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Exact Travelling Wave Solutions of the Nonlinear Klein-Gordon Equation Using the Modified Tanh-Function Expansion Method and a Direct Approach

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ABSTRACT

In this paper, we will use the tanh-function method with the aid of Maple software to find new and exact travelling wave solutions of the nonlinear Klein-Gordon equation. We will also rely on a direct algebraic method based on the Liénard equation to discover other new and exact solutions. The results include soliton solutions, periodic solutions, and rational function solutions. The new results obtained in this paper will be compared with the well-known results. Additionally, some 2D and 3D graphs of the obtained exact traveling wave solutions will be presented. Finally, the tanh-function expansion method used in this paper is straightforward and concise, and it can be applied to other nonlinear partial differential equations in mathematical physics.

Keywords: Exact travelling wave solutions, Nonlinear PDEs, Nonlinear Klein-Gordon equation, tanh-function method, Liénard equation

1. Introduction

In recent years, various new natural phenomena have emerged in mathematical physics and other fields such as plasma physics, biology, chemistry, engineering, quantum mechanics, fluid mechanics, optical fibers, hydrodynamic waves, and more, which can be explained using nonlinear partial differential equations (PDEs). There are numerous analytical methods available to obtain precise wave solutions for nonlinear PDEs in mathematical physics, such as the modified extended tanh-function method [1-5], the $\left(\frac{G'}{G}\right)$ -expansion method [6-8], the Exp-function method [9], the Jacobi elliptic function method [10], the auxiliary equation method [11], the generalized $\left(\frac{G'}{G}, \frac{1}{G}\right)$ −expansion method [12], the generalized projective Riccati equations method [13], the enhanced algebraic method [14], the new mapping method [15], and others.

The tanh-function method depends on adding integration constants to the resulting nonlinear ODEs from the nonlinear PDEs using wave transformation.

The aim of this paper is to use the modified tanh-function expansion method [1, 2] and the direct method supported by the Liénard equation [12] to discover new exact travelling wave solutions for the nonlinear Klein-Gordon equation given below [16]:

$$u_{tt} - k^2 u_{xx} + \alpha u - \beta u^3 + \gamma u^5 = 0, \qquad (1.1)$$

where k, α, β and γ are nonzero real-valued constants.

Equation (1.1) models nonlinear longitudinal wave propagation in elastic rods, as previously discussed in [16] using the (G'/G,1/G)-expansion method.

This article is organized as follows. In Section 2, we provide a description of the modified tanh-function expansion method. In Section 3, we apply this method to the nonlinear Klein-Gordon equation. In Section 4, we present further results for the nonlinear Klein-Gordon equation using a direct approach. In Section 5, we offer our conclusions.

2. Description of the modified tanh-function expansion method

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \qquad (2.1)$$

where P is a polynomial in its arguments. The essence of the modified tanh-function expansion method can be presented in the following steps [1, 2]:

Step 1: Seek travelling wave solutions of Eq. (2.1) by takin

$$u(x,t) = u(\xi), \xi = x - ct, \qquad (2.2)$$

The transformation (2.2) converts Eq. (2.1) to the ordinary differential equation (ODE):

$$Q(u, u', u'', \ldots) = 0, \qquad (2.3)$$

where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate Eq. (2.3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integral constant(s) may be zero.

Step 3: We assume that Eq. (2.3) has the formula solution:

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \phi^i(\xi) + \sum_{i=1}^{n} b_i \phi^{-i}(\xi), \qquad (2.4)$$

where n is a positive integer that can be determined by balancing the highest-order derivative term with the highest nonlinear term in Eq. (2.4), a_0, a_i, b_i, $i = 1, 2, \ldots, n$ are parameters to be determined such that $a_n \neq 0$ or $b_n \neq 0$ and $\phi'(\xi)$ is a solution of the following Riccati equation:

$$\phi'(\xi) = b + \phi^2(\xi), \qquad (2.5)$$

where b is a constant. It is well-known that Eq. (2.5) has three types of exact solutions [1, 2]. In some nonlinear equations the balance number n is not a positive integer. In this case, we make the following transformations [16]:

(a) When $n = q/p$, where q/p is a fraction in the lowest terms, we let

$$u\,(\xi) = v^{\frac{q}{p}}(\xi)\,, \qquad (2.6)$$

then substitute (2.6) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number;

(b) When n is a negative number, we let

$$u\,(\xi) = v^{n}(\xi), \qquad (2.7)$$

and substitute (2.7) into (2.3) to get a new equation in the new function $v\,(\xi)$ with a positive integer balance number.

