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A Simulation Study to Assess the Performance of Three Normality Tests

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Abstract

In this paper, Anderson-Darling (AD), Lilliefors (LI), and Jarque-Bera (JB) tests were compared for Type I error and for power of the tests. The simulation was run 100,000 times for different situations and for different types of departures from normality. For all different sample sizes and distributions, AD gave the most powerful results, followed by the JB test.

Keywords: Type I error; power of the test; Anderson-Darling; Lilliefors; Jarque-Bera tests.

المستخلص

في هذه الورقة، تمت مقارنة اختبار أندرسون–دارلنج (AD)، واختبار ليليفورس (LI) و اختبار جاركي–بيرا (JB) بالنسبة لخطأ النوع الأول وقوة الاختبار. تم إجراء عدد 100000 محاكاة لحالات مختلفة وأنواع مختلفة بعيدة عن الشكل الطبيعي. لجميع أحجام العينات والتوزيعات، وقذ أعطى اختبار AD نتائج أقوى، يليه اختبار JB.

Introduction

Most statistical tests such as *t*-tests, linear regression analysis and Analysis of Variance (ANOVA) require the normality assumptions. Data can be viewed with graphical methods to roughly assess normality. However, graphical methods do not test if the differences between normal distribution and the sample distribution are significant. Tests used for assessing normality are Chi-square, Kolmogorov-Smirnov, Shapiro-Wilks, Anderson Darling, Lilliefors and Jarque-Bera.

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The last three are the most frequently used tests. In most situations, data deviates from normality. Previous studies did not attempt to determine which testing method gives higher power for different cases of sample sizes and distributions and they also had low simulation runs (Ohta and Arizono, 1989; Lin and Mudholkar, 1980).

The main objective of this paper was to evaluate Anderson-Darling (Anderson and Darling, 1952), Lilliefors (Lilliefors, 1967), and Jarque-Bera (Jarque and Bera,1987) tests for Type I error rates and for power of the tests.

The main three tests that assess assumption of normality are Anderson-Darling (AD), Lilliefors (LI), and Jarque-Bera (JB).

Anderson Darling Test

This test for normality, developed by Anderson and Darling (1952), is a popular normality test based on empirical distribution function (EDF) statistics. The Anderson-Darling test is commonly used to test whether a data sample comes from a normal distribution. Let $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ be the order statistics of a random sample $x_1, x_2, \cdots, x_n$ comes from a distribution with cumulative distribution function $F(x)$. The Anderson–Darling test statistic (AD) is defined by

$$\mathrm{AD} = \frac{-1}{n}\left[\sum_{i=1}^{n}(2i-1)\left(\ln \hat{u}_{(i)} + \ln(1-\hat{u}_{(n-i+1)})\right)\right] - n;$$

where

$$\hat{u}_{(i)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\hat{z}_{(i)}} e^{-\frac{t^2}{2}}dt, \qquad \hat{z}_{(i)} = \frac{x_{(i)} - \bar{x}}{s};$$

$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$ are the sample mean and variance (see Gan and Koehler, 1990).

If the resulting AD statistic is significant, then the null hypotheses (H_0) that the sample comes from a normally distributed population is rejected.

Lilliefors Test

Lilliefors test (LI) is a modification of the Kolmogorov-Smirnov test (Lilliefors, 1967). It is suitable when the unknown parameters of the null distribution must be estimated from the sample data. The LI statistic is defined by

$$\mathrm{LI} = \max\left[\max_{1\le i\le n}\left(\frac{i}{n} - \hat{u}_{(i)}\right), \max_{1\le i\le n}\left(\hat{u}_{(i)} - \frac{i-1}{n}\right)\right];$$

where

$$\hat{u}_{(i)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\hat{z}_{(i)}} e^{-\frac{t^2}{2}}\,dt, \qquad \hat{z}_{(i)} = \frac{x_{(i)} - \bar{x}}{s};$$

$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$ are the sample mean and variance.

This test is available when *n* is greater than or equal to 3.

Jarque-Bera Test

The Jarque–Bera (JB) test is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution. Let $x_1, x_2, \cdots, x_n$ be a random sample. The third standardized sample moment of $x_1, x_2, \cdots, x_n$ is called sample skewness and is denoted by

$$\sqrt{b_1} = g_3 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^{3/2}}.$$

The fourth standardized sample moment of $x_1, x_2, \cdots, x_n$ is called sample kurtosis and is denoted by

$$b_2 = g_4 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^{2}}$$

The test statistic denoted by

$$\mathrm{JB} = \frac{n}{6}\left(\left(\sqrt{b_1}\right)^2 + \frac{(b_2 - 3)^2}{4} \right),$$

is called Jarque-Bera statistic or just JB statistic, the corresponding test for normality is called Jarque-Bera test or shortly JB test. The test was defined and treated in Jarque and Bera (1987) and their earlier papers. JB is asymptotically chi-squared distributed with two degrees of freedom because JB is just the sum of squares of two asymptotically independent standardized normals, (see Bowman and Shenton, 1975). The Jarque-Bera test is probably the best known normality test to economists and is often used as a test of the normality of residuals.

Materials and Methods

A computer simulation program was used to study Monte Carlo techniques to evaluate the power of Anderson Darling (AD), Lilliefors (LI), and Jarque–Bera (JB) test statistics in testing whether a random sample of *n* independent observations come from a population with a normal distribution. The null and alternative hypotheses are:

H_0: The distribution is normal

H_1: The distribution is not normal.

