Transient Solution of an M/M/1 Queue with Balking, Feedback, Catastrophe and Repair Using the Probability Generating Function

إعداد: د. معمر الاخضر

جامعة طرابلس، كلية الاقتصاد والعلوم السياسية، قسم الإحصاء والاقتصاد القياسي

Msukkss2010 @gmail.com
ORCID 0009-0002-2471-0053

الملخص

تقدم هذه الورقة حل عابر للطابور ***M/M/1*** الواقع تحت تأثير الامتناع، التغذية المرتدة، الكارثة و اصلاح الفشل مستخدمين في ذلك الدالة المولدة للاحتمال، بهدف الوصول لاستنباط احتمال وجود ***n*** من الوحدات، الحصول على بعض الحالات الخاصة وربطها بالدراسات السابقة.

الكلمات المفتاحية: تحليل عابر، تغذية مرتدة، امتناع، نظرية الطوابير، كارثة، اصلاح، دالة مولدة للاحتمال.

ABSTRACT

This paper presents a transient solution which is obtained analytically through processing the probability-generating function regarding the system size in an M/M/1 queue with feedback, balking, possibility of catastrophes at services failures and repairs.

Keywords: Transient analysis, feedback, balking, queueing theory, catastrophe, repair, probability generating function.

1. Introduction

In recent times, there has been a great and widespread development in communications systems, especially large-scale communications networks, which now contain repeaters in transportation systems, in which messages are sent through specific servers in the form of data packets at a specific time. However, sometimes the receiving device refuses to receive these messages due to a transmission error from the source, and then the transmitting device retransmits it again using repeaters in the transmission systems, so those messages take the form of a queue linked to time. This matter requires searching for simplified mathematical methods for the temporary solution of this type of queue models that are subject to the influence of abstention and feedback, catastrophes and repair.

In the year 2000, both Kumar and Arivudainambi used the probability generating function to derive a solution to the queue M/M/1 with the catastrophes, so they obtain the probability of the presence of n customers in the system and obtain performance measures. They also derived from their solution a solution to the model in case stability [6]. In 2009, Thangaraj and Vanitha transient solution for M/M/1 with the feedback and catastrophes using the continuous function method [8]. Also in the year 2012, Chandrasekaran and Saravanarajan solved the queue M/M/1 with feedback, catastrophe, and repair using Laplace transforms. They obtained the probability of having n customers in the system and the performance measures [2]. In 2016, Shanmugasundaram and Chitra solved the queue M/M/C with the feedback and catastrophes using the probability generating function to facilitate access to the true roots and thus obtained the probability that there are n customers in the system [7]. In 2017, Kotb and Akhdar arrived at the queue solution M/M/1 under the influence of feedback, catastrophe, and repair, using Rauch's method of complex analysis [4]. Finally, in the year 2020, Akhdar disbanded the M/M/2 queue that was under the influence of feedback, disasters, and reform, while preserving the customers when they escaped, using Rauch's method of complex analysis [1].

In this paper, the researcher will work to derive the solution for the M/M/1 queue that is under the influence of abstinence, feedback, catastrophes, and repair, using the method of the probability generating function and real roots to arrive at the probability that there is no customer in the system, and the probability that there are *n* customers in the system. As well, Extract some special cases and link them to previous studies.

2. Basic Notations and Assumptions

To construct the system of this paper, we define the following parameters:

Probability generating function. $P(s,t) =$

Laplace transform of $P(s,t)$. $P^*(s,z) =$

Transient state probability that there are exactly *n* customers in the system. $p_n(t) =$

Laplace transform of $p_n(t)$. $p_n^*(z) =$

Probability that no customers are in the service department at time *t*. $p_0(t) =$

Laplace transform of $p_0(t)$. $p_0^*(z) =$

Probability that the server is under repair at time *t*. $Q(t) =$

Laplace transform of $Q(t)$. $Q^*(z) =$

Mean arrival rate. $\lambda =$

Mean service rate per service representative. $\mu =$

= Probability that a customer joins the queue. β

Probability that a customer joins the departure process. $q =$

Probability that a customer joins the end of the original queue. $1-q =$

Catastrophe rate. $\upsilon =$

Repair rate. $\eta =$

n = Number of customers in the system.