Step 4: We substitute (2.4) with (2.5) into Eq. (2.3) yields a set of algebraic equations involving a_0, a_i, b_i, $i = 1, 2, \ldots, n$ and c, which can be solved using Maple or Mathematica to obtain analytic exact solutions of the nonlinear PDE (2.1) in closed form.

In the next sections, we will find the exact solutions of Eq. (1.1) using the modified tanh-function expansion method and a direct method with the help of Lienard equation.

3. Exact travelling wave solutions of Eq. $(\mathbf{1.1})$ using the modified tanh-function expansion method

In this section, we will apply the modified tanh-function expansion method described in Sec. 2 to construct new exact travelling wave solutions of the nonlinear Klein-Gordon equation (1.1). To this aim, we use the wave transform (2.2) to convert Eq. (1.1) to the following nonlinear ODE:

$$c^2u'' - k^2u'' + \alpha u - \beta u^3 + \gamma u^5 = 0. \qquad (3.1)$$

Integrating (3.1) w. r. to ξ twice, we have

$$(c^2 - k^2)u'' + \alpha u - \beta u^3 + \gamma u^5 = 0, \qquad (3.2)$$

By balancing u^5 with u'' in Eq. (3.2), we get $n = \frac{1}{2}$. Therefore, we use the new transformation [16]:

$$u(\xi) = v^{\frac{1}{2}}(\xi), \qquad (3.3)$$

where $v(\xi)$ is a new function of ξ . Substituting (3.3) into Eq. (3.2) ,we get the new nonlinear ODE:

$$(c^2 - k^2)\left(\frac{1}{2}vv'' - \frac{1}{4}(v')^2\right) + \alpha v^2 - \beta v^3 + \gamma v^4 = 0. \qquad (3.4)$$

we balance the variables vv'' with v^4 in Eq. (3.4) giving $N = 1$. Thus, we obtain the corresponding solution:

$$v(\xi) = \alpha_0 + \alpha_1\phi + \alpha_{-1}\phi^{-1}, \qquad (3.5)$$

where $\alpha_0, \alpha_1, \alpha_{-1}$ are constants to be detemined, such that $\alpha_1 \neq 0$, or $\alpha_{-1} \neq 0$, while ϕ satisfies the Riccati Eq. (1.5).

Substituting (3.5) into (3.4) and using (1.5), the left-hand side of (3.4) becomes a polynomial in ϕ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations as follows:

$$\phi^4: \gamma\alpha_1^4 + \frac{3}{4}\alpha_1^2c^2 - \frac{3}{4}\alpha_1^2k^2 = 0,$$

$$\phi^3: 4\gamma\alpha^{0\alpha_1^3} - \beta\alpha_1^3 + c^2\alpha^0\alpha^1 - k^2\alpha^0\alpha^1 = 0,$$

$$\phi^2: \tfrac{2}{3}\alpha_{-1}\alpha_1c^2 - \tfrac{2}{3}\alpha_{-1}\alpha_1k^2 + \tfrac{1}{2}\alpha_1^2bc^2 - \tfrac{1}{2}\alpha_1^2bk^2 - 3\beta\alpha_0\alpha_1^2 + 4\gamma\alpha^{-1}\alpha^{13}6\gamma\alpha_0^2\alpha_1^2 + \alpha\alpha_1^2 = 0,$$

$$\phi^1: bc^2\alpha_0\alpha_1 - bk^2\alpha_0\alpha_1 + 12\gamma\alpha_0\alpha_{-1}\alpha_1^2 + 4\gamma\alpha_1\alpha_0^3 - 3\beta\alpha_{-1}\alpha_1^2 - 3\beta\alpha_0^2\alpha_1 + 2\alpha\alpha^0\alpha^1 = 0,$$

