nonnegative Borel function $\rho \in \boldsymbol{L}^p(X)$ such that $\int_\gamma \rho \, ds = \infty$.

In Shanmugalingam [4], upper gradients have been used to define Sobolev type spaces on metric spaces. We will use the following equivalent definition.

Definition 2.6
Let $u \in \boldsymbol{L}^p(X)$, then we define

$$\| u \|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p} \tag{4}$$

where the g_u is the minimal p—weak upper gradient of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u: \| u \|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\| u - v \|_{N^{1,p}(X)} = 0$.

We also have the following lemma about minimal p–weak upper gradients, see Björn–Björn [7], Corollary 3.4.

Lemma 2.7
If $u, v \in N^{1,p}(X)$, then $g_{u=} g_v$ a.e. on $\{x \in X: u(x) = v(x)\}$. Moreover, if $c \in \boldsymbol{R}$ is a constant, then $g_u = 0$ a.e. on $\{x \in X: u(x) = c\}$.

Theorem 2.8 (Theorem 1.56 in [6])
If $u \in N^{1,p}(X)$, then $u \in ACC_p(X)$, i.e. u is absolutely continuous on $p-a.e.$ curve in the sense that $u \circ \gamma : [0, l\gamma] \to \boldsymbol{R}$ is absolutely continuous for p–a.e. curve in γ in X.

Lemma 2.9 (Lemma 2.14 in [6]) Assume that $u \in ACC_p(X)$ and that $g \in \mathbf{L}^p(X)$ is a p–weak upper gradient of u then for p—a.e. curve $\gamma : [0, l_\gamma] \longrightarrow X$ we have

$$|(u \circ \gamma)'(t)| \le g(\gamma(t)) \tag{5}$$

for a.e. $t \in [0, l_\gamma]$. Conversely, if $g \ge 0$ is measurable, $u \in ACC_p(X)$ and (5) holds for p–a.e. curve $\gamma: [0, l\gamma] \to X$, then g is a p–weak upper gradient of u.

Theorem 2.10 [6] The space $N^{1,p}(X)$ is a linear normed space.

Proof. That the $N^{1,p}(X)$ is a vector space follows directly from Lemma 2.2. The only difficulty is to proof the triangle inequality. To prove this let $u, v \in N^{1,p}(X)$ and $\varepsilon > 0$ be arbitrary. Find upper gradients $g, g' \in L^p(X)$ of u and v, respectively, so that

$$\begin{gathered}\left(\| u \|_{L^p(X)}^p + \| g \|_{L^p(X)}^p\right)^{\frac{1}{p}} \le \| u \|_{N^{1,p}(X)} + \varepsilon \\ \left(\| v \|_{L^p(X)}^p + \| g' \|_{L^p(X)}^p\right)^{1/p} \le \| v \|_{N^{1,p}(X)} + \varepsilon\end{gathered} \tag{6}$$

We know that $g + g'$ is an upper gradient of $u + v$. Now, note that the left-hand sides of (6) are the L^p-norms (on $\boldsymbol{R}^2$) of

$$\left(\| u \|_{L^p(X)}, \| g \|_{L^p(X)}\right) \quad \text{and} \quad \left(\| v \|_{L^p(X)}, \| g' \|_{L^p(X)}\right),$$

respectively. Similarly,

$$\| u + v \|_{N^{1,p}(X)}^p \le \left(\| u + v \|_{L^p(X)}^p + \| g + g' \|_{L^p(X)}^p\right)^{1/p}$$

$$\le \left(\left(\| u \|_{L^p(X)} + \| v \|_{L^p(X)}\right)^p + \left(\| g \|_{L^p(X)} + \| g' \|_{L^p(X)}\right)^p\right)^{1/p},$$

which is the $\boldsymbol{L}^p$ –norm of

$$\left(\| u \|_{\boldsymbol{L}^p(X)} + \| v \|_{\boldsymbol{L}^p(X)}, \ \| g \|_{\boldsymbol{L}^p(X)} + \| g' \|_{\boldsymbol{L}^p(X)}\right).$$

The triangle inequality of the L^p –norm now implies that

$$\| u+v \|_{N^{1,p}(X)}^p \leq \left(\| u \|_{L^p(X)}^p + \| g \|_{L^p(X)}^p\right)^{1/p} + \left(\| v \|_{L^p(X)}^p + \| g' \|_{L^p(X)}^p\right)^{1/p}$$

$$\leq \| u \|_{N^{1,p}(X)}^p + \varepsilon + \| v \|_{N^{1,p}(X)}^p + \varepsilon$$

And letting $\varepsilon \to 0$ proves the triangle inequality for the $\|\cdot\|_{N^{1,p}(X)}$.

In Shanmugalingam [4], Theorem 3.7 and p. 249 it has been shown that the space $N^{1,p}(X)$ is a Banach space and lattice.