The assumptions of this model are listed as follows:

(1) Customers arrive at the server one by one according to Poisson process with rate λ_n .

(2) Assume $(1-\beta)$ be the probability that a unit balks (does not enter the queue),

where: $\lambda_n = \begin{cases} \lambda; & n=0, \beta=1 \\ \beta\lambda; & n \ge 1, 0 \le \beta < 1 \end{cases}$

(3) Service times of the customers are independent and identically distributed *(iid)* exponential random variables with rate μ_n . The customers are served according to FCFS discipline.

(4) After completion of each service the customer either joins at the end of the original queue as a feedback customer with probability $(1-q)$ or departure the system with probability q .

(5) The catastrophe occurs at the service department according to Poisson process with rate υ when the system is not empty or empty. The occurrence of a catastrophe destroys all the customers in the instants and affects the system as well.

(6) The repair times of the failed server after catastrophe are *iid* exponential random variables with rate η . After a repair on the server is completed, the server immediately returns to its working position for service when a new customer arrives.

3. Model Formulation and Analysis

From the above notations and assumptions and applying Markove conditions, we obtain the following probability differential-difference equations as:

$$\varphi'(t) = -(\eta+\upsilon)\varphi(t)+\upsilon \quad (1)$$

$$p_0'(t) = -(\lambda+\upsilon)p_0(t)+q\mu p_1(t)+\eta\varphi(t), \qquad n=0$$

(2)

$$p_1'(t) = -(\beta\lambda+q\mu+\upsilon)p_1(t)+\lambda p_0(t)+q\mu p_2(t), \qquad n=1$$

(3)

$$p_n'(t) = -(\beta\lambda+q\mu+\upsilon)p_n(t)+\beta\lambda p_{n-1}(t)$$

$$+q\mu p_{n+1}(t), \qquad n \geq 2$$

(4)

Here we use a simple and direct approach. We assume that the initially there are $n(0)=a$ customers. Define:

$$\frac{\partial H(s,t)}{\partial t} = -(\lambda+\upsilon)p_0(t) + q\mu p_1(t) + \eta\varphi(t) - (\beta\lambda+q\mu+\upsilon)sp_1(t) + \lambda sp_0(t)$$

$$+qsp_2(t)[\beta\lambda+q\mu+\upsilon]\sum_{n=0}^{\infty} p_n(t)s^n + [\beta\lambda+q\mu+\upsilon]p_0(t)$$

$$+(\beta\lambda+q\mu+\upsilon)sp_1(t) + \beta\lambda s\sum_{n=1}^{\infty} p_{n-1}(t)s^{n-1} - \beta\lambda sp_0(t)$$

$$+q\mu/s\sum_{n=-1}^{\infty} p_{n+1}(t)s^{n+1} - q\mu/s\; p_0(t) - q\mu p_1(t)$$

$$-q\mu sp_2(t) - \lambda p_0(t) - \upsilon p_0(t) + q\mu p_1(t) + \eta\varphi(t)$$

$$-(\beta\lambda+q\mu+\upsilon)sp_1(t) + \lambda sp_0(t)q\mu sp_2(t)$$

(5)

Using some algebra, it obtains:

$$\frac{\partial H(s,t)}{\partial t} - \{\beta\lambda s + q\mu/s - [\beta\lambda+q\mu+\upsilon]\}H(s,t) = q\mu(1-1/s)p_0(t)$$

$$-\lambda(1-\beta)(1-s)p_0(t) + \eta\varphi(t) \qquad (6)$$

This is linear differential equation. And its solution is:

$$H(s,t) = C\,e^{[\beta\lambda s+q\mu/s]t}.e^{-(\beta\lambda+q\mu+\upsilon)t}$$

$$+q\mu(1-1/s)\int_0^t p_0(u)e^{[\beta\lambda s+q\mu/s](t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du$$

$$-\lambda(1-\beta)(1-s)\int_0^t p_0(u)e^{[\beta\lambda s+q\mu/s](t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du$$

$$+\eta\int_0^t \varphi(u)e^{[\beta\lambda s+q\mu/s](t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du \tag{7}$$