$$\phi^0: 3\alpha_{-1}\alpha_1bc^2 - 3\alpha_{-1}\alpha_0bk^2 - 6\beta\alpha_{-1}\alpha_0\alpha_1 + 12\gamma\alpha_{-1}\alpha_0^2\alpha_1 + \gamma\alpha_0^4 + 6\gamma\alpha_{-1}^2\alpha_1^2 + \alpha\alpha_0^2 - \beta\alpha_0^3 - \tfrac{1}{4}\alpha_{-1}^2c^2 + \tfrac{1}{4}\alpha_{-1}^2k^2 - \tfrac{1}{4}\alpha_1^2b^2c^2 + \tfrac{1}{4}\alpha_1^2b^2k^2 + 2\alpha\alpha^{-1}\alpha^1 = 0,$$

$$\phi^{-1}: bc^2\alpha_{-1}\alpha_0 - bk^2\alpha_{-1}\alpha_0 + 12\gamma\alpha_{-1}^2\alpha_1\alpha_0 + 4\gamma\alpha_{-1}\alpha_0^3 - 3\beta\alpha_{-1}^2\alpha_1 - 3\beta\alpha^{-1\alpha_0^2} + 2\alpha\alpha^{-1}\alpha^0 = 0,$$

ϕ^{-2}: $\frac{1}{2}\alpha_{-1}^2 bc^2 - \frac{1}{2}\alpha_{-1}^2 bk^2 - 3\beta\alpha_0\alpha_{-1}^2 + 6\gamma\alpha_{-1}^2\alpha_{-1}^3 + 4\gamma\alpha_1\alpha_{-1}^3 + \alpha\alpha_{-1}^2 + \frac{2}{3}\alpha^{-1}\alpha^1 b^2c^2 - \frac{2}{3}\alpha^{-1}\alpha^1 b^2k^2 = 0,$

ϕ^{-3}: $b^2c^2\alpha^{-1}\alpha^0 - b^2k^2\alpha^{-1}\alpha^0 + 4\gamma\alpha^{0\alpha_{-1}^3} - \beta\alpha_{-1}^3 = 0,$

ϕ^{-4}: $\frac{3}{4}\alpha_{-1}^2 b^2c^2 - \frac{3}{4}\alpha_{-1}^2 b^2k^2 + \gamma\alpha_{-1}^4 = 0.$

On solving the above system using Maple or Matimatica, we have the following results:

Result 1.

$$b = \frac{\alpha}{(c^2-k^2)},\ \beta = 4\sqrt{\frac{\gamma\alpha}{3}},\ c = c,\ \alpha_{-1} = 0,\ \alpha_0 = \frac{1}{2}\sqrt{\frac{3\alpha}{\gamma}},\ \alpha_1 = \sqrt{-\frac{3(c^2-k^2)}{4\gamma}}, \tag{3.6}$$

where $\gamma\alpha > 0$ and $\gamma(c^2-k^2) < 0$.

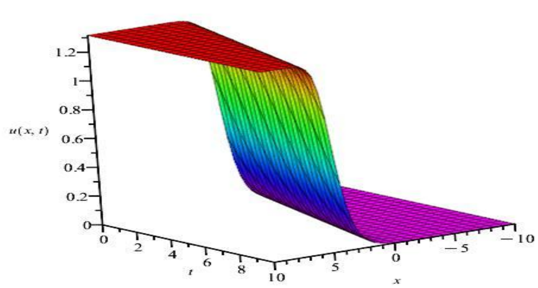
From (3.5), and (3.6), we deduce the exact traveling wave solutions of Eq. (1.1) as follows:

Case 1.

If $\alpha(c^2-k^2) < 0$, then $b < 0$, and from (3.6), we obtain the dark soliton solution

$$u(\xi) = \left[\sqrt{\frac{3\alpha}{4\gamma}}\left(1 \pm \tanh\left(\sqrt{-\frac{\alpha}{(c^2-k^2)}}\xi\right)\right)\right]^{\frac{1}{2}}, \tag{3.7}$$

where $\xi = x - ct$.