Matlab R2014a was used to write the program. Type I error rates and statistical power of Anderson and Darling, Lilliefors, and Jarque–Bera tests were measured for different situations. The level of significance $\alpha = 0.05$ was used to investigate the power of the tests. Samples with various sample sizes were taken from the $N(0,1)$, *t*(30), χ^2(30), Gamma(2, 3), χ^2(3), Beta (2, 5), Wiebull(1.5, 1), and Exp (0.50) distributions (Fig. 1).

Random numbers were generated using generators from Matlab R2014a (functions normrnd, trnd, chi2rnd, gamrnd, wblrnd, exprnd, and betarnd) (Moonjung and Wendy, 2015). Sample sizes were chosen as *n* = 7, 8, 9, 10 (5) 95, 100 (50) 450, 500 (250) 2000 for each distribution. This allowed assessment of the Type I error rates and power of statistical tests under small, moderate and large sample size conditions. In each case 100,000 pairs of data sets were generated. Each pair was then compared by each of the three tests. The populations were standardized because they have different means and variances. When samples were taken from $N(0,1)$ populations, the number of rejected H_0 hypotheses was declared as the probability for Type I error. When samples were taken from populations with non-normal distributions, the number of rejected H_0 hypotheses was declared as the test's power. Accordingly, to compute empirical Type I error rate and test power, the program ran each condition 100,000 times and kept tract of proportion of significant statistics.

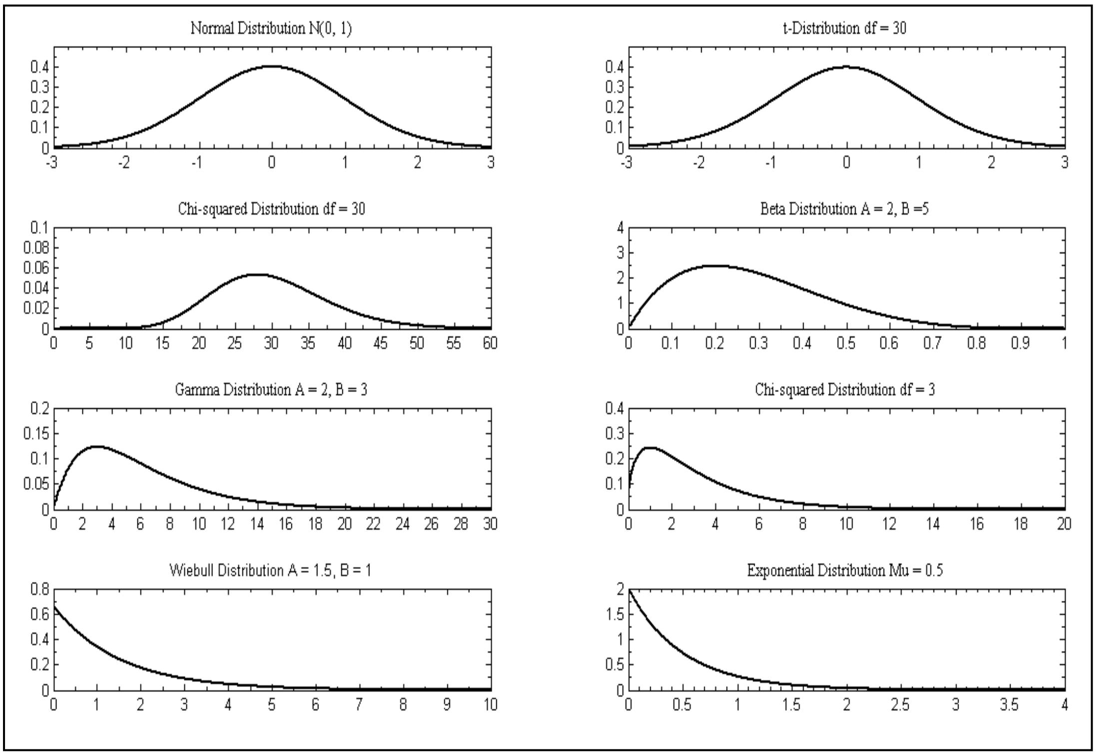

Fig. 1. Probability density functions of the eight distributions used in the simulation.

Results and Discussion

The power of the tests varies with the significance level, α, sample size and alternative distributions. However, only the results of power for $\alpha = 0.05$, several sample sizes and selected distributions were presented in this paper due to space limitation. The sample sizes presented in the figures were selected at the point which the power dramatically changed.

Empirical results of 100,000 simulation runs are given in the Appendix. Figure 1 below shows the plot of simulated type I error rates for all three tests against the standard normal distribution for $\alpha = 0.05$. From figure 1 it can be clearly seen that all three tests had similar type I error rates.

Figure 2 below shows the plot of simulated power for all three tests against the *t*-distribution with 30 degrees of freedom for $\alpha = 0.05$. Referring to figure 1 and figure 2 we note that the number of rejected null hypotheses (power of the tests) were similar to the number of rejected null hypotheses (type I error rate) for standard normal distribution when using AD and LI test for sample size less than 100.