Now, we find the constant C at t = 0, obtained as:

$$H(s,\ 0)=C$$

(8)

But $H(s,\ 0)=s^k\sum_{n=0}^{\infty}p_{n(0)}=s^k$

(9)

From equations (8) and (9), it find:

$$C=s^k$$

(10)

Then equation (8) becomes:

$$H(s,t)=\sum_{k=0}^{\infty}p_k s^k\, e^{(\beta\lambda s+q\mu/s)t}.e^{-(\beta\lambda+q\mu+\upsilon)t}$$

$$+q\mu(1-1/s)\int_0^t p_0(u)e^{(\beta\lambda s+q\mu/s)(t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du$$

$$-\lambda(1-\beta)(1-s)\int_0^t p_0(u)e^{[\beta\lambda s+\mu/s](t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du$$

$$+\eta\int_0^t \varphi(u)e^{(\beta\lambda s+q\mu/s)(t-u)}.e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}du$$

(11)

Since the generating function of the modified Bessel's function $I_n(x)$ is given by:

$$e^{(\beta\lambda s+q\mu/s)\,t}=\sum_{n=-\infty}^{\infty}I_n(at)(bs)^n \tag{12}$$

From equations (12) in equation (11), it finds:

$$H(s,t)=\sum_{k=0}^{\infty}p_k s^k \sum_{n=-\infty}^{\infty}I_n(at)(bs)^n e^{-(\beta\lambda s+q\mu/s)t}$$
$$+q\mu\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](bs)^n du$$
$$-q\mu b\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](bs)^{n-1} du$$
$$-\lambda(1-\beta)\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](bs)^n du$$
$$+\lambda(1-\beta)b^{-1}\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](bs)^{n+1} du$$
$$+\eta\int_0^t \varphi(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](bs)^n du$$
(13)

But $H(s,t)=\sum_{n=0}^{\infty}p_n(t)s^n$, and using $I_n(x)s^{n+m}=I_{n-m}(x)s^n$, therefore:

$$\sum_{n=0}^{\infty}b^{-n}p_n(t)s^n=\sum_{k=0}^{\infty}p_k b^{-k}e^{-(\beta\lambda s+q\mu/s)t}\sum_{n=-\infty}^{\infty}I_{n-k}(at)(s)^n$$
$$+q\mu\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](s)^n du$$
$$-q\mu b\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_{n+1}[a(t-u)](s)^n du$$
$$-\lambda(1-\beta)\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty}I_n[a(t-u)](s)^n du$$

$$+\lambda(1-\beta)b^{-1}\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty} I_{-1n}[a(t-u)](s)^n du$$

$$+\eta\int_0^t \varphi(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\sum_{n=-\infty}^{\infty} I_n[a(t-u)](s)^n du$$

(14)

Comparing the coefficients of s^n on both sides of equation (14) for $n \geq 0$, it find:

$$p_n(t)=\sum_{k=0}^{\infty} p_k b^{n-k}\, I_{n-k}(at)e^{-(\beta\lambda s+q\mu/s)t}$$

$$+q\mu\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\{I_n[a(t-u)]-bI_{n+1}[a(t-u)]\}du$$

$$-\lambda(1-\beta)\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\{I_n[a(t-u)]-b^{-1}I_{n-1}[a(t-u)]\}du$$

$$+\eta\int_0^t \varphi(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}I_n[a(t-u)]du \qquad (15)$$

Let n=0 in equation (15) obtained:

$$p_0(t)=\sum_{k=0}^{\infty} p_k b^{-k}\, I_{-k}(at)e^{-(\beta\lambda s+q\mu/s)t}$$

$$+q\mu\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\{I_0[a(t-u)]-bI_1[a(t-u)]\}du$$

$$-\lambda(1-\beta)\int_0^t p_0(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}\{I_0[a(t-u)]-b^{-1}I_{-1}[a(t-u)]\}du$$

$$+\eta\int_0^t \varphi(u)e^{-(\beta\lambda s+q\mu/s)(t-u)}I_0[a(t-u)]du \qquad (16)$$