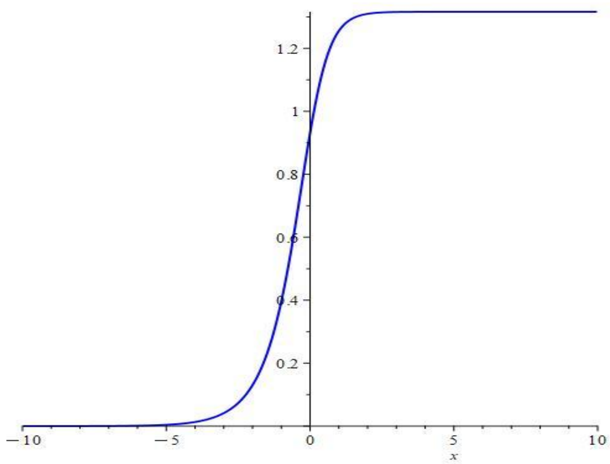


Figure 1. The profile of the dark-soliton solution (3.7) with $k = \gamma = \alpha = 1,\ c = \frac{1}{2}$.

Case 2.

If $\alpha(c^2-k^2) = 0$, then $b = 0$, and from (3.6), we obtain the rational exact solution

$$u(\xi) = \left[-\sqrt{-\frac{3c^2-3k^2}{4\gamma}}\left(-\frac{1}{\xi}\right)\right]^{\frac{1}{2}}, \tag{3.8}$$

where $\xi = x - ct$.

Result 2.

$$b = \frac{\alpha}{(c^2-k^2)},\ \beta = 4\sqrt{\frac{\gamma\alpha}{3}},\ c = c,\ \alpha_{-1} = \sqrt{-\frac{3\alpha^2}{4(c^2-k^2)\gamma}},\ \alpha_0 = \frac{1}{2}\sqrt{\frac{3\alpha}{\gamma}},\ \alpha_1 = 0, \tag{3.9}$$

where $\gamma\alpha > 0$ and $\gamma(c^2-k^2) < 0$.

In this result, if $\alpha(c^2-k^2) < 0$, then $b < 0$, and from (3.9), we obtain the singular soliton solution

$$u(\xi) = \left[\sqrt{\frac{3\alpha}{4\gamma}}\left(1 \pm \coth\left(\sqrt{-\frac{\alpha}{(c^2-k^2)}}\xi\right)\right)\right]^{\frac{1}{2}}, \tag{3.10}$$

where $\xi = x - ct$.

Result 3.

$$b = -\frac{\alpha}{5(c^2-k^2)},\ \beta = \frac{8}{15}\sqrt{15\gamma\alpha},\ c = c,\ \alpha_{-1} = \frac{3}{10}\sqrt{\frac{\alpha^2}{-\gamma 3(c^2-k^2)}},\ \alpha_0 = \frac{1}{2}\sqrt{\frac{3\alpha}{\gamma}},$$

$$\alpha_1 = \frac{1}{2}\sqrt{-\frac{3(c^2-k^2)}{\gamma}},$$

(3.11)

where $\gamma\alpha > 0$ and $\gamma(c^2-k^2) < 0$.

In this result, we have the follow two cases of exact solutions:

Case 1.

If $\alpha(c^2-k^2) < 0$, then $b > 0$, and from (3.11), we obtain the periodic solution

$$u(\xi) = \left[\sqrt{\frac{3\alpha}{5\gamma}}\left(1 \pm \frac{1}{2}\tan(\sqrt{\frac{-\alpha}{(5c^2-5k^2)}}\xi) \pm \frac{1}{2}\cot(\sqrt{\frac{-\alpha}{(5c^2-5k^2)}}\xi)\right)\right]^{\frac{1}{2}}, \quad (3.12)$$

where $\xi = x - ct$.

Case 2.

If $\alpha(c^2-k^2) = 0$, then $b = 0$, and from (3.11), we obtain the rational exact solution

$$u(\xi) = \left[-\frac{1}{2}\sqrt{\frac{-3c^2+3k^2}{\gamma}}\left(-\frac{1}{\xi}\right)\right]^{\frac{1}{2}}, \quad (3.13)$$

Where $\xi = x - ct$.