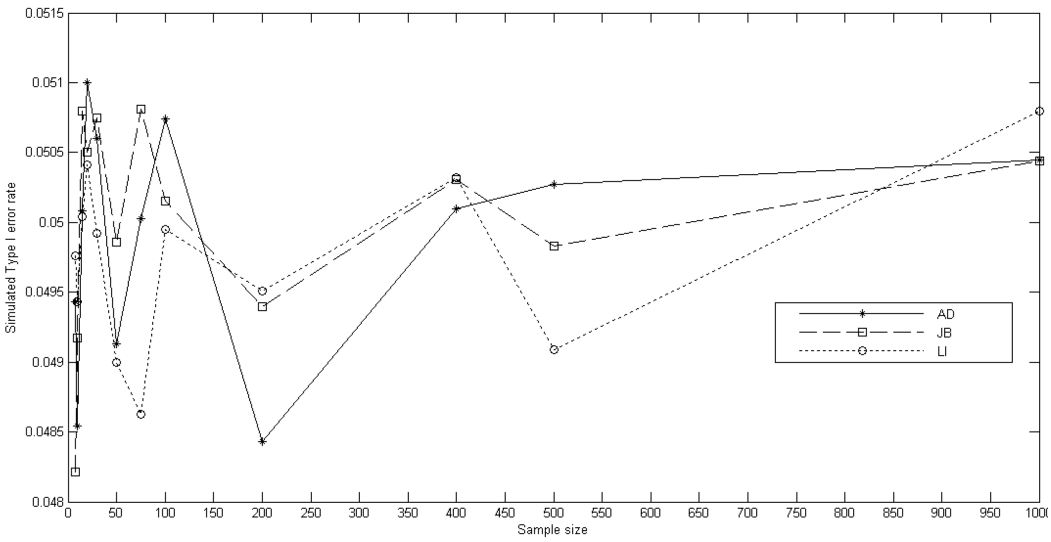

Figure 1. Comparison of simulated type I rates for three normality tests against standard normal distribution (skewness = 0.00, kurtosis = 3.00).

Figure 3 displayes the plot of power for the three tests against $\chi^2(30)$ distribution for 5% significance level. It is clear from Figure 3 that the performance of all three tests is low (less than 0.6) for sample sizes less than 150 but JB test performs better than AD and LI tests. JB and AD reached good power (0.6 or more) for sample size of at least 200 while LI requires sample size of at least 300 to reach good power. LI is the weakest test and requires much larger sample size to achieve comparable power with the other two tests.

Figure 4 represents the change in power for different sample sizes for all three tests against Beta(2, 5) distribution. Again the performance of all three tests is low for small sample sizes but in general AD test performs better than JB and LI tests. AD attained good power (0.6 or more) for sample size of 75 or more while JB and LI tests require sample size of 100 or more in order to attain good power.

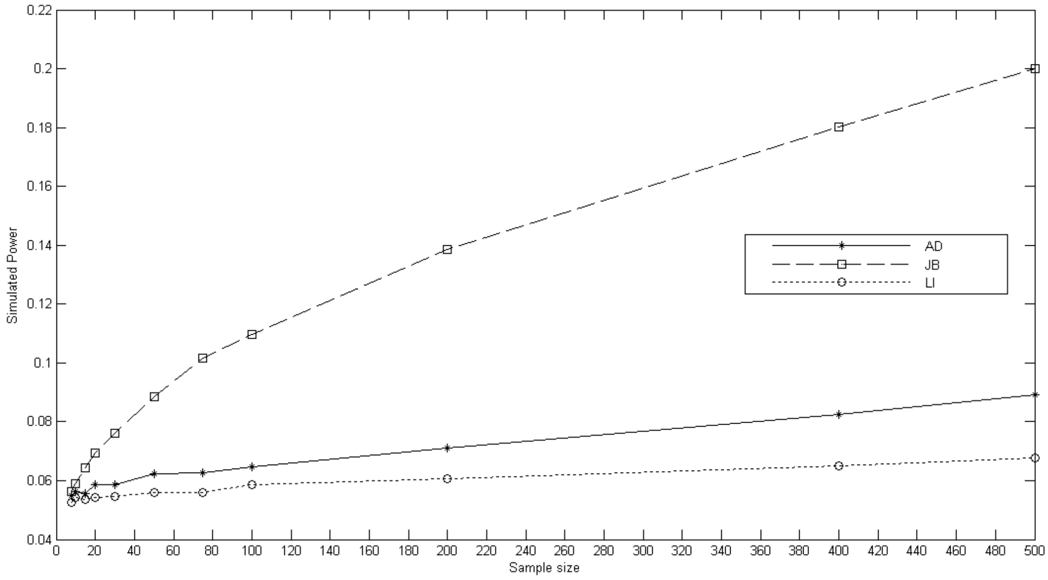

Figure 2. Comparison of simulated power for three normality tests against *t*-distribution with 30 degrees of freedom (skewness = 0.00, kurtosis = 3.23).

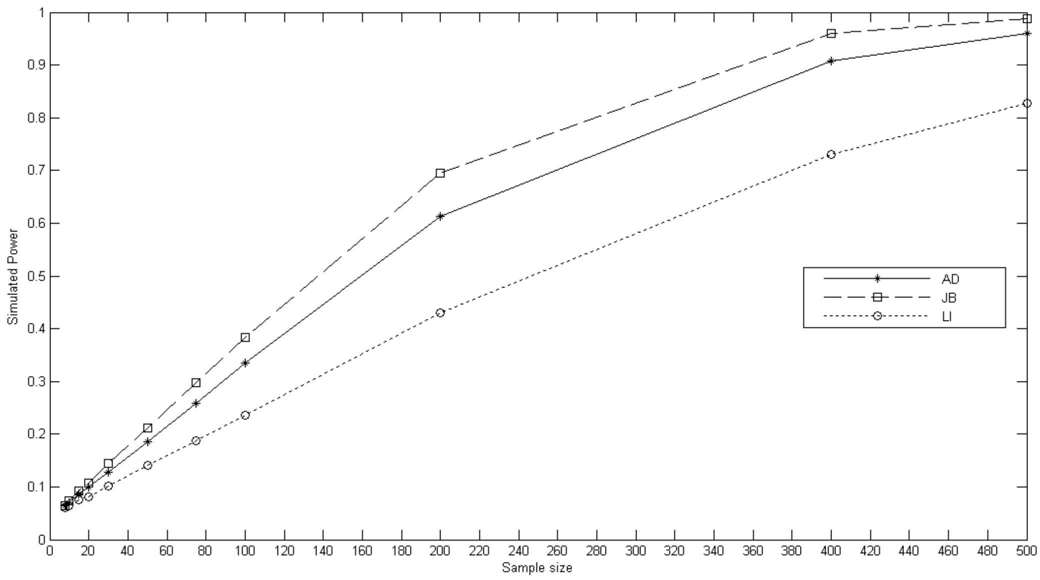

Figure 3. Comparison of simulated power for three normality tests against chi-squared distribution with 30 degrees of freedom ((skewness = 0.51, kurtosis = 3.40).