Using the Laplace transform in equation (16), it obtains:

$$p^*_0(z)=\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} (q\mu)^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n+1}$$

$$-\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} [\lambda(1-\beta)]^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n-1}$$

$$+\eta\varphi^*(z)\sum_{n=k}^{\infty} (q\mu)^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n+1}$$

$$-\eta\varphi^*(z)\sum_{n=k}^{\infty} [\lambda(1-\beta)]^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n-1}$$

(17)

Using the Laplace transform in equation (1), it obtained:

$$\varphi^*(z)=\frac{\upsilon}{z(z+\eta+\upsilon)} \tag{18}$$

From equations (17) and (18) it finds:

$$p^*_0(z)=\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} (q\mu)^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n+1}$$

$$-\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} [\lambda(1-\beta)]^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n-1}$$

$$+\frac{\upsilon}{z(z+\eta+\upsilon)}\sum_{n=k}^{\infty} (q\mu)^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n+1}$$

$$-\frac{\upsilon}{z(z+\eta+\upsilon)}\sum_{n=k}^{\infty} [\lambda(1-\beta)]^n \left[\frac{w-\sqrt{w^2-a^2}}{2\beta\lambda}\right]^{n-1}$$

(19)

By Inverse $\varphi^{*}(z)$, $p_0^{*}(z)$ in equations (18) and (19), to get the explicit expression for $\varphi(t)$, $p_0(t)$ as:

$$Q(t)=\frac{\upsilon}{(\eta+\upsilon)}\left[1-e^{-(\eta+\upsilon)t}\right], \quad (20)$$

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty}p_k\sum_{n=k}^{\infty}\left\{\frac{(n+1)(q\mu)^n I_{n+1}\left(2\sqrt{\beta\lambda q\mu t}\right)}{(\beta\lambda/q\mu)^{\frac{n+1}{2}}}-\frac{(n-1)(q\mu)^n I_{n-1}\left(2\sqrt{\beta\lambda q\mu t}\right)}{(\beta\lambda/q\mu)^{\frac{n-1}{2}}}\right\}$$

$$*\,e^{-(\beta\lambda+q\mu+\upsilon)t}+\frac{\eta\upsilon\left(1-e^{-(\eta+\upsilon)t}\right)}{(\eta+\upsilon)(t-u)}\sum_{n=k}^{\infty}\left\{\frac{(n+1)(q\mu)^n I_{n+1}\left[2\sqrt{\beta\lambda q\mu}(t-u)\right]}{(\beta\lambda/q\mu)^{\frac{n+1}{2}}}\right.$$

$$\left.-\frac{(n-1)[\lambda(1-\beta)]^n I_{n-1}\left[2\sqrt{\beta\lambda q\mu}(t-u)\right]}{(\beta\lambda/q\mu)^{\frac{n-1}{2}}}\right\}e^{-(\beta\lambda+q\mu+\upsilon)(t-u)}$$

(21)

Using the Laplace transform equation (15), it obtained:

$$p^{*}{}_n(z)=\sum_{k=0}^{\infty}p_k b^{n-k}\frac{\left[w-\sqrt{w^2-a^2}\right]^{n-k}}{a^{n-k}\sqrt{w^2-a^2}}$$

$$+q\mu p_0^{*}(z)b^n\left\{\frac{\left[w-\sqrt{w^2-a^2}\right]^{n}}{a^{n}\sqrt{w^2-a^2}}-b\frac{\left[w-\sqrt{w^2-a^2}\right]^{n+1}}{a^{n+1}\sqrt{w^2-a^2}}\right\}$$

$$-\lambda(1-\beta)p_0^{*}(z)b^n\left\{\frac{\left[w-\sqrt{w^2-a^2}\right]^{n}}{a^{n}\sqrt{w^2-a^2}}-b^{-1}\frac{\left[w-\sqrt{w^2-a^2}\right]^{n-1}}{a^{n-1}\sqrt{w^2-a^2}}\right\}$$

$$+\frac{\upsilon}{z(z+\eta+\upsilon)}b^{n}\frac{\left[w-\sqrt{w^{2}-a^{2}}\right]^{n}}{a^{n}\sqrt{w^{2}-a^{2}}} \qquad (22)$$