Result 4.

$$b = \frac{\alpha}{4(c^2-k^2)}, \quad \beta = 4\sqrt{\frac{\alpha\,\gamma}{3}}, \quad c = c,\ \alpha_{-1} = \frac{3}{8}\sqrt{\frac{\alpha^2}{-\gamma 3(c^2-k^2)}},\ \alpha_0 = \frac{1}{2}\sqrt{\frac{3\alpha}{\gamma}},$$

$$\alpha_1 = \frac{1}{2}\sqrt{-\frac{3(c^2-k^2)}{\gamma}}, \quad (3.14)$$

where $\gamma\alpha > 0$ and $\gamma(c^2-k^2) < 0$.

In this result, if $\alpha(c^2-k^2) < 0$, then $b < 0$, and from (3.14), we obtain the straddled soliton solution

$$u(\xi) = \left[\sqrt{\frac{3\alpha}{4\gamma}}\left(1 \pm \frac{1}{2}\tanh(\sqrt{\frac{-\alpha}{(4c^2-4k^2)}}\xi) \pm \frac{1}{2}\coth(\sqrt{\frac{-\alpha}{(4c^2-4k^2)}}\xi)\right)\right]^{\frac{1}{2}}, \quad (3.15)$$

where $\xi = x - ct$.

Result 5.

$$b = \frac{\alpha}{(c^2-k^2)},\ \beta = 0,\ c = c,\ \alpha_{-1} = \frac{1}{2}\sqrt{-\frac{3\alpha^2}{\gamma(c^2-k^2)}},\ \alpha_0 = 0,\ \alpha_1 = \frac{1}{2}\sqrt{-\frac{3(c^2+k^2)}{\gamma}}, \quad (3.16)$$

where $\gamma\alpha > 0$ and $\gamma(c^2-k^2) < 0$

In this result, if $\alpha(c^2-k^2) < 0$, then $b < 0$, and from (3.16), we obtain the straddled solitons solution

$$u(\xi) = \left[\sqrt{\frac{3\alpha}{4\gamma}}\left(1 \pm \frac{1}{2}\tanh(\sqrt{\frac{-\alpha}{(c^2-k^2)}}\xi) \pm \frac{1}{2}\coth(\sqrt{\frac{-\alpha}{(c^2-k^2)}}\xi)\right)\right]^{\frac{1}{2}}, \quad (3.17)$$

where $\xi = x - ct$.

4. Further results for Equation (3.1)

In this section, we will utilize a direct method with the aid of the Lienard equation to solve Eq. (3.2) and obtain new exact traveling wave solutions for Eq. (1.1) which are different of the results obtained in Sec. (3). to achieve this, we rewrite the equation (3.2) in the following form:

$$u'' + \frac{\alpha}{(c^2-k^2)}u - \frac{\beta}{(c^2-k^2)}u^3 + \frac{\gamma}{(c^2-k^2)}u^5 = 0, \quad (4.1)$$

if we set

$$l_1 = \frac{\alpha}{(c^2-k^2)}, \quad l_3 = -\frac{\beta}{(c^2-k^2)}, \quad l_5 = \frac{\gamma}{(c^2-k^2)},$$

in Eq. (4.1), then we obtain the famous Lienard equation

$$u'' + l_1 u - l_3 u^3 + l_5 u^5 = 0, \quad (4.2)$$

It is well-known that equation (4.2) has many solutions [12] with the aid of these solutions. we have the following solitary wave solution of Eq. (4.1):

(I)

$$u(x,t) = \pm\left\{\frac{4\left(\frac{\alpha}{k^2-c^2}\right)}{\frac{\beta}{(k^2-c^2)}\pm\sqrt{\frac{\beta^2}{(c^2-k^2)^2}-\frac{16\gamma\alpha}{3(c^2-k^2)^2}}\cosh\left(2\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)}\right\}^{\frac{1}{2}}, \quad (4.3)$$

where $\frac{\alpha}{(c^2-k^2)} < 0$ and $\frac{\beta^2}{(c^2-k^2)^2} - \frac{16\gamma\alpha}{3(c^2-k^2)^2} > 0$.

