Libyan Journal of Science (An International Journal): Vol. 18, 2014/015

Unique Integrable Solution For The Coupled System Of Hammerstein Integral Equations

Sokina Mohamed Ramadan El-Jerbi
Faculty of Arts and Science, University of Al-Mergeb, Al Khums,
Misallata Branch, Libya.
E-.mail: www.smrmrj3020@Yahoo.com

Abstract

We are concerned here with the coupled system of Hammerstein integral equations. The existence of a unique integrable solution will be proved under the lipschitz condition. Also, the monotonicity solution for the coupled system is provided.

المستخلص

يهتم البحث بالنظام المزدوج لمعادلات هامرشتين التكاملية، حيث يتم إثبات وجود حل تكاملي وحيد تحت شرط لبشز، وعرض اضطرادية الحل لهذا النظام المزدوج أيضا.

Keywords: Integration; Hammerstein; coupled system; lipschitz condition.

Introduction and Preliminaries

The most frequently investigated for Hammerstein nonlinear integral equation [1][3]:

$$x(t) = \varphi(t) + \int_0^1 k(t,s) f\big(s, x(s)\big) \ ds \qquad , \ t \in [0,1]$$

Here, we prove the existence of a unique integrable solution for the coupled system of Hammerstein integral equations.

Accepted for publication: 27/12/2015

$$x(t) = g_1(t) + \int_0^1 k_1(t,s) f_1\big(s, y(s)\big) \; ds$$

$$(1)$$

$$y(t) = g_2(t) + \int_0^1 k_2(t,s) f_2\big(s, x(s)\big) \; ds \qquad\qquad t \in [0,1]$$

Which is proved by applying Banach fixed point principle for Contraction Theorem when the two functions f_1 and f_2 satisfy the lipschitz condition:

$$||f_i(t,x) - f_i(t,y)||_{L_1} \; < c_i ||x - y||_{L_1}$$

Under certain monotonicity condition by using the technique of measure of noncompactness.

Here, we relax the monotonicity and prove the existence of $L_1 -$ solution of the coupled system of Hammerstein integral equations (1).

Let $L_1 = L_1(I)$ be the class of Lebesgue integrable functions on I = [0; 1] with the standard norm:

$$\big||x|\big| = \int_0^1 |x(t)| \; dt$$

Let $L_1^* = L_1^*(I)$ be the class of all column vectors of integrable functions [2]:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, x_i \in \; L_1 \;, i = 1,2 \quad , and \qquad ||x||_{L_1} = \sum_{i=1}^{2} ||x_{\mathrm{i}}||_{L_i} \; = \sum_{i=1}^{2} \int_0^1 |x_i(t)| dt$$

Theorem (1.1) Banach Contraction Mapping Principle[4]:
Let (X, ρ) be a complete metric space and let $T : X \; \rightarrow X$ be a contraction map. Then T has a unique fixed point in X . Moreover, for any $x_0 \in X$ the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ converges to the fixed point.

Theorem 1.2: [5]
Let $K : I \times I \rightarrow R_+$ be measurable with respect to both variables and such that the integral operator K generated by k maps L_1 into itself. If for every $p \in (0,1)$ and for all $t_1, t_2 \in I$ the implication:

$$t_1 \leq t_2 \Rightarrow \int_0^p k(t_1, s)\, ds \leq \int_0^p k(t_2, s)\, ds$$

holds, then the operator K transforms the set of positive and nondecreasing functions from L_1 into itself.

Existing Unique Integrable Solution

Consider the following assumptions:

(i) $g_i : I \rightarrow R$ are integrable on I;

(ii) $f_i : I \times R \rightarrow R$ are measurable in $t \in I$ for any $x \in R$ and lipschitz in $x \in D \subset R$ for any $t \in I$

i.e. $||f_i(t,x) - f_i(t,y)||_{L_1} < c_i||x-y||_{L_1}$ and $|f_i(t,0)| \leq |a_i(t)|$, $a_i(t) \in L_1$;