Substituting equations (20) into (22) and taking the inverse Laplace transform by using some properties of Bessel functions, we gained $p_n(t)$ explicitly ofas: $t$and $\beta, q, \upsilon, \eta\ \lambda, \mu$,

$$p_n(t)=\sum_{k=0}^{\infty}p_k\left(\beta\lambda/q\mu\right)^{n-k}I_{n-k}\left(2\sqrt{\beta\lambda q\mu t}\right)e^{-(\beta\lambda+q\mu+\upsilon)t}$$
$$+q\mu p_0(z)(\beta\lambda/q\mu)^{n}\sum_{n=0}^{\infty}\left\{nI_n\left(2\sqrt{\beta\lambda q\mu t}\right)\right.$$
$$\left.-(n+1)(\beta\lambda/q\mu)I_{n+1}\left(2\sqrt{\beta\lambda q\mu t}\right)\right\}$$
$$-\lambda(1-\beta)p_0(z)(\beta\lambda/q\mu)^{n}\sum_{n=0}^{\infty}\left\{nI_n\left(2\sqrt{\beta\lambda q\mu t}\right)\right.$$
$$\left.-(n-1)(\beta\lambda/q\mu)^{-1}I_{n-1}\left(2\sqrt{\beta\lambda q\mu t}\right)\right\}$$
$$+\frac{\eta\upsilon}{(\eta+\upsilon)}\left[1-e^{-(\eta+\upsilon)t}\right](\beta\lambda/q\mu)^{n}\sum_{n=0}^{\infty}nI_n\left(2\sqrt{\beta\lambda q\mu t}\right)$$

(23)

4. Cases Special

Some queuing systems can be obtained as special cases of this system:
Case (1): let $\beta=1$, this is the queue: M/M/1 with feedback, catastrophe and repair. Then relations (21) and (23) are given by:

The probability that there are *n* customers in the system at time *t* is:

$$p_n(t)=\sum_{k=0}^{\infty}p_k\left(\lambda/q\mu\right)^{n-k}I_{n-k}\left(2\sqrt{\lambda q\mu t}\right)e^{-(\lambda+q\mu+\upsilon)t}$$

$$+ q\mu p_0(z)(\lambda/q\mu)^n \sum_{n=0}^{\infty}\{nI_n(2\sqrt{\lambda q\mu}t)$$
$$-(n+1)(\lambda/q\mu)I_{n+1}(2\sqrt{\lambda q\mu}t)\}$$
$$-(n-1)(\lambda/q\mu)^{-1} I_{n-1}(2\sqrt{\lambda q\mu}t)\}$$
$$+\frac{\eta\upsilon}{(\eta+\upsilon)}\left[1-e^{-(\eta+\upsilon)t}\right](\lambda/q\mu)^n \sum_{n=0}^{\infty} nI_n(2\sqrt{\lambda q\mu}t), \qquad n \ge 1$$

(24)

The probability that no customers in the system department is:

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty}\left\{\frac{(n+1)(q\mu)^n I_{n+1}(2\sqrt{\lambda q\mu}t)}{(\lambda/q\mu)^{\frac{n+1}{2}}}-\frac{(n-1)(q\mu)^n I_{n-1}(2\sqrt{\lambda q\mu}t)}{(\lambda/q\mu)^{\frac{n-1}{2}}}\right\}$$

$$* e^{-(\lambda+q\mu+\upsilon)t}+\frac{\eta\upsilon(1-e^{-(\eta+\upsilon)t})}{(\eta+\upsilon)(t-u)}\sum_{n=k}^{\infty}\left\{\frac{(n+1)(q\mu)^n I_{n+1}\left[2\sqrt{\lambda q\mu}(t-u)\right]}{(\lambda/q\mu)^{\frac{n+1}{2}}}\right.$$

(25)

Relations (24) and (25) are the same results as Akhdar [1] if $\mu_1 = \mu_2 = \mu$.