Theorem 2.11

The space $N^{1,p}(X)$ is a lattice, i.e. if $u, v \in N^{1,p}(X)$, then $max\{u, v\}, min\{u, v\} \in N^{1,p}(X)$.

Definition 2.12

The Capacity of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \| u \|_{N^{1,p}(X)}$$

where the infimnm is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

We say that a property holds *quasieverywhere* (q.e.) in X, if it holds everywhere except a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u,\ v \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [4] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

The following proposition gives the relation between small set and small curve family.

Proposition 2.13

Let $E \subset$ X. Then $C_p(E) = 0$ if and only if $\mu(E) = \mathrm{Mod}_p(\Gamma_E) = 0$

Where

$$\Gamma_E = \{\gamma \in \Gamma(X) : \gamma^{-1}(E) \neq \phi\}.$$

We also have the following property for Newtonian functions, see Theorem 1.1 in Björn–Björn–Shanmugalingam [10].

Lemma 2.14

Every function $u \in N^{1,p}(X)$ is quasicontinuous, i.e., for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{X \setminus G}$ is continuous.

3. $N^{1,p}$ (Ω) and $N_0^{1,p}(\Omega)$

For $E \subset X$ we define the space $N^{1,p}(E)$ with respect to the restrictions of the metric d and the measure μ to E. A function $f \in N^{1,p}(X)$ clearly has a restriction $f|_E$ which belongs to the $N^{1,p}(E)$ and any p −weak upper gradient of it remains a p–weak upper gradient when restricted. However, the restriction of a minimal p–weak upper gradient is not always minimal. If $E = \Omega$ is open then the restriction of a minimal p–weak upper gradient remains minimal.

Lemma 3.1

Assume that $f \in N^{1,p}(X)$ with a minimal p–weak upper gradient g_f (with respect to X). Then $g_f|_\Omega$ is a minimal p−weak upper gradient of f with respect to Ω.

From now on we assume that X supports a p–Poincaré inequality, i.e. there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(z,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(\mathrm{B})}\int_{B(z,r)} |u - u_{\mathrm{B(z,r)}}| \; d\mu \leq C\, r \left(\frac{1}{\mu(B)}\int_{B(z,\lambda r)} g^p \, d\mu\right)^{1/p},$$

where $u_{B(z,r)} := \frac{1}{\mu(B)}\int_{B(z,r)} u \; d\mu$.

To be able to compare the boundary values of Newtonian functions we need to define a Newtonian space with zero boundary values outside of Ω as follows.

Definition 3.2

Let $\Omega \subset X$ be open, then the Newtonian space with zero boundary values $N_0^{1,p}(\Omega)$ is defined by

$$N_0^{1,p}(\Omega) = \{f|_\Omega : \; f \in X \;\; and \; f = 0 \; q.e. \quad in \quad X \setminus \Omega\}.$$

Under our assumptions, Lipschitz functions with compact support are dense in $N_0^{1,p}(\Omega)$, see Shanmugalingam [5]. Moreover, the proof of this result in [6] shows that if $0 \leq u \in N_0^{1,p}(\Omega)$, then we can choose the Lipschitz approximation to be nonnegative. The following Poincaré type inequality is useful, for a proof, see e.g Kinnunen–Shanmugalingam [9], Lemma 2.1.

Lemma 3.3

Assume that $\Omega \subset X$ is a nonempty bounded open set with $\mathrm{C_p}(\mathrm{X} \setminus \Omega) > 0$. There exists a constant $\mathrm{C} > 0$ such that for all $u \in N_0^{1,p}(\Omega)$ we have

$$\int_\Omega |u|^p \, d\mu \leq C\int_\Omega g_u^p \; d\mu$$

The following lemma is useful for proving that a function belongs to the $N_0^{1,p}(\Omega)$, see Lemma 5.3 in Björn–Björn [8].

Lemma 3.4

Let $u \in N^{1,p}(\Omega)$ be such that $v \leq u \leq w$ q.e. in Ω for some $v, w \in N_0^{1,p}(\Omega)$. Then $u \in N_0^{1,p}(\Omega)$.

Proposition 3.5 (Proposition 2.38 in [6])

For $N_0^{1,p}(\Omega)$ we have

$$\| f \|_{N^{1,p}(X)} = \| f \|_{N^{1,p}(\Omega)}.$$

Proof. We may assume that $f = 0$ out side of Ω. Let gf be a minimal p–weak upper gradient of f with respect to X. By Lemma 3.1, $g_f|_\Omega$ is a minimal p–weak upper gradient of f with respect to Ω. On the other hand by Lemma 2.7 $g_f = 0$ a.e. outside of Ω. Hence

$$\| f \|_{N^{1,p}(X)}^p = \| f \|_{L^p(X)}^p + \| g_f \|_{L^p(X)}^p$$

$$= \| f \|_{L^p(\Omega)}^p + \| g_f \|_{L^p(\Omega)}^p = \| f \|_{N^{1,p}(\Omega)}^p.$$

4. The Newtonian space $N^{1,p}(\Omega)$ in the Euclidean spaces

In this section we see that, when restricted to $\boldsymbol{R}^n$, the Newtonian space is the refined Sobolev space $W^{1,p}(\boldsymbol{R}^n)$, as defined in Chapter 4 in Heinonen– Kilpeläinen–Martio [11].