(II)

$$u(x,t) = \pm\left\{\frac{4\left(\frac{\alpha}{k^2-c^2}\right)}{\frac{\beta}{(k^2-c^2)}\pm\sqrt{\frac{16\gamma\alpha}{3(c^2-k^2)^2}-\frac{\beta^2}{(c^2-k^2)^2}}\sinh\left(2\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)}\right\}^{\frac{1}{2}}, \quad (4.4)$$

where $\frac{\alpha}{(c^2-k^2)} < 0$ and $\frac{\beta^2}{(c^2-k^2)^2} - \frac{16\gamma\alpha}{3(c^2-k^2)^2} < 0$.

(III)

$$u(x,t) = \pm\left\{\frac{2\alpha}{\beta}\left(1 \pm \tanh\left(\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)\right)\right\}^{\frac{1}{2}}, \quad (4.5)$$

and

$$u(x,t) = \pm\left\{\frac{2\alpha}{\beta}\left(1 \pm \coth\left(\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)\right)\right\}^{\frac{1}{2}}, \quad (4.6)$$

where $\frac{\beta^2}{(c^2-k^2)^2} - \frac{16\gamma\alpha}{3(c^2-k^2)^2} = 0, -\frac{\beta^2}{(c^2-k^2)} > 0, \frac{\gamma}{(c^2-k^2)} < 0$ and $\frac{\alpha}{(c^2-k^2)} < 0$.

Also, we have the periodic wave solution of Eq. (4.1) as follows:

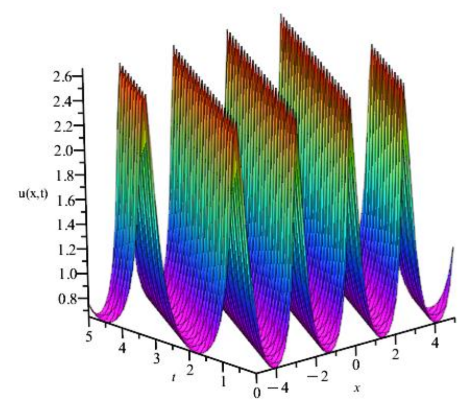
(IV)

$$u(x,t) = \pm\left\{\frac{4\left(\frac{\alpha}{k^2-c^2}\right)}{\frac{\beta}{(k^2-c^2)}\pm\sqrt{\frac{\beta^2}{(c^2-k^2)^2}-\frac{16\gamma\alpha}{3(c^2-k^2)^2}}\cos\left(2\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)}\right\}^{\frac{1}{2}}, \quad (4.7)$$

and

$$u(x,t) = \pm\left\{\frac{4\left(\frac{\alpha}{k^2-c^2}\right)}{\frac{\beta}{(k^2-c^2)}\pm\sqrt{\frac{\beta^2}{(c^2-k^2)^2}-\frac{16\gamma\alpha}{3(c^2-k^2)^2}}\sin\left(2\sqrt{\frac{\alpha}{k^2-c^2}}\xi\right)}\right\}^{\frac{1}{2}}, \quad (4.8)$$

where $\frac{\alpha}{(c^2-k^2)} > 0$ and $\frac{\beta^2}{(c^2-k^2)^2} - \frac{16\gamma\alpha}{3(c^2-k^2)^2} > 0$ and $\xi = x - ct$.



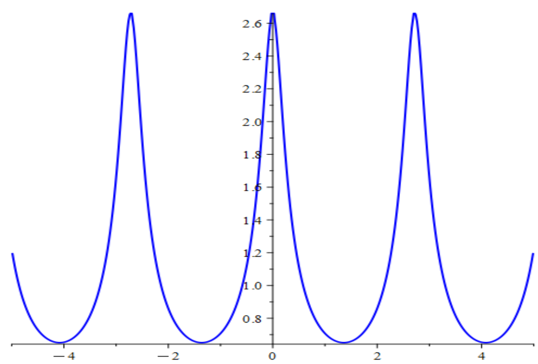


Figure 2. The profile of the periodic solution (4.7) with $c = \gamma = \alpha = 1$, $k = \frac{1}{2}$, $\beta = 5$.