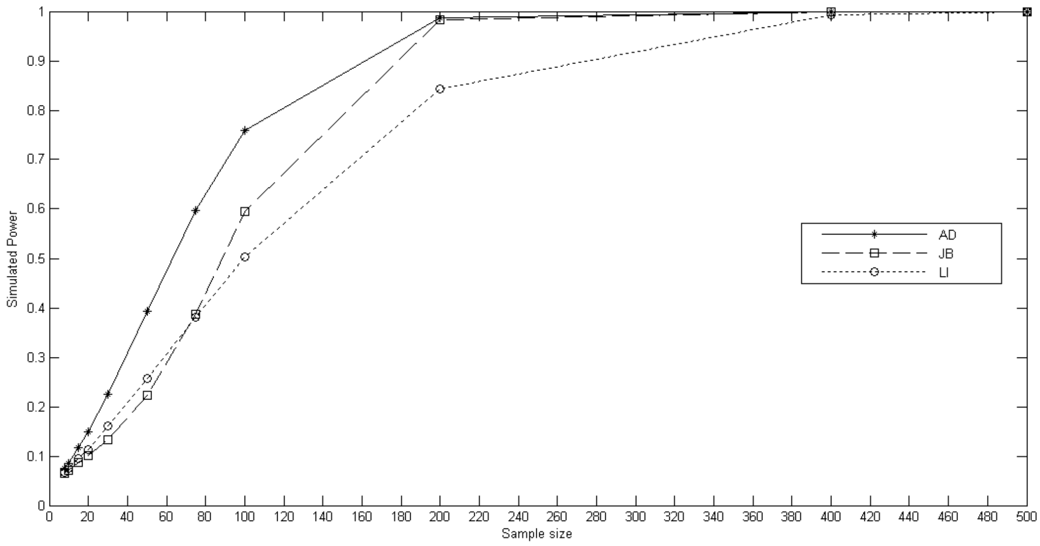

Figure 4. Comparison of simulated power for three normality tests against Beta (2, 5) distribution ((skewness = 0.6, kurtosis = 2.88).

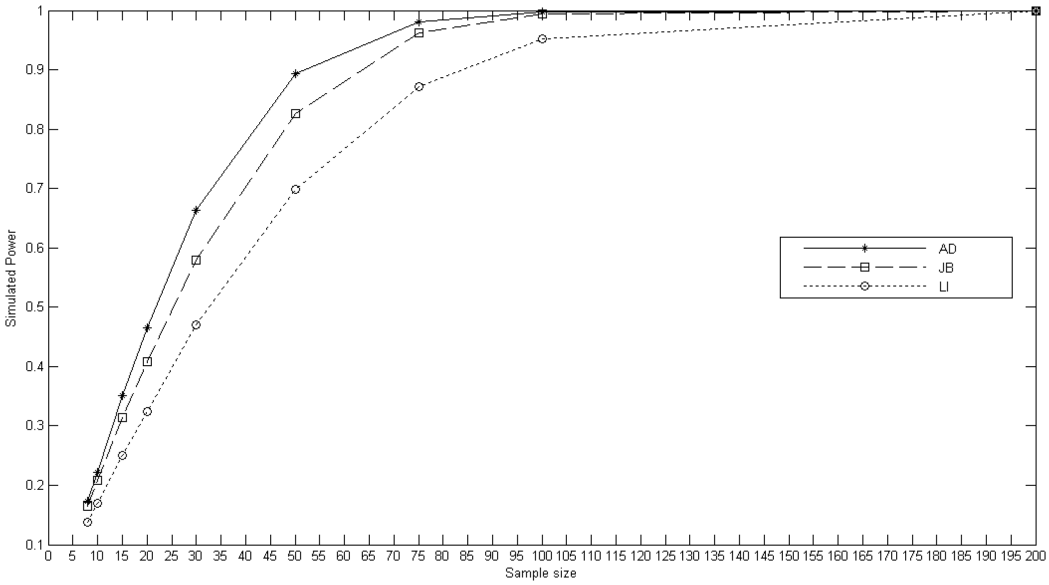

Figure 5. Comparison of simulated power for three normality tests against gamma (2, 3) distribution ((skewness = 1.41, kurtosis = 6.00).

Figure 5 demonstrates the plot of power for the three tests against Gamma(2, 3) distribution for $\alpha = 0.05$. It is obvious that the performance of all tests is low for small sample sizes but AD and JB test perform better than LI test. AD and JB together attained good power (0.6 or more) for sample size of at least 30 while LI requires sample size of at least 40 to attain good power.

Figure 6 shows the plot of power for the three tests against $\chi^2(3)$ distribution for $\alpha = 0.05$. It is clear that for small sample sizes (15 or less) the performance of all three tests is low (less than 0.6). All three tests attained good power (0.6 or more) for sample size of at least 25. Figure 7 above displays the plot of power for the three tests against Wiebull(1.5, 1) distribution for $\alpha = 0.05$ while figure 8 below displays the plot of power for the three tests against Exponential(0.5) distribution for $\alpha = 0.05$. It is clear from these plots that the performance of all tests is low for sample sizes less than 15 but in general AD test perform better than JB and LI tests. AD attained good power (0.6 or more) for sample size of 15 or more while JB and LI tests require sample size of 20 or more to attain good power.