Case (2): let $\beta = 1$ and $q = 1$, this is the queue: M/M/1 with catastrophe and repair. Then relations (21) and (23) are given by:

The probability that there are *n* customers in the system at time *t* is:

$$p_n(t)=\sum_{k=0}^{\infty} p_k (\lambda/\mu)^{n-k} I_{n-k}(2\sqrt{\lambda\mu}t)e^{-(\lambda+\mu+\upsilon)t}$$
$$+\mu p_0(z)(\lambda/\mu)^n \sum_{n=0}^{\infty}\{nI_n(2\sqrt{\lambda\mu}t)$$
$$-(n+1)(\lambda/\mu)I_{n+1}(2\sqrt{\lambda\mu}t)\}$$
$$-(n-1)(\lambda/\mu)^{-1} I_{n-1}(2\sqrt{\lambda\mu}t)\}$$

$$+\frac{\eta\upsilon}{(\eta+\upsilon)}\left[1-e^{-(\eta+\upsilon)t}\right](\beta/\mu)^n\sum_{n=0}^{\infty}nI_n\left(2\sqrt{\lambda\mu t}\right), \qquad n\geq 1$$

(26)

The probability that no customers in the system department is:

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty}p_k\sum_{n=k}^{\infty}\left\{\frac{(n+1)(\mu)^n I_{n+1}\left(2\sqrt{\lambda\mu t}\right)}{(\lambda/\mu)^{\frac{n+1}{2}}}-\frac{(n-1)(\mu)^n I_{n-1}\left(2\sqrt{\lambda\mu t}\right)}{(\lambda/\mu)^{\frac{n-1}{2}}}\right\}$$

$$*\, e^{-(\lambda+\mu+\upsilon)t}+\frac{\eta\upsilon\left(1-e^{-(\eta+\upsilon)t}\right)}{(\eta+\upsilon)(t-u)}\sum_{n=k}^{\infty}\left\{\frac{(n+1)(\mu)^n I_{n+1}\left[2\sqrt{\lambda\mu}(t-u)\right]}{(\lambda/\mu)^{\frac{n+1}{2}}}\right.$$

(27)

Relations (26) and (27) are the same results as Kumar and Krishnamoorthy [5].

Case (3): let $\beta=1$ and $\eta=0$, this is the queue: M/M/1 with feedback and catastrophe. Then relations (20), (21) and (23) are given by:

The probability that the server is under repair at time t is:

$$Q(t)=1-e^{-\upsilon t},$$

(28)

The probability that there are *n* customers in the system at time *t* is:

$$p_n(t)=\sum_{k=0}^{\infty}p_k(\lambda/q\mu)^{n-k}I_{n-k}\left(2\sqrt{\lambda q\mu t}\right)e^{-(\lambda+q\mu+\upsilon)t}$$

$$+q\mu p_0(z)(\lambda/q\mu)^n\sum_{n=0}^{\infty}\left\{nI_n\left(2\sqrt{\lambda q\mu t}\right)\right.$$

$$-(n+1)(\lambda/q\mu)I_{n+1}\left(2\sqrt{\lambda q\mu t}\right)\}$$

$$\left.-(n-1)(\lambda/q\mu)^{-1}I_{n-1}\left(2\sqrt{\lambda q\mu t}\right)\right\}, \qquad n\geq 1$$

(29)

The probability that no customers in the system department is:

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty}p_k\sum_{n=k}^{\infty}\left\{\frac{(n+1)(q\mu)^n I_{n+1}\left(2\sqrt{\lambda q\mu t}\right)}{(\lambda/q\mu)^{\frac{n+1}{2}}}\right.$$

$$\left.-\frac{(n-1)(q\mu)^n I_{n-1}\left(2\sqrt{\lambda q\mu t}\right)}{(\lambda/q\mu)^{\frac{n-1}{2}}}\right\}e^{-(\lambda+q\mu+\upsilon)t}$$