Now, we deduce new exact solutions of Eq. (3.1) in terms of Jacobi elliptic functions as follows:

(V)

$$u(x,t) = \pm\left\{\frac{3\beta}{8\gamma}\left(1 \pm \text{sn}\left(\frac{\sqrt{3}\left(\frac{\beta}{k^2-c^2}\right)}{4r\sqrt{\frac{\gamma}{(k^2-c^2)}}}\xi, r\right)\right)\right\}^{\frac{1}{2}}, \quad (4.9)$$

where $\alpha = \frac{3\beta^2(4r^2+1)}{64\gamma r^2}$, $\frac{\beta}{(c^2-k^2)} > 0, \frac{\gamma}{(c^2-k^2)} < 0.$

When the modulus $r \to 1$, then we have the dark soliton solutions:

$$u(x,t) = \left\{\frac{3\beta}{8\gamma}\left(1 + \tanh\left(\frac{1}{4}\frac{\sqrt{3}\beta\xi}{(-c^2+k^2)\sqrt{\frac{\gamma}{(c^2-k^2)}}}\right)\right)\right\}^{\frac{1}{2}}. \quad (4.10)$$

(VI)

$$u(x,t) = \pm\left\{\frac{3\beta}{8\gamma}\left(1 \pm \text{cn}\left(\frac{\sqrt{3}\left(\frac{\beta}{k^2-c^2}\right)}{4r\sqrt{\frac{\gamma}{(c^2-k^2)}}}\xi, r\right)\right)\right\}^{\frac{1}{2}}, \quad (4.11)$$

where $\alpha = \frac{3\beta^2(4r^2+1)}{64\gamma r^2}$, $\frac{\beta}{(c^2-k^2)} < 0, \frac{\gamma}{(c^2-k^2)} > 0.$

When the modulus $r \to 1$, then we have the bright soliton solutions:

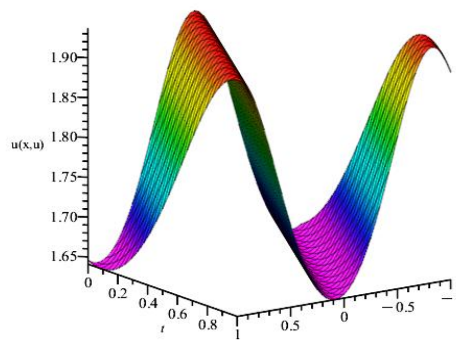
$$u(x,t) = \left\{\frac{3\beta}{8\gamma}\left(1 + \text{sech}\left(\frac{1}{4}\frac{\sqrt{3}\beta\xi}{(-c^2+k^2)\sqrt{\frac{\gamma}{(c^2-k^2)}}}\right)\right)\right\}^{\frac{1}{2}}. \quad (4.12)$$

(VII)

$$u(x,t) = \pm\left\{\frac{3\beta}{8\gamma}\left(1 \pm \text{dn}\left(\frac{\sqrt{3}\left(\frac{\beta}{k^2-c^2}\right)}{4\sqrt{\frac{\gamma}{(c^2-k^2)}}}\xi, r\right)\right)\right\}^{\frac{1}{2}}, \quad (4.13)$$

where $\alpha = \frac{3\beta^2(4r^2+1)}{64\gamma r^2}$, $\frac{\beta}{(c^2-k^2)} < 0, \frac{\gamma}{(c^2-k^2)} > 0$ and $\xi = x - ct$.

When the modulus $r \to 1$, then we have the same bright soliton solutions (4.10).



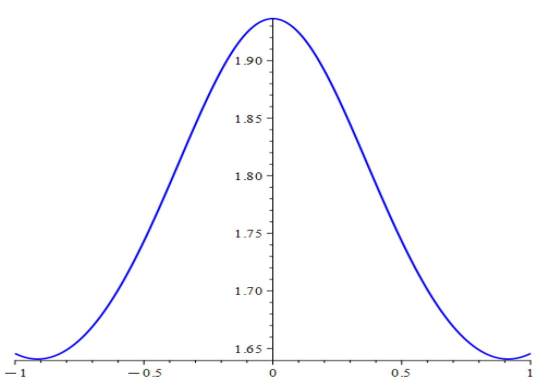


Figure 3. The profile of the Jacobi elliptic function solution (4.13) with $c = \gamma = \alpha = 1$, $k = \frac{1}{2}$, $\beta = 5$, $r = 0.9$.