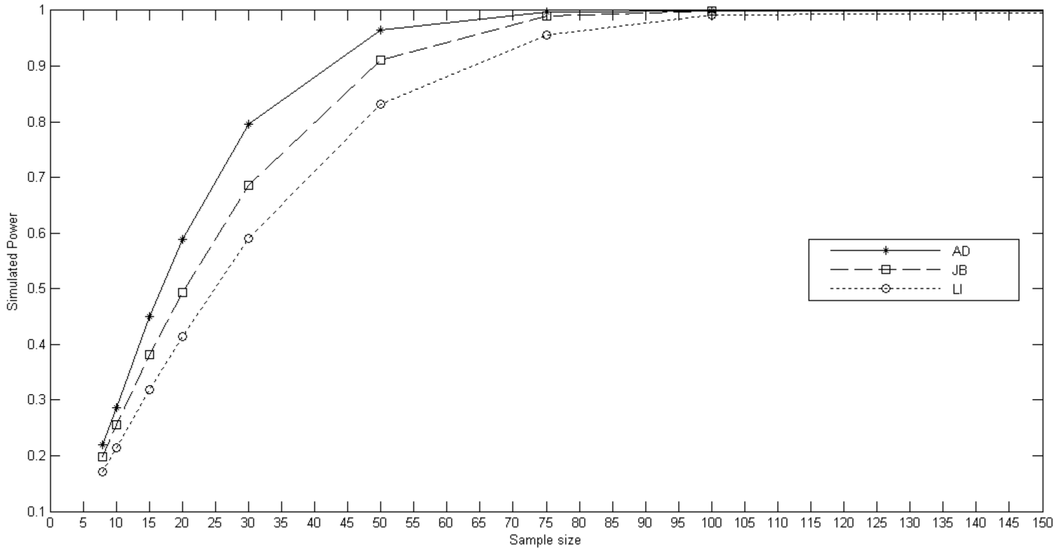

Figure 6. Comparison of simulated power for three normality tests against chi-squared distribution with 3 degrees of freedom ((skewness = 1.63, kurtosis = 7.00).

Conclusion

The results of 100,000 simulation runs showed that;

1. When the distribution is standard normal, any of the three tests can be used to compare Type I error rates.

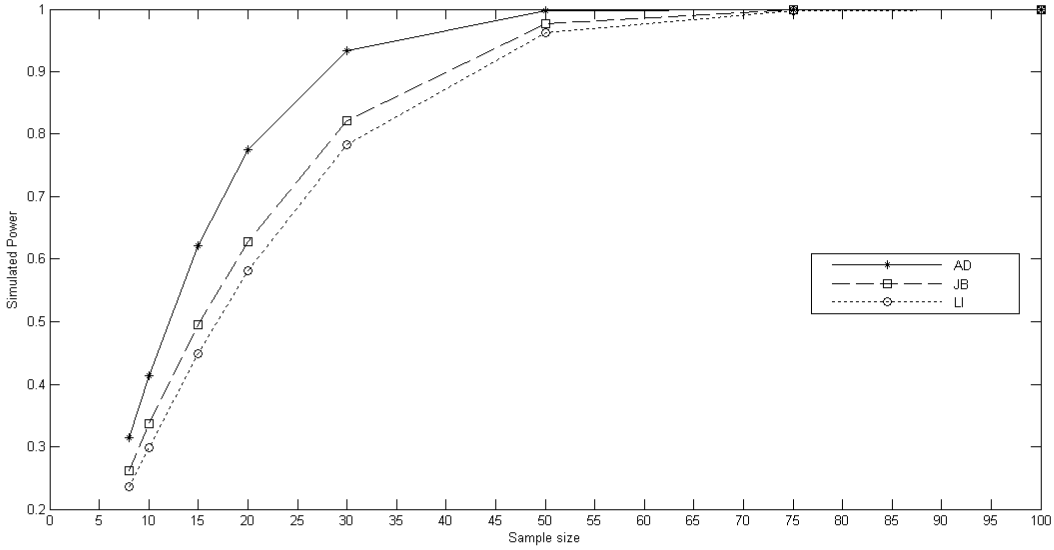

Figure 7. Comparison of simulated power for three normality tests against Wiebull (1.5, 1) distribution ((skewness = 2.00, kurtosis = 9.00).