(30)

Relations (28), (29) and (30) are the same results as Thangaraj and Vanitha [8]. And Shanmugasundaram and Chitra [7] if $C=1$

Case (4): let $\beta=1, q=1$ and $\eta=0$, this is the queue: M/M/1 with catastrophe. Then relations (20), (21) and (23) are given by:

The probability that the server is under repair at time t is:

$$Q(t)=1-e^{-\upsilon t},$$

(31)

The probability that there are *n* customers in the system at time *t* is:

$$p_n(t)=\sum_{k=0}^{\infty}p_k(\lambda/\mu)^{n-k} I_{n-k}\left(2\sqrt{\lambda\mu t}\right)e^{-(\lambda+\mu+\upsilon)t}$$

$$+\mu p_0(z)(\lambda/\mu)^n\sum_{n=0}^{\infty}\left\{nI_n\left(2\sqrt{\lambda\mu t}\right)\right.$$

$$-(n+1)(\lambda/\mu)I_{n+1}\left(2\sqrt{\lambda\mu t}\right)\Big\}$$

$$-(n-1)(\lambda/\mu)^{-1}I_{n-1}\left(2\sqrt{\lambda\mu t}\right)\Big\},\qquad n\geq 1 \qquad (32)$$

The probability that no customers in the system department is:

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty}p_k\sum_{n=k}^{\infty}\left\{\frac{(n+1)(\mu)^n I_{n+1}\left(2\sqrt{\lambda\mu t}\right)}{(\lambda/\mu)^{\frac{n+1}{2}}}\right.$$

$$-\frac{(n-1)(\mu)^n I_{n-1}\left(2\sqrt{\lambda\mu}t\right)}{(\lambda/\mu)^{\frac{n-1}{2}}}\Bigg\}e^{-(\lambda+\mu+\upsilon)t}$$

(33)

Relations (31), (32) and (33) are the same results as Kumar and Arivudainambi [6].

Case (5): let $\beta=1, q=1, \upsilon=0$ and $\eta=0$, this is the queue: M/M/1 without any concepts. Then relations (20), (21) and (23) are given by:

The probability that the server is under repair at time t is:

$$Q(t)=0,$$

(34)

The probability that there are *n* customers in the system at time *t* is:

$$p_n(t)=\sum_{k=0}^{\infty} p_k(\lambda/\mu)^{n-k} I_{n-k}\left(2\sqrt{\lambda\mu}t\right)e^{-(\lambda+\mu)t}$$

$$+\mu p_0(\text{z})(\lambda/\mu)^n \sum_{n=0}^{\infty}\Big\{nI_n\left(2\sqrt{\lambda\mu}t\right)$$

$$-(n+1)(\lambda/\mu)I_{n+1}\left(2\sqrt{\lambda\mu}t\right)\Big\}$$

$$-(n-1)(\lambda/\mu)^{-1} I_{n-1}\left(2\sqrt{\lambda\mu}t\right)\Big\}, \qquad n\geq 1$$

(35)

The probability that no customers in the system department is:

$$p_0(t)=\frac{1}{t}\sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty}\Bigg\{\frac{(n+1)(\mu)^n I_{n+1}\left(2\sqrt{\lambda\mu}t\right)}{(\lambda/\mu)^{\frac{n+1}{2}}}$$

$$-\frac{(n-1)(\mu)^n I_{n-1}\left(2\sqrt{\lambda\mu}t\right)}{(\lambda/\mu)^{\frac{n-1}{2}}}\Bigg\}e^{-(\lambda+\mu)t} \qquad (36)$$

Relations (34), (35) and (36) are the same results as Groos and Harris [3].

5. Conclusion

The aim of this paper was to obtain the transient solution of M/M/1 queue with balking, feedback, and catastrophe and repair using the Probability Generating Function. The transient state probabilities and some special case were obtained, linking the results to previous studies.

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