(iii) $k_i : I \times I \rightarrow R$ are integrable in t for any $s \in I$ and measurable in s for any $t \in I$ such that:

$$\int_0^1 |k_i(t,s)| ds < M_i \qquad where \quad i = 1,2$$

Consider the operator T defined by:

$$TW = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_0^1 k_1(t,s) f_1\big(s, y(s)\big)\ ds \\ g_2(t) + \int_0^1 k_2(t,s) f_2\big(s, x(s)\big)\ ds \end{pmatrix}$$

Lemma 2.1 :
The operator T maps the space L_1^* into itself $T : L_1^* \rightarrow L_1^*$

Proof: From the assumption (ii) we have:

$$||f_i(t,x) - f_i(t,y)||_{L_1} \; < \; c_i||x-y||_{L_1}$$

and $|f_i(t,0)| \le |a_i(t)| \quad , a_i(t) \in L_1 \Rightarrow$

$$||f_i(t,x)||_{L_1} \; - \; ||a_i(t)||_{L_1} \; \le \; ||f_i(t,x) - f_i(t,0)||_{L_1} \;\; < c_i||x||_{L_1}$$

i.e. $||f_i(t,x)||_{L_1} \; - \; ||a_i(t)||_{L_1} \; < c_i||x||_{L_1}$

Therefore,

$$||f_i(t,x)||_{L_1} \; \le \; ||a_i||_{L_1} \; < c_i||x||_{L_1} \qquad (2)$$

Now, Let $\begin{pmatrix} x \\ y \end{pmatrix} \in L_1^*$:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_1(t) + \int_0^1 k_1(t,s) f_1\big(s, y(s)\big) \;\; ds \\ g_2(t) + \int_0^1 k_2(t,s) f_2\big(s, x(s)\big) \;\; ds \end{pmatrix}$$

$$||T\begin{pmatrix} x \\ y \end{pmatrix}||_{L_1}^* = \left\| g_1(t) + \int_0^1 k_1(t,s) f_1\big(s, y(s)\big) \;\; ds \right\|_{L_1}$$

$$+ \left\| \; g_2(t) + \int_0^1 k_2(t,s) f_2\big(s, x(s)\big) \;\; ds \; \right\|_{L_1}$$

$$= \int_0^1 |g_1(t) + \int_0^1 \; k_1(t,s) f_1\big(s, y(s)\big) \,|\, ds \, dt$$
$$+ \int_0^1 |g_2(t) \;\; + \int_0^1 \; k_2(t,s) f_2\big(s, x(s)\big) \,|\, ds \, dt$$

$$\le \int_0^1 |g_1(t)\,|\, dt \; + \int_0^1 \int_0^1 |k_1(t,s)| \;\; |f_1\big(s, y(s)\big)| \;\; ds \, dt$$
$$+ \int_0^1 |g_2(t)\,|\, dt \; + \int_0^1 \int_0^1 |k_2(t,s)| \;\; |f_2\big(s, x(s)\big)| \;\; ds \, dt$$

Interchange the order of integration we get:

$$||T\begin{pmatrix} x \\ y \end{pmatrix}||^{*}_{L_1} \leq \left||g_1|\right|_{L_1} + \left||g_2|\right|_{L_1} + \int_0^1 |f_1\big(s, y(s)\big)| \,(\int_0^1 |k_1(t,s)| \;\; dt\,)\, ds$$

$$+ \int_0^1 |f_2\big(s, x(s)\big)| \,(\int_0^1 |k_2(t,s)| \;\; dt\,)\, ds$$

$$\leq \left||g_1|\right|_{L_1} + \left||g_2|\right|_{L_1} + M_1 \int_0^1 |f_1\big(s, y(s)\big)|\, ds + M_2 \int_0^1 |f_2\big(s, x(s)\big)|\, ds$$

$$\leq \left||g_1|\right|_{L_1} + \left||g_2|\right|_{L_1} + M_1 \left||\, f_1(s,y)|\right|_{L_1} + \; M_2 \left||\, f_2(s,x)|\right|_{L_1}$$