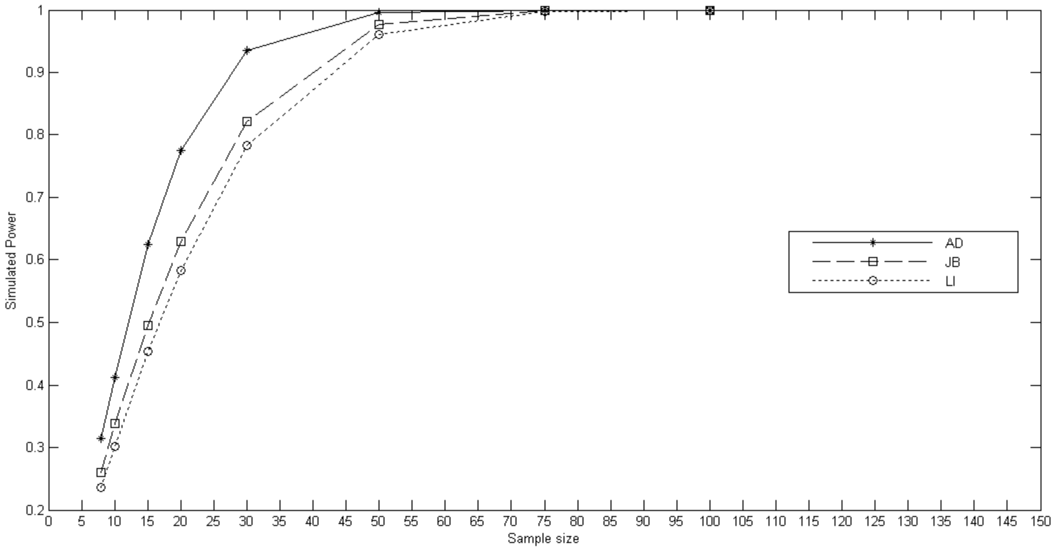

Figure 8. Comparison of simulated power for three normality tests against Exponential (0.5) distribution ((skewness = 2.00, kurtosis = 9.00).

1. Regardless of the distribution and sample size, AD test gaves higher power levels than the other two tests.
2. In all situations, LI test achieved smallest power levels.

3. JB and LI tests performed similar in samples with all distributions except for *t*(30) distribution.
4. All tests were more powerful when used on data with exponential distribution (0.50) and Weibull distribution (1.5, 1) for sample sizes 30 or more.

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Appendix

Type I error rate and power of tests for different distributions and sample sizes

Sample size	Distribution					
	N(0, 1) (Skewness = 0.00, Kurtosis = 3.00)			t(30) (Skewness = 0.00, Kurtosis = 3.23)		
	Type I error rate			Power of test		
	AD	JB	LI	AD	JB	LI
7	0.0496	0.0490	0.0502	0.0535	0.0547	0.0531
8	0.0494	0.0482	0.0498	0.0551	0.0563	0.0526
9	0.0502	0.0491	0.0492	0.0561	0.0587	0.0553
10	0.0485	0.0492	0.0494	0.0561	0.0591	0.0544
15	0.0501	0.0508	0.0500	0.0556	0.0643	0.0537
20	0.0510	0.0505	0.0504	0.0585	0.0695	0.0544
25	0.0507	0.0508	0.0494	0.0592	0.0732	0.0559
30	0.0506	0.0508	0.0499	0.0588	0.0759	0.0547
35	0.0496	0.0499	0.0501	0.0617	0.0809	0.0560
40	0.0496	0.0501	0.0499	0.0606	0.0833	0.0558
45	0.0496	0.0505	0.0501	0.0623	0.0857	0.0562
50	0.0491	0.0499	0.0490	0.0624	0.0885	0.0558
55	0.0506	0.0499	0.0498	0.0630	0.0905	0.0565
60	0.0506	0.0505	0.0505	0.0638	0.0954	0.0575
65	0.0497	0.0495	0.0500	0.0631	0.0943	0.0571
70	0.0498	0.0489	0.0495	0.0645	0.0971	0.0568
75	0.0500	0.0508	0.0486	0.0627	0.1017	0.0561
80	0.0501	0.0495	0.0493	0.0633	0.1008	0.0566
85	0.0498	0.0507	0.0495	0.0633	0.1034	0.0567
90	0.0493	0.0505	0.0507	0.0653	0.1075	0.0579
95	0.0502	0.0501	0.0496	0.0661	0.1085	0.0580
100	0.0507	0.0502	0.0500	0.0645	0.1098	0.0588
150	0.0495	0.0495	0.0484	0.0690	0.1254	0.0598
200	0.0484	0.0494	0.0495	0.0711	0.1385	0.0607
250	0.0502	0.0494	0.0504	0.0738	0.1498	0.0616
300	0.0503	0.0500	0.0506	0.0778	0.1594	0.0633
350	0.0503	0.0502	0.0506	0.0800	0.1732	0.0648
400	0.0501	0.0503	0.0503	0.0825	0.1803	0.0652
450	0.0498	0.0505	0.0499	0.0839	0.1889	0.0656
500	0.0503	0.0498	0.0491	0.0892	0.2000	0.0677
750	0.0489	0.0503	0.0490	0.1055	0.2488	0.0754
1000	0.0505	0.0504	0.0508	0.1206	0.2858	0.0822
1250	0.0512	0.0506	0.0506	0.1394	0.3281	0.0927
1500	0.0507	0.0500	0.0512	0.1568	0.3682	0.0977
1750	0.0505	0.0504	0.0486	0.1740	0.4099	0.1073
2000	0.0497	0.0502	0.0510	0.1952	0.4439	0.1165