Lemma 4.1

Let Γ be a collection of curves in $\boldsymbol{R}^n$. If $Mod_p\Gamma = 0$, then a.e. (with respect to the $(n-1)$ −dimensional Lebesgue measure) line parallel to the x_1-axis contains no curve from Γ.

The following Theorem was obtained by Shanmugalingam [4], we use the proof in [6].

Theorem 4.2

If $\Omega \subset R^n$, then $N^{1,p}(\Omega) = W^{1,p}(\Omega)$, as a Banach spaces, with equivalent norms. More precisely, if $u \in N^{1,p}(\Omega)$, then $u \in W^{1,p}(\Omega)$ and conversely, for every $u \in W^{1,p}(\Omega)$ then there exists $\bar{u} \in N^{1,p}(\Omega)$such that $\bar{u} = u$ a.e., in Ω. Moreover, if $u \in W^{1,p}(\Omega)$ is quasicontinuous, then $u \in N^{1,p}(\Omega)$.

Proof. Let $u \in N^{1,p}(\Omega)$ with a p −weak upper gradient $g \in L^P(\Omega)$. By Lemma 2.9 and Proposition 2.5, $u \in ACC_p(\Omega)$ and $\int_\gamma g\, ds < \infty$ for $p\text{-}a.e.$ curve γ in Ω. Lemma 2.8 and Lemma 4.1 now imply that u is absolutely continuous on a.e. (with respect to the $(n -$

1)−dimensional Lebesgue measure) line segment l in Ω parallel to the x_1 axis and for a.e. $x \in l$,

$$\left|\frac{\partial u(x)}{\partial x_1}\right| \leq g(x).$$

The Fubini theorem then shows that $|\partial u/\partial x_1| \leq g$ a.e. in Ω and hence $\partial u/\partial x_1 \in \boldsymbol{L}^p(\Omega) \subset \boldsymbol{L}^1_{loc}(\Omega)$. The absolute continuity of u at l implies that $\partial u/\partial x_1$ is the one—dimensional distributional of u in l.

Another application of the Fubini theorem to the integrals

$$\int_\Omega \frac{\partial\varphi(x)}{\partial x_1} u(x)\ dx = -\int_\Omega \varphi(x)\frac{\partial u(x)}{\partial x_1}\ dx$$

with $\varphi \in C_0^\infty(\Omega)$, shows that $\partial u/\partial x_1$ is the distributional derivative of u in Ω. The other partial derivative are handled similarly. Hence $u \in W^{1,p}_{loc}(\Omega)$ and

$$\| u \|_{W^{1,p}(\Omega)} \leq n \| u \|_{N^{1,p}(\Omega)}.$$

Conversely, let $u \in W^{1,p}(\Omega)$. By e.g. Theorem 2.3.2 in Ziemer [12] there exist $u_j \in C^\infty(\Omega)$ such that $u_j \to u$ in $W^{1,p}(\Omega)$, as $j \to \infty$. Lemma 2.3 shows that $|\nabla u_j|$ are upper gradients of u_j. Hence $u_j \in N^{1,p}(\Omega)$ and $\| u_j \|_{N^{1,p}(\Omega)} \leq \| u_j \|_{W^{1,p}(\Omega)}$, $j = 1,2,\ldots$. Proposition 2.3 in [6] provides us with a function $\bar{u} \in N^{1,p}(\Omega)$ such that $\bar{u} = u$ a.e. and $|\nabla u|$ is a p—weak upper gradient of $\bar{u}$. Moreover, $\| \bar{u} \|_{N^{1,p}(\Omega)} \leq \| u \|_{W^{1,p}(\Omega)}$. Thus $u \in N^{1,p}(\Omega)$.

If $u \in W^{1,p}(\Omega)$ is quasicontinuous, then $\bar{u} = u$ q.e., by Proposition 5.23 in [6], and hence $u \in N^{1,p}(\Omega)$.

Proposition 4.3

Let $\Omega \subset \boldsymbol{R}^n$ and let u be locally Lipschitz in Ω. Then $g_u = |\nabla u|$ a.e., in Ω.

Corollary 4.4

For every $u \in N^{1,p}(\Omega)$, we have

$$\| u \|_{N^{1,p}(\Omega)} = \| u \|_{W^{1,p}(\Omega)}.$$

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