By inequality (2)

$$||T\begin{pmatrix} x \\ y \end{pmatrix}||^{*}_{L_1} \leq \left||g_1|\right|_{L_1} + \left||g_2|\right|_{L_1} + \; M_1 \left||a_1|\right|_{L_1} + M_1 C_1 \,||y||_{L_1}$$

$$+ M_2 \left||a_2|\right|_{L_1} + M_2 C_2 \,||x||_{L_1}$$

$$= \left[\; \left||g_1|\right|_{L_1} + \left||g_2|\right|_{L_1} \right] + M\, [\, \left||a_1|\right|_{L_1} + \left||a_2|\right|_{L_1}]$$

$$+ MC\, \left[\, \left||x|\right|_{L_1} + \left||y|\right|_{L_1} \right]$$

Where, $M = \max\{ M_1 \,, M_2 \}$ and, $MC = \max\, \{ M_1 C_1 \,, M_2 C_2 \}$

Thus,

$$||T\begin{pmatrix} x \\ y \end{pmatrix}||^{*}_{L_1} \leq ||\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}||^{*}_{L_1} + \; M||\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}||^{*}_{L_1} \; + MC||\begin{pmatrix} x \\ y \end{pmatrix}||^{*}_{L_1}$$

Therefore,

$$||\mathrm{TW}||^{*}_{\mathrm{L}_1} \leq \; ||\mathrm{G}||^{*}_{\mathrm{L}_1} + \mathrm{M}||\mathrm{A}||^{*}_{\mathrm{L}_1} + \; \mathrm{CM}||\mathrm{W}||^{*}_{\mathrm{L}_1}$$

The last estimate shows that the operator T maps the space L_1^* into itself.

Theorem 2.2:

Let the assumptions $(i)-(iii)$ be satisfied then the coupled system (1) has a unique integrable solution in $\begin{pmatrix} x \\ y \end{pmatrix} \in L_1^*$

Proof:

$$U=\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} g_1(t)+\int_0^1 k_1(t,s)f_1\big(s,y_1(s)\big)\ ds \\ g_2(t)+\int_0^1 k_2(t,s)f_2\big(s,x_1(s)\big)\ ds \end{pmatrix}$$

$$V=\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} g_1(t)+\int_0^1 k_1(t,s)f_1\big(s,y_2(s)\big)\ ds \\ g_2(t)+\int_0^1 k_2(t,s)f_2\big(s,x_2(s)\big)\ ds \end{pmatrix}$$

Let,

$$TU=\begin{pmatrix} g_1(t)+\int_0^1 k_1(t,s)f_1\big(s,y_1(s)\big)\ ds \\ g_2(t)+\int_0^1 k_2(t,s)f_2\big(s,x_1(s)\big)\ ds \end{pmatrix}$$

Indeed, $T : L_1^* \rightarrow L_1^*$

$$TU-TV=\begin{pmatrix} \int_0^1 k_1(t,s)\ [f_1\big(s,y_1(s)\big)-f_1\big(s,y_2(s)\big)]\ ds \\ \int_0^1 k_2(t,s)\ [f_2\big(s,x_1(s)\big)-f_2\big(s,x_2(s)\big)]\ ds \end{pmatrix}$$

$$\begin{aligned} ||TU-TV||_{L_1}^* &= \left|\left|\int_0^1 k_1(t,s)\ [f_1\big(s,y_1(s)\big)-f_1\big(s,y_2(s)\big)]\ ds\right|\right| \\ &+\left|\left|\int_0^1 k_2(t,s)\ [f_2\big(s,x_1(s)\big)-f_2\big(s,x_2(s)\big)]\ ds\right|\right| \end{aligned}$$

$$||TU-TV||_{L_1}^* = \int_0^1 |TU-TV|dt$$

$$||TU - TV||^*_{L_1} \leq \int_0^1 |\int_0^1 k_1(t,s)\,[f_1(s,y_1(s)) - f_1(s,y_2(s))]\,|\,ds$$

$$+\int_0^1 |\int_0^1 k_2(t,s)\,[f_2(s,x_1(s)) - f_2(s,x_2(s))]\,|\,ds$$

$$\leq \int_0^1 \int_0^1 |k_1(t,s)|\,|f_1(s,y_1(s)) - f_1(s,y_2(s))|\; ds$$

$$+\int_0^1 \int_0^1 |k_2(t,s)|\,|f_2(s,x_1(s)) - f_2(s,x_2(s))\,|\,ds$$

Interchange the order of integration we get:

$$||TU - TV||^*_{L_1} \leq \int_0^1 |f_1(s,y_1(s)) - f_1(s,y_2(s))|\,[\int_0^1 |k_1(t,s)|dt]\; ds$$
$$+\int_0^1 |f_2(s,x_1(s)) - f_2(s,x_2(s))|\,[\int_0^1 |k_2(t,s)|dt]\; ds$$

$$\leq\; M_1 \int_0^1 |f_1(s,y_1(s)) - f_1(s,y_2(s))|\,ds$$
$$+\;\; M_2 \int_0^1 |f_2(s,x_1(s)) - f_2(s,x_2(s))|\,ds$$

$$\leq\; M_1||\,f_1(s,y_1(s)) - f_1(s,y_2(s))\,||_{L_1}$$

$$+\;\; M_2||\,f_2(s,x_1(s)) - f_2(s,x_2(s))\,||_{L_1}$$

$$\leq M_1C_1\left||y_1 - y_2|\right|_{L_1} + M_2C_2\left||x_1 - x_2|\right|_{L_1}$$

$$\leq\; \mathrm{MC}\;\left(\;\left||\mathrm{y}_1 - \mathrm{y}_2|\right|_{\mathrm{L}_1} + \left||\mathrm{x}_1 - \mathrm{x}_2|\right|_{\mathrm{L}_1}\;\right)$$
$$||TU - TV||^*_{L_1} \;\leq\; MC\;||U - V||_{L_1}$$

Where $MC = \max\,\{\,M_1C_1\;,\;M_2C_2\,\}$

Then,

If $MC < 1$ then T is a contraction.

By using fixed point Theorem, $\exists\ x , y \in L_1$ such that $TU = U$.

Existing Monotonic Integrable Solution

Corollary 3.1 :

Let the assumptions of Theorem**(2.2)** be satisfied if k_1 and g_1 are monotonic in t ,then x is monotonic solution in I and if k_2 and g_2 are monotonic in t , then y is monotonic solution in I .

Proof:

let $t_2 > t_1$, $t_1 , t_2 \in I$, then:

(1) $$x(t) = g_1(t) + \int_0^1 k_1(t,s)\, f_1(\ s , y\ (\varphi_1(s))\ )\ ds$$

$$x(t_2) = g_1(t_2) + \int_0^1 k_1(t_2,s)\, f_1(\ s, y\ (\varphi_1(s))\ )\ ds$$

$$\geq g_1(t_1) + \int_0^1 k_1(t_1,s)\, f_1(\ s, y\ (\varphi_1(s))\ )\ ds$$

$$= x(t_1)$$

Then, $x(t)$ is monotonical nondecreasing .

Also,

(2) $$y(t) = g_2(t) + \int_0^1 k_2(t,s)\, f_2(\ s , y\ (\varphi_2(s))\ )\ ds$$

$$y(t_2) = g_2(t_2) + \int_0^1 k_2(t_2,s)\, f_2(\ s, y\ (\varphi_2(s))\ )\ ds$$

$$\geq g_2(t_1) + \int_0^1 k_2(t_1, s)\, f_2(\; s, y\;(\varphi_2(s))\;\;)\; ds$$

$$= y(t_1)$$

Then, $y(t)$ is monotonical nondecreasing .

References

[1] Banas, J . (1989) Integrable solutions of Hammerstein and Urysohn integral equations, J. Austral. Math. Soc., (Series A) 46, 61-68.
[2] Banas, J. and El-sayed, W. G. (1993). Solvability of Functional and Integral Equations In Some Classes of Integrable Functions, Politechniki. Rzeszowskiej, Rzeszow.
[3] Emmanuele, G. (1993). Integrable Solution of Hammerstein Integral equation, Applicable Analysis, 50, 277-284.
[4] Goeble, K. and Krik, W. A. (1990). Topics in metric fixed point theory, Cambridge University Press,
[5] Zabrejko, P. P., Koshelev, A. I., Krasnoselskii, M. A., Mikhlin, S. G. Rakovshchik, L. S. and Stecenko, V.J. (1975) Integral Equations, Nauka, Moscow [English Translation: Noordhoff, Leyden]