Sample size	Distribution					
	χ^2 (30) (Skewness = 0.51, Kurtosis = 3.40) Power of test			Beta(2, 5) (Skewness = 0.60, Kurtosis = 2.88) Power of test		
	AD	JB	LI	AD	JB	LI
7	0.0610	0.0614	0.0586	0.0684	0.0614	0.0624
8	0.0644	0.0647	0.0612	0.0745	0.0643	0.0674
9	0.0676	0.0693	0.0637	0.0793	0.0666	0.0712
10	0.0699	0.0738	0.0641	0.0859	0.0705	0.0758
15	0.0862	0.0921	0.0747	0.1164	0.0860	0.0956
20	0.0991	0.1076	0.0820	0.1480	0.1015	0.1139
25	0.1139	0.1265	0.0933	0.1844	0.1185	0.1357
30	0.1282	0.1452	0.1023	0.2245	0.1339	0.1604
35	0.1418	0.1615	0.1102	0.2639	0.1533	0.1826
40	0.1563	0.1788	0.1202	0.3080	0.1754	0.2085
45	0.1708	0.1943	0.1302	0.3510	0.1963	0.2300
50	0.1857	0.2113	0.1403	0.3931	0.2224	0.2573
55	0.2001	0.2289	0.1485	0.4375	0.2517	0.2820
60	0.2156	0.2475	0.1583	0.4782	0.2824	0.3070
65	0.2292	0.2623	0.1687	0.5209	0.3148	0.3338
70	0.2464	0.2810	0.1791	0.5578	0.3491	0.3576
75	0.2583	0.2982	0.1882	0.5966	0.3878	0.3810
80	0.2768	0.3145	0.1976	0.6311	0.4262	0.4057
85	0.2905	0.3310	0.2071	0.6680	0.4700	0.4323
90	0.3050	0.3485	0.2166	0.7020	0.5121	0.4580
95	0.3228	0.3673	0.2285	0.7266	0.5505	0.4759
100	0.3357	0.3835	0.2363	0.7593	0.5956	0.5037
150	0.4838	0.5558	0.3363	0.9324	0.8930	0.7035
200	0.6131	0.6941	0.4302	0.9860	0.9836	0.8421
250	0.7174	0.8038	0.5198	0.9977	0.9983	0.9216
300	0.8001	0.8804	0.5980	0.9997	0.9998	0.9634
350	0.8626	0.9288	0.6682	0.9999	1.0000	0.9845
400	0.9086	0.9605	0.7304	1.0000	1.0000	0.9938
450	0.9394	0.9775	0.7805	1.0000	1.0000	0.9978
500	0.9595	0.9875	0.8265	1.0000	1.0000	0.9992
750	0.9968	0.9997	0.9515	1.0000	1.0000	1.0000
1000	0.9997	1.0000	0.9877	1.0000	1.0000	1.0000
1250	1.0000	1.0000	0.9975	1.0000	1.0000	1.0000
1500	1.0000	1.0000	0.9995	1.0000	1.0000	1.0000
1750	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000
2000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Sample size	Distribution					
	Gamma(2, 3) (Skewness = 1.41, Kurtosis = 6.00) Power of test			χ^2 (3) (Skewness = 1.63, Kurtosis = 7.00) Power of test		
	AD	**JB**	**LI**	**AD**	**JB**	**LI**
7	**0.1455**	**0.1424**	**0.1205**	**0.1858**	**0.1730**	**0.1482**
8	**0.1731**	**0.1650**	**0.1384**	**0.2189**	**0.1973**	**0.1703**
9	**0.1993**	**0.1858**	**0.1564**	**0.2556**	**0.2256**	**0.1930**
10	**0.2212**	**0.2085**	**0.1689**	**0.2867**	**0.2553**	**0.2143**
15	**0.3516**	**0.3138**	**0.2502**	**0.4495**	**0.3822**	**0.3191**
20	**0.4647**	**0.4085**	**0.3235**	**0.5878**	**0.4932**	**0.4145**
25	**0.5695**	**0.4955**	**0.3942**	**0.7029**	**0.5919**	**0.5027**
30	**0.6640**	**0.5794**	**0.4696**	**0.7963**	**0.6864**	**0.5904**
35	**0.7381**	**0.6518**	**0.5323**	**0.8625**	**0.7571**	**0.6638**
40	**0.8026**	**0.7188**	**0.5925**	**0.9110**	**0.8213**	**0.7309**
45	**0.8518**	**0.7763**	**0.6495**	**0.9421**	**0.8705**	**0.7844**
50	**0.8930**	**0.8259**	**0.6989**	**0.9643**	**0.9101**	**0.8316**
55	**0.9208**	**0.8661**	**0.7402**	**0.9777**	**0.9381**	**0.8659**
60	**0.9440**	**0.9000**	**0.7826**	**0.9866**	**0.9599**	**0.8982**
65	**0.9612**	**0.9272**	**0.8172**	**0.9922**	**0.9737**	**0.9210**
70	**0.9731**	**0.9469**	**0.8467**	**0.9955**	**0.9840**	**0.9417**
75	**0.9811**	**0.9623**	**0.8721**	**0.9972**	**0.9903**	**0.9562**
80	**0.9865**	**0.9738**	**0.8933**	**0.9984**	**0.9939**	**0.9674**
85	**0.9911**	**0.9819**	**0.9115**	**0.9992**	**0.9967**	**0.9764**
90	**0.9938**	**0.9882**	**0.9270**	**0.9996**	**0.9980**	**0.9839**
95	**0.9960**	**0.9921**	**0.9431**	**0.9998**	**0.9990**	**0.9876**
100	**0.9973**	**0.9949**	**0.9524**	**0.9999**	**0.9996**	**0.9918**
150	**1.0000**	**1.0000**	**0.9960**	**1.0000**	**1.0000**	**0.9999**
200	**1.0000**	**1.0000**	**0.9998**	**1.0000**	**1.0000**	**1.0000**
250	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
300	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
350	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
400	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
450	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
500	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1250	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1500	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Sample size	Distribution					
	Wiebull(1.5, 1) (Skewness = 2.00, Kurtosis = 9.00) Power of test			Exp(0.5) (Skewness = 2.00, Kurtosis = 9.00) Power of test		
	AD	**JB**	**LI**	**AD**	**JB**	**LI**
7	**0.2638**	**0.2293**	**0.2046**	**0.2662**	**0.2302**	**0.2062**
8	**0.3138**	**0.2610**	**0.2355**	**0.3140**	**0.2602**	**0.2361**
9	**0.3649**	**0.2962**	**0.2701**	**0.3618**	**0.2963**	**0.2686**
10	**0.4144**	**0.3365**	**0.2991**	**0.4126**	**0.3391**	**0.3017**
15	**0.6222**	**0.4947**	**0.4486**	**0.6251**	**0.4958**	**0.4530**
20	**0.7749**	**0.6286**	**0.5815**	**0.7754**	**0.6295**	**0.5830**
25	**0.8759**	**0.7382**	**0.6949**	**0.8758**	**0.7387**	**0.6930**
30	**0.9343**	**0.8223**	**0.7835**	**0.9346**	**0.8217**	**0.7830**
35	**0.9663**	**0.8826**	**0.8520**	**0.9673**	**0.8845**	**0.8538**
40	**0.9849**	**0.9280**	**0.9027**	**0.9847**	**0.9295**	**0.9025**
45	**0.9928**	**0.9588**	**0.9385**	**0.9922**	**0.9573**	**0.9371**
50	**0.9970**	**0.9769**	**0.9616**	**0.9967**	**0.9763**	**0.9611**
55	**0.9988**	**0.9877**	**0.9769**	**0.9986**	**0.9878**	**0.9772**
60	**0.9995**	**0.9938**	**0.9862**	**0.9993**	**0.9939**	**0.9867**
65	**0.9998**	**0.9971**	**0.9923**	**0.9997**	**0.9969**	**0.9927**
70	**0.9999**	**0.9987**	**0.9956**	**0.9999**	**0.9987**	**0.9963**
75	**1.0000**	**0.9994**	**0.9976**	**1.0000**	**0.9994**	**0.9977**
80	**1.0000**	**0.9997**	**0.9988**	**1.0000**	**0.9998**	**0.9990**
85	**1.0000**	**0.9998**	**0.9995**	**1.0000**	**0.9999**	**0.9993**
90	**1.0000**	**1.0000**	**0.9997**	**1.0000**	**1.0000**	**0.9997**
95	**1.0000**	**1.0000**	**0.9999**	**1.0000**	**1.0000**	**0.9998**
100	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**0.9999**
150	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
200	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
250	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
300	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
350	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
400	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
450	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
500	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1250	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1500	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**