5. Conclusion

The tanh-function method and a direct algebraic method based on the Liénard equation are used in this article to obtain many new exact wave solutions to the nonlinear Klein-Gordon equation. Comparing our new results obtained in this paper with the well-known results in [16], we conclude that all results obtained in article are new and not found elsewhere. 2D and 3D graphs of certain selected solutions were depicted to show the physical structure of different solutions types. The method employed in this article is effective and can be applied to other nonlinear models in the field of mathematical physics. Furthermore, with the aid of Maple software, we have demonstrated that all the solutions obtained in this paper satisfy the original governing equations.

6. References

1. Zayed, E. M. E., Alurrfi, K. A. E., (2013), The modified extended tanh-function method and its applications to the generalized KdV-mKdV equation with any-order nonlinear terms. Int. J. Environ. Eng. Sci. Technol. Res., 8, 165-170.
2. Zayed, E. M., Alurrfi, K. A., (2015), On solving the nonlinear Biswas-Milovic equation with dual-power law nonlinearity using the extended tanh-function method. J. Adv. Phys., 11, 3002-3012.
3. Rabie, W. B., Ahmed, H. M., Darwish, A., Hussein, H. H., (2023), Construction of new solitons and other wave solutions for a concatenation model using modified extended tanh-function method. Alex. Eng. J., 74, 445-451.
4. Ananna, S. N., Gharami, P. P., An, T., Asaduzzaman, M., (2022), The improved modified extended tanh-function method to develop the exact travelling wave solutions of a family of 3D fractional WBBM equations. Results Phys., 41, 105969.
5. El-Wakil, S. A., Abdou, M. A., (2007), Modified extended tanh-function method for solving nonlinear partial differential equations. Chaos Solitons & Fractals, 31, 1256-1264.
6. Shahoot, A. M., Alurrfi, K. A., Elmrid, M. O., Almsiri, A. M., Arwiniya, A. M., (2019), The (G′/G)-expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines. J. Taibah Univ. Sci., 13, 63-70.
7. Zayed, E. M., Alurrfi, K. A., Arnous, A. H., Hashemi, M. S., Bayram, M., (2024), Effects of high dispersion and generalized non-local laws on optical soliton perturbations in magneto-optic waveguides with sextic-power law refractive index. Nonlinear Dyn., 112, 8507-8525.
8. Wang, M., Li, X., Zhang, J., (2008), The (G′/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A, 372, 417-423.
9. He, J. H., Wu, X. H., (2006), Exp-function method for nonlinear wave equations. Chaos, Solitons Fractals, 30, 700-708.
10. Zayed, E. M. E, , Alurrfi, K. A. E., (2015), A new Jacobi elliptic function expansion method for solving a nonlinear PDE describing pulse narrowing nonlinear transmission lines. J. Part. Diff. Eq., 28, 128-138.
11. Du, Y., Yin, T., Pang, J., (2024), The exact solutions of Schrödinger–Hirota equation based on the auxiliary equation method. Opt. Quantum Electron., 56, 712.
12. Alurrfi, K. A., Shahoot, A. M., Elhasadi, O. I., (2024), Exact solutions for the GKdV–mKdV

equation with higher-order nonlinear terms using the generalized (G′/G,1/G)-expansion method and the generalized Liénard equation. Ric. Mat., 73, 887–905.

13. Zayed, E. M. E, Alurrfi, K. A. E., (2015), The generalized projective Riccati equations method and its applications for solving two nonlinear PDEs describing microtubules, Int. J. Phys. Sci., 13, 391-402.
14. Arnous, A. H., Hashemi, M. S., Nisar, K. S., Shakeel, M., Ahmad, J., Ahmad, I., Shah, N. A., (2024), Investigating solitary wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics. Results in Phys., 57, 107369.
15. Zayed, E. M., Alurrfi, K. A., Alshbear, R. A., (2023), On application of the new mapping method to magneto-optic waveguides having Kudryashov's law of refractive index. Optik, 287, 171072.
16. Zayed, E. M., Alurrfi, K. A., (2014), The (G' /G,1/G)-expansion method and its applications for solving two higher order nonlinear evolution equations, Math. Probl. Eng., Volume 2014, Article ID 746538, 20 pages.