Sample size	Distribution					
	Beta(2, 5) (Skewness = 0.60, Kurtosis = 2.88) Power of test			χ^2 (3) (Skewness = 1.63, Kurtosis = 7.00) Power of test		
	AD	**JB**	**LI**	**AD**	**JB**	**LI**
7	**0.0684**	**0.0614**	**0.0624**	**0.1858**	**0.1730**	**0.1482**
8	**0.0745**	**0.0643**	**0.0674**	**0.2189**	**0.1973**	**0.1703**
9	**0.0793**	**0.0666**	**0.0712**	**0.2556**	**0.2256**	**0.1930**
10	**0.0859**	**0.0705**	**0.0758**	**0.2867**	**0.2553**	**0.2143**
15	**0.1164**	**0.0860**	**0.0956**	**0.4495**	**0.3822**	**0.3191**
20	**0.1480**	**0.1015**	**0.1139**	**0.5878**	**0.4932**	**0.4145**
25	**0.1844**	**0.1185**	**0.1357**	**0.7029**	**0.5919**	**0.5027**
30	**0.2245**	**0.1339**	**0.1604**	**0.7963**	**0.6864**	**0.5904**
35	**0.2639**	**0.1533**	**0.1826**	**0.8625**	**0.7571**	**0.6638**
40	**0.3080**	**0.1754**	**0.2085**	**0.9110**	**0.8213**	**0.7309**
45	**0.3510**	**0.1963**	**0.2300**	**0.9421**	**0.8705**	**0.7844**
50	**0.3931**	**0.2224**	**0.2573**	**0.9643**	**0.9101**	**0.8316**
55	**0.4375**	**0.2517**	**0.2820**	**0.9777**	**0.9381**	**0.8659**
60	**0.4782**	**0.2824**	**0.3070**	**0.9866**	**0.9599**	**0.8982**
65	**0.5209**	**0.3148**	**0.3338**	**0.9922**	**0.9737**	**0.9210**
70	**0.5578**	**0.3491**	**0.3576**	**0.9955**	**0.9840**	**0.9417**
75	**0.5966**	**0.3878**	**0.3810**	**0.9972**	**0.9903**	**0.9562**
80	**0.6311**	**0.4262**	**0.4057**	**0.9984**	**0.9939**	**0.9674**
85	**0.6680**	**0.4700**	**0.4323**	**0.9992**	**0.9967**	**0.9764**
90	**0.7020**	**0.5121**	**0.4580**	**0.9996**	**0.9980**	**0.9839**
95	**0.7266**	**0.5505**	**0.4759**	**0.9998**	**0.9990**	**0.9876**
100	**0.7593**	**0.5956**	**0.5037**	**0.9999**	**0.9996**	**0.9918**
150	**0.9324**	**0.8930**	**0.7035**	**1.0000**	**1.0000**	**0.9999**
200	**0.9860**	**0.9836**	**0.8421**	**1.0000**	**1.0000**	**1.0000**
250	**0.9977**	**0.9983**	**0.9216**	**1.0000**	**1.0000**	**1.0000**
300	**0.9997**	**0.9998**	**0.9634**	**1.0000**	**1.0000**	**1.0000**
350	**0.9999**	**1.0000**	**0.9845**	**1.0000**	**1.0000**	**1.0000**
400	**1.0000**	**1.0000**	**0.9938**	**1.0000**	**1.0000**	**1.0000**
450	**1.0000**	**1.0000**	**0.9978**	**1.0000**	**1.0000**	**1.0000**
500	**1.0000**	**1.0000**	**0.9992**	**1.0000**	**1.0000**	**1.0000**
750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1250	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1500	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
1750	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2